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Tron Omland

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HOW MANY PYTHAGOREAN TRIPLES WITH A GIVEN INRADIUS?

TRON OMLAND

ABSTRACT. We present a very short proof to answer the question of the title.

If a circle with radius $r > 0$ is inscribed in a right-angled triangle with catheti a and b (whose lengths must be $> 2r$), then $2A = rP$, where A and P are the triangle's area and perimeter, that is, $ab = r(a + b + \sqrt{a^2 + b^2})$. For a given radius r we find all the possible right-angled triangles with inradius r by manipulating this equation, and the solution is:

For each radius $r > 0$ and each cathetus $a > 2r$, the other cathetus b is given by

$$b = 2r \cdot \frac{a - r}{a - 2r}. \tag{1}$$

Now set $a = 2r + m$ for $m > 0$ so that $b = 2r + n$ for $n > 0$ with $mn = 2r^2$. Then all right-angled triangles with inradius r has edges with lengths $(2r + m, 2r + n, 2r + (m + n))$ for some $m, n > 0$ with $mn = 2r^2$.

Therefore, given a natural number r, the possible Pythagorean triples with inradius r coincide with the possible ways of factoring $2r^2$ into a product of two numbers m and n.

Lemma. *Let $r = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ for distinct odd primes $p_1, p_2, \dots, p_n$ and integers $\alpha_0 \geq 0$ and $\alpha_1, \alpha_2, \dots, \alpha_n \geq 1$. Then there are*

$$(\alpha_0 + 1)(2\alpha_1 + 1)(2\alpha_2 + 1) \cdots (2\alpha_n + 1)$$

Pythagorean triples with inradius r, and 2^n of these triples are primitive.
The formula also holds for $n = 0$, i.e., when $r = 2^{\alpha_0}$ for some $\alpha_0 \geq 0$.

Proof. To distinguish between m and n, we factor $2r^2$ as $m \cdot n$ such the highest power of 2 that divides m and n is odd and even, respectively. That is, $m = 2^{\beta_0} p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$, where $\beta_0 \in \{1, 3, \dots, 2\alpha_0 + 1\}$ and $\beta_i \in \{0, 1, \dots 2\alpha_i\}$ for all i.

The primitive triples appear precisely when $\gcd(m, n) = 1$, i.e. for $m = 2^{2\alpha_0+1} p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$, where $\beta_i \in \{0, 2\alpha_i\}$ for all i. $\square$

The result for *primitive* triples is well-known [1], but our proof is simpler also in this case.

Finally, we remark that by solving (1) with respect to r, we get that the inradius r and catheti a, b of a right-angled triangle satisfy

$$r = \frac{a + b - \sqrt{a^2 + b^2}}{2}.$$

If $a, b, \sqrt{a^2 + b^2}$ are all natural numbers, then either none or two of these are odd, so r is also a natural number. Hence, the technique of the above proof generates all Pythagorean triples.

REFERENCES

[1] Neville Robbins. On the number of primitive Pythagorean triangles with a given inradius. *Fibonacci Quart.*, 44(4):368–369, 2006.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, P.O.BOX 1053 BLINDERN, NO-0316 OSLO, NORWAY
E-mail address: trono@math.uio.no

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