

Accepted Manuscript

How many Pythagorean triples with a given inradius?

Tron Omland

PII: S0022-314X(16)30160-3
DOI: <http://dx.doi.org/10.1016/j.jnt.2016.06.009>
Reference: YJNTH 5506

To appear in: *Journal of Number Theory*

Received date: 22 May 2016

Accepted date: 26 June 2016

Please cite this article in press as: T. Omland, How many Pythagorean triples with a given inradius?, *J. Number Theory* (2016), <http://dx.doi.org/10.1016/j.jnt.2016.06.009>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



HOW MANY PYTHAGOREAN TRIPLES WITH A GIVEN INRADIUS?

TRON OMLAND

ABSTRACT. We present a very short proof to answer the question of the title.

If a circle with radius $r > 0$ is inscribed in a right-angled triangle with catheti a and b (whose lengths must be $> 2r$), then $2A = rP$, where A and P are the triangle's area and perimeter, that is, $ab = r(a + b + \sqrt{a^2 + b^2})$. For a given radius r we find all the possible right-angled triangles with inradius r by manipulating this equation, and the solution is:

For each radius $r > 0$ and each cathetus $a > 2r$, the other cathetus b is given by

$$(1) \quad b = 2r \cdot \frac{a - r}{a - 2r}.$$

Now set $a = 2r + m$ for $m > 0$ so that $b = 2r + n$ for $n > 0$ with $mn = 2r^2$. Then all right-angled triangles with inradius r has edges with lengths $(2r + m, 2r + n, 2r + (m + n))$ for some $m, n > 0$ with $mn = 2r^2$.

Therefore, given a natural number r , the possible Pythagorean triples with inradius r coincide with the possible ways of factoring $2r^2$ into a product of two numbers m and n .

Lemma. Let $r = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ for distinct odd primes p_1, p_2, \dots, p_n and integers $\alpha_0 \geq 0$ and $\alpha_1, \alpha_2, \dots, \alpha_n \geq 1$. Then there are

$$(\alpha_0 + 1)(2\alpha_1 + 1)(2\alpha_2 + 1) \cdots (2\alpha_n + 1)$$

Pythagorean triples with inradius r , and 2^n of these triples are primitive.

The formula also holds for $n = 0$, i.e., when $r = 2^{\alpha_0}$ for some $\alpha_0 \geq 0$.

Proof. To distinguish between m and n , we factor $2r^2$ as $m \cdot n$ such the highest power of 2 that divides m and n is odd and even, respectively. That is, $m = 2^{\beta_0} p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$, where $\beta_0 \in \{1, 3, \dots, 2\alpha_0 + 1\}$ and $\beta_i \in \{0, 1, \dots, 2\alpha_i\}$ for all i .

The primitive triples appear precisely when $\gcd(m, n) = 1$, i.e. for $m = 2^{2\alpha_0+1} p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$, where $\beta_i \in \{0, 2\alpha_i\}$ for all i . \square

The result for *primitive* triples is well-known [1], but our proof is simpler also in this case.

Finally, we remark that by solving (1) with respect to r , we get that the inradius r and catheti a, b of a right-angled triangle satisfy

$$r = \frac{a + b - \sqrt{a^2 + b^2}}{2}.$$

If $a, b, \sqrt{a^2 + b^2}$ are all natural numbers, then either none or two of these are odd, so r is also a natural number. Hence, the technique of the above proof generates all Pythagorean triples.

REFERENCES

- [1] Neville Robbins. On the number of primitive Pythagorean triangles with a given inradius. *Fibonacci Quart.*, 44(4):368–369, 2006.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, P.O.BOX 1053 BLINDERN, NO-0316 OSLO, NORWAY
E-mail address: trono@math.uio.no

Date: July 29, 2016.

2010 Mathematics Subject Classification. 11A05.