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## HOW MANY PYTHAGOREAN TRIPLES WITH A GIVEN INRADIUS?

TRON OMLAND

ABSTRACT. We present a very short proof to answer the question of the title.

If a circle with radius  $r > 0$  is inscribed in a right-angled triangle with catheti  $a$  and  $b$  (whose lengths must be  $> 2r$ ), then  $2A = rP$ , where  $A$  and  $P$  are the triangle's area and perimeter, that is,  $ab = r(a + b + \sqrt{a^2 + b^2})$ . For a given radius  $r$  we find all the possible right-angled triangles with inradius  $r$  by manipulating this equation, and the solution is:

For each radius  $r > 0$  and each cathetus  $a > 2r$ , the other cathetus  $b$  is given by

$$(1) \quad b = 2r \cdot \frac{a - r}{a - 2r}.$$

Now set  $a = 2r + m$  for  $m > 0$  so that  $b = 2r + n$  for  $n > 0$  with  $mn = 2r^2$ . Then all right-angled triangles with inradius  $r$  has edges with lengths  $(2r + m, 2r + n, 2r + (m + n))$  for some  $m, n > 0$  with  $mn = 2r^2$ .

Therefore, given a natural number  $r$ , the possible Pythagorean triples with inradius  $r$  coincide with the possible ways of factoring  $2r^2$  into a product of two numbers  $m$  and  $n$ .

**Lemma.** Let  $r = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$  for distinct odd primes  $p_1, p_2, \dots, p_n$  and integers  $\alpha_0 \geq 0$  and  $\alpha_1, \alpha_2, \dots, \alpha_n \geq 1$ . Then there are

$$(\alpha_0 + 1)(2\alpha_1 + 1)(2\alpha_2 + 1) \cdots (2\alpha_n + 1)$$

Pythagorean triples with inradius  $r$ , and  $2^n$  of these triples are primitive. The formula also holds for  $n = 0$ , i.e., when  $r = 2^{\alpha_0}$  for some  $\alpha_0 \geq 0$ .

*Proof.* To distinguish between  $m$  and  $n$ , we factor  $2r^2$  as  $m \cdot n$  such the highest power of 2 that divides  $m$  and  $n$  is odd and even, respectively. That is,  $m = 2^{\beta_0} p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$ , where  $\beta_0 \in \{1, 3, \dots, 2\alpha_0 + 1\}$  and  $\beta_i \in \{0, 1, \dots, 2\alpha_i\}$  for all  $i$ .

The primitive triples appear precisely when  $\gcd(m, n) = 1$ , i.e. for  $m = 2^{2\alpha_0 + 1} p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$ , where  $\beta_i \in \{0, 2\alpha_i\}$  for all  $i$ .  $\square$

The result for *primitive* triples is well-known [1], but our proof is simpler also in this case.

Finally, we remark that by solving (1) with respect to  $r$ , we get that the inradius  $r$  and catheti  $a, b$  of a right-angled triangle satisfy

$$r = \frac{a + b - \sqrt{a^2 + b^2}}{2}.$$

If  $a, b, \sqrt{a^2 + b^2}$  are all natural numbers, then either none or two of these are odd, so  $r$  is also a natural number. Hence, the technique of the above proof generates all Pythagorean triples.

## REFERENCES

- [1] Neville Robbins. On the number of primitive Pythagorean triangles with a given inradius. *Fibonacci Quart.*, 44(4):368–369, 2006.

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