Proof of some divisibility results on sums involving binomial coefficients

Ji-Cai Liu

PII: S0022-314X(17)30208-1
DOI: http://dx.doi.org/10.1016/j.jnt.2017.05.006
Reference: YJNTH 5771

To appear in: *Journal of Number Theory*

Received date: 15 January 2017
Revised date: 26 April 2017
Accepted date: 11 May 2017

Proof of some divisibility results on sums involving binomial coefficients

Ji-Cai Liu

College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, PR China
jc2051@163.com

Abstract. By using the Rodriguez-Villegas-Mortenson supercongruences, we prove four supercongruences on sums involving binomial coefficients, which were originally conjectured by Sun. We also confirm a related conjecture of Guo on integer-valued polynomials.

Keywords: Supercongruences; Delannoy number; Legendre symbol; Zeilberger algorithm

MR Subject Classifications: Primary 11A07; Secondary 33C05

1 Introduction

In 2003, Rodriguez-Villegas [11] conjectured 22 supercongruences for hypergeometric Calabi-Yau manifolds of dimension $d \le 3$. For manifolds of dimension $d = 1$, associated to certain elliptic curves, four conjectural supercongruences were posed. Mortenson [8, 9] first proved these four supercongruences by using the Gross-Koblitz formula.

Theorem 1.1 *(Rodriguez-Villegas-Mortenson) Suppose $p \ge 5$ is a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(1/2)_k^2}{(1)_k^2} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} \frac{(1/3)_k(2/3)_k}{(1)_k^2} \equiv \left(\frac{-3}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{(1/4)_k(3/4)_k}{(1)_k^2} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} \frac{(1/6)_k(5/6)_k}{(1)_k^2} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol and $(x)_k = x(x+1)\cdots(x+k-1)$.

Sun [12] introduced the following two kinds of polynomials:

$$d_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} 2^k \quad \text{and} \quad s_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}\binom{x+k}{k}.$$

Note that $d_n(m)$ are the Delannoy numbers, which count the number of paths from $(0, 0)$ to (m, n), only using steps $(1, 0), (0, 1)$ and $(1, 1)$. For more information on Delannoy numbers, one can refer to [2].

The first aim of this paper is to prove the following result, which was originally conjectured by Sun [12, Conjecture 6.11].

Theorem 1.2 *Suppose $p \geq 5$ is a prime. Then*

$$\sum_{k=0}^{p-1}(2k+1)s_k\left(-\frac{1}{2}\right)^2 \equiv \frac{3}{4}\left(\frac{-1}{p}\right)p^2 \pmod{p^4}, \tag{1.1}$$

$$\sum_{k=0}^{p-1}(2k+1)s_k\left(-\frac{1}{3}\right)^2 \equiv \frac{7}{9}\left(\frac{-3}{p}\right)p^2 \pmod{p^4}, \tag{1.2}$$

$$\sum_{k=0}^{p-1}(2k+1)s_k\left(-\frac{1}{4}\right)^2 \equiv \frac{13}{16}\left(\frac{-2}{p}\right)p^2 \pmod{p^4}, \tag{1.3}$$

$$\sum_{k=0}^{p-1}(2k+1)s_k\left(-\frac{1}{6}\right)^2 \equiv \frac{31}{36}\left(\frac{-1}{p}\right)p^2 \pmod{p^4}. \tag{1.4}$$

Recently, Guo [5, Theorem 5.1] showed that for any odd prime p and p-adic integer x,

$$\sum_{k=0}^{p-1}(2k+1)s_k(x)^2 \equiv p^2\sum_{k=0}^{p-1}\sum_{j=0}^{k}\frac{(-1)^k}{k+1}\binom{x+k}{2k}\binom{x}{j}\binom{x+j}{j}\binom{2k}{j+k} \pmod{p^4}. \tag{1.5}$$

Recall that a polynomial $P(x)$ with real coefficients is called *integer-valued*, if $P(x)$ takes integer values for all $x \in \mathbb{Z}$. The second aim of this paper is the prove the following result, which was originally conjectured by Guo [5, Conjecture 5.5].

Theorem 1.3 *Let n and m be positive integers and $\varepsilon = \pm 1$. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1}\varepsilon^k(2k+1)d_k(x)^m s_k(x)^m \tag{1.6}$$

is integer-valued.

2 Proof of Theorem 1.2

We need the following lemma (see [7, Theorem 1.3]).

Lemma 2.1 *Suppose $p \geq 5$ is a prime. Then*

$$\sum_{k=0}^{2p-1}\frac{(1/2)_k^2}{(1)_k^2} \equiv \frac{5}{4}\left(\frac{-1}{p}\right) \pmod{p^2}, \quad \sum_{k=0}^{2p-1}\frac{(1/3)_k(2/3)_k}{(1)_k^2} \equiv \frac{11}{9}\left(\frac{-3}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{2p-1}\frac{(1/4)_k(3/4)_k}{(1)_k^2} \equiv \frac{19}{16}\left(\frac{-2}{p}\right) \pmod{p^2}, \quad \sum_{k=0}^{2p-1}\frac{(1/6)_k(5/6)_k}{(1)_k^2} \equiv \frac{41}{36}\left(\frac{-1}{p}\right) \pmod{p^2}.$$

Proof of Theorem 1.2. We begin with the following identity [4, (2.5)]:

$$\binom{x}{k}\binom{x+k}{k}\binom{x}{j}\binom{x+j}{j}=\sum_{s=0}^{j+k}\binom{j+k}{s}\binom{s}{j}\binom{s}{k}\binom{x}{s}\binom{x+s}{s}. \tag{2.1}$$

Note that

$$\binom{x+k}{2k}\binom{x}{j}\binom{x+j}{j}=\binom{x}{k}\binom{x+k}{k}\binom{x}{j}\binom{x+j}{j}\Big/\binom{2k}{k}. \tag{2.2}$$

Substituting (2.1) and (2.2) into the right-hand side of (1.5) and then exchanging the summation order gives

$$\begin{aligned}&\sum_{k=0}^{p-1}(2k+1)s_k(x)^2\\&\equiv p^2\sum_{s=0}^{2p-2}\sum_{k=0}^{p-1}\sum_{j=0}^{k}\frac{(-1)^k}{k+1}\binom{2k}{j+k}\binom{j+k}{s}\binom{s}{j}\binom{s}{k}\binom{x}{s}\binom{x+s}{s}\Big/\binom{2k}{k} \pmod{p^4}.\end{aligned} \tag{2.3}$$

By the Chu-Vandermonde identity, we have

$$\sum_{j=0}^{k}\binom{2k}{j+k}\binom{j+k}{s}\binom{s}{j}=\sum_{j=0}^{k}\binom{2k}{s}\binom{s}{j}\binom{2k-s}{k-j}=\binom{2k}{s}\binom{2k}{k}. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\sum_{k=0}^{p-1}(2k+1)s_k(x)^2\equiv p^2\sum_{s=0}^{2p-2}\sum_{k=0}^{p-1}\frac{(-1)^k}{k+1}\binom{2k}{s}\binom{s}{k}\binom{x}{s}\binom{x+s}{s} \pmod{p^4}. \tag{2.5}$$

Applying the following identity [6, (2.6)]:

$$\sum_{k=0}^{s}\frac{(-1)^k}{k+1}\binom{2k}{s}\binom{s}{k}=(-1)^s,$$

we obtain

$$p^2\sum_{s=0}^{p-1}\sum_{k=0}^{p-1}\frac{(-1)^k}{k+1}\binom{2k}{s}\binom{s}{k}\binom{x}{s}\binom{x+s}{s}=p^2\sum_{s=0}^{p-1}(-1)^s\binom{x}{s}\binom{x+s}{s}. \tag{2.6}$$

On the other hand, we have the following supercongruence [6, (3.2)]:

$$\sum_{k=0}^{p-1}\frac{(-1)^k}{k+1}\binom{2k}{s}\binom{s}{k}\equiv(-1)^s\left(-1+\frac{2p}{s+1}\right) \pmod{p^2} \tag{2.7}$$

for $p \le s \le 2p-2$. It follows that

$$
\begin{aligned}
&p^2 \sum_{s=p}^{2p-2} \sum_{k=0}^{p-1} \frac{(-1)^k}{k+1} \binom{2k}{s} \binom{s}{k} \binom{x}{s} \binom{x+s}{s} \\
&\equiv 2p^3 \sum_{s=p}^{2p-1} \frac{(-1)^s}{s+1} \binom{x}{s} \binom{x+s}{s} - p^2 \sum_{s=p}^{2p-1} (-1)^s \binom{x}{s} \binom{x+s}{s} \pmod{p^4}, \qquad (2.8)
\end{aligned}
$$

where we have used the fact that $\binom{x}{s}\binom{x+s}{s} \equiv 0 \pmod{p^2}$ for $s = 2p-1$ and $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$.

Note that

$$
\sum_{s=0}^{n-1} \frac{(-1)^s}{s+1} \binom{x}{s} \binom{x+s}{s} = \frac{n(-1)^{n+1}}{x(x+1)} \binom{x}{n} \binom{x+n}{n},
$$

which can be easily proved by induction on n. So we have

$$
\begin{aligned}
\sum_{s=p}^{2p-1} \frac{(-1)^s}{s+1} \binom{x}{s} \binom{x+s}{s} &= \sum_{s=0}^{2p-1} \frac{(-1)^s}{s+1} \binom{x}{s} \binom{x+s}{s} - \sum_{s=0}^{p-1} \frac{(-1)^s}{s+1} \binom{x}{s} \binom{x+s}{s} \\
&\equiv 0 \pmod{p} \qquad (2.9)
\end{aligned}
$$

for $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$.

Furthermore, combining (2.5), (2.6), (2.8) and (2.9), we get

$$
\begin{aligned}
&\sum_{k=0}^{p-1} (2k+1) s_k(x)^2 \\
&\equiv p^2 \left(2 \sum_{s=0}^{p-1} (-1)^s \binom{x}{s} \binom{x+s}{s} - \sum_{s=0}^{2p-1} (-1)^s \binom{x}{s} \binom{x+s}{s} \right) \pmod{p^4} \qquad (2.10)
\end{aligned}
$$

for $x = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$.

Finally, noting that

$$
(-1)^s \binom{x}{s} \binom{x+s}{s} = \frac{(-x)_s (1+x)_s}{(1)_s^2},
$$

and then using Theorem 1.1 and Lemma 2.1, we obtain (1.1)-(1.4). This completes the proof. □

3 Proof of Theorem 1.3

Chen and Guo [3] defined the following multi-variable Schmidt polynomials:

$$
S_n(x_0, \cdots, x_n) = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x_k.
$$

The following lemma is a special case of [3, Theorem 1.1]. It has already been used by Guo [5] to prove Sun's conjectures on integer-valued polynomials.

Lemma 3.1 *Let n and m be positive integers. Then all the coefficients in*

$$\sum_{k=0}^{n-1} \varepsilon^k (2k+1) S_k(x_0, \cdots, x_k)^m$$

are multiples of n.

Lemma 3.2 *For any non-negative integer n, we have*

$$d_n(x)s_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k} \sum_{j=0}^{k}\sum_{i=0}^{j} \binom{x+j}{k+j}\binom{x}{i}\binom{k}{j}\binom{j}{i} 2^i. \tag{3.1}$$

Proof. Since both sides of (3.1) are polynomials in x of degree $3n$, it suffices to prove that for any non-negative integers n and m,

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{j=0}^{n} \binom{n}{i}\binom{m}{i}\binom{n}{j}\binom{m}{j}\binom{m+j}{j} 2^i \\
&= \sum_{k=0}^{n}\sum_{j=0}^{n}\sum_{i=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\binom{m+j}{k+j}\binom{m}{i}\binom{k}{j}\binom{j}{i} 2^i.
\end{aligned} \tag{3.2}$$

Clearly, (3.2) is equivalent to

$$\begin{aligned}
&\sum_{i=0}^{m}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{i}\binom{n}{j}\binom{m}{j}\binom{m+j}{j} 2^i \\
&= \sum_{k=0}^{m}\sum_{j=0}^{m}\sum_{i=0}^{m} \binom{n+k}{2k}\binom{2k}{k}\binom{m+j}{k+j}\binom{m}{i}\binom{k}{j}\binom{j}{i} 2^i.
\end{aligned} \tag{3.3}$$

Let $A_m^{(1)}(n)$ and $A_m^{(2)}(n)$ denote the left-hand side and the right-hand side of (3.3), respectively. Applying the multi-Zeilberger algorithm [1, 10], we find that $A_m^{(1)}(n)$ and $A_m^{(2)}(n)$ satisfy the same recurrence of order 4:

$$\begin{aligned}
&(m+1)^3(m+2)(3m^2+18m+26)A_m^{(s)}(n) - 2(m+2)(12m^3n^2+12m^3n+90m^2n^2 \\
&+ 3m^3+90m^2n+212mn^2+23m^2+212mn+156n^2+55m+156n+41)A_{m+1}^{(s)}(n) \\
&- 2(3m^6+30m^4n^2+45m^5+30m^4n+300m^3n^2+287m^4+300m^3n+1094m^2n^2 \\
&+ 995m^3+1094m^2n+1720mn^2+1964m^2+1720mn+978n^2+2070m+978n \\
&+ 898)A_{m+2}^{(s)}(n) - 2(m+3)(12m^3n^2+12m^3n+90m^2n^2+3m^3+90m^2n+212mn^2 \\
&+ 22m^2+212mn+154n^2+50m+154n+34)A_{m+3}^{(s)}(n) + (m+3)(m+4)^3 \\
&\times (3m^2+12m+11)A_{m+4}^{(s)}(n) = 0, \quad \text{for } s = 1, 2.
\end{aligned}$$

It is easily checked that $A_m^{(1)}(n) = A_m^{(2)}(n)$ for $0 \le m \le 3$. This proves (3.3). $\square$

Proof of Theorem 1.3. We can rewrite (3.1) as

$$d_n(x)s_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k} f_k(x), \tag{3.4}$$

where

$$f_k(x) = \sum_{j=0}^{k}\sum_{i=0}^{j} \binom{x+j}{k+j}\binom{x}{i}\binom{k}{j}\binom{j}{i} 2^i.$$

Clearly, these $f_k(x)$ are integer-valued polynomials for $0 \le k \le n$. From Lemma 3.1 and the identity (3.4), we immediately conclude that (1.6) is integer-valued. $\square$

Acknowledgments. The author would like to thank the anonymous referee for careful reading of this manuscript and helpful comments, which make the paper more readable.

References

[1] M. Apagodu and D. Zeilberger, Multi-variable Zeilberger and Almkvist–Zeilberger algorithms and the sharpening of Wilf–Zeilberger theory, Adv. Appl. Math. 37 (2006), 139–152.

[2] C. Banderier and S.R. Schwer, Why Delannoy's numbers? J. Stat. Plan. Infer. 135 (2005), 40–54.

[3] Q.-F. Chen and V.J.W. Guo, On the divisibility of sums involving powers of multi-variable Schmidt polynomials, Int. J. Number Theory, in press. (arXiv:1412.5734)

[4] V.J.W. Guo, Some congruences involving powers of Legendre polynomials, Integral Transforms Spec. Funct. 26 (2015), 660–666.

[5] V.J.W. Guo, Proof of Sun's conjectures on integer-valued polynomials, J. Math. Anal. Appl. 444 (2016), 182–191.

[6] J.-C. Liu, A supercongruence involving Delannoy numbers and Schröder numbers, J. Number Theory 168 (2016), 117–127.

[7] J.-C. Liu, Congruences for truncated hypergeometric series ${}_2F_1$, Bull. Aust. Math. Soc., online, doi: 10.1017/S0004972717000181.

[8] E. Mortenson, A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function, J. Number Theory 99 (2003), 139–147.

[9] E. Mortenson, Supercongruences between truncated ${}_2F_1$ hypergeometric functions and their Gaussian analogs, Trans. Amer. Math. Soc. 355 (2003), 987–1007.

[10] M. Petkovšek, H. S. Wilf and D. Zeilberger, $A = B$, A K Peters, Ltd., Wellesley, MA, 1996.

[11] F. Rodriguez-Villegas, Hypergeometric families of Calabi-Yau manifolds, in: Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), Fields Inst. Commun., 38, Amer. Math. Soc., Providence, RI, 2003, 223–231.

[12] Z.-W. Sun, Supercongruences involving dual sequences, Finite Fields Appl. 46 (2017), 179–216.