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Generalized additive bases, König's lemma, and the Erdős–Turán conjecture

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Abstract

Let A be a set of nonnegative integers. For every nonnegative integer n and positive integer h, let $r_A(n,h)$ denote the number of representations of n in the form $n = a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, \ldots, a_h \in A$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_h$. The infinite set A is called a *basis of order h* if $r_A(n,h) \geqslant 1$ for every nonnegative integer n. Erdős and Turán conjectured that $\limsup_{n\to\infty} r_A(n,2) = \infty$ for every basis A of order 2. This paper introduces a new class of additive bases and a general additive problem, a special case of which is the Erdős–Turán conjecture. König's lemma on the existence of infinite paths in certain graphs is used to prove that this general problem is equivalent to a related problem about finite sets of nonnegative integers.
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1. Representation functions and the Erdős–Turán conjecture

Let $\mathbf{Z}$ denote the integers and $\mathbf{N}_0$ the nonnegative integers. Let A be a finite set of integers. We denote the largest element of A by $\max(A)$ and the cardinality of A by

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card(A). For any real numbers a and b, we denote by $[a,b]$ the finite set of integers n such that $a \leqslant n \leqslant b$.

For any set A of integers, we denote by $r_A(n,h)$ the number of representations of n in the form $n = a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, \ldots, a_h \in A$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_h$. The function r_A is called the *unordered representation function* of the set A, or, simply, the *representation function* of A.

The set A of nonnegative integers is called a *basis of order h* if every nonnegative integer can be represented as the sum of h not necessarily distinct elements of A. If A is a basis of order h with representation function r_A, then

$$1 \leqslant r_A(n,h) < \infty$$

for all nonnegative integers n. We call A an *asymptotic basis of order h* if $r_A(n,h) = 0$ for only finitely many $n \in \mathbf{N}_0$. If $h \geqslant 2$ and $f : \mathbf{N}_0 \to \mathbf{N}_0$ is any function such that card$(f^{-1}(0)) < \infty$, then the *representation function problem* is to determine if there exists a set A of nonnegative integers such that $r_A(n,h) = f(n)$ for all $n \geqslant 0$. In this case, of course, A is an asymptotic basis of order h.

Similarly, the set A of integers is called a *basis of order h* for $\mathbf{Z}$ if every integer can be represented as the sum of h not necessarily distinct elements of A, and an *asymptotic basis of order h* for $\mathbf{Z}$ if only finitely many integers cannot be represented as the sum of h not necessarily distinct elements of A. In the case of additive bases for $\mathbf{Z}$, Nathanson [5] proved that *every* function with only finitely many zeros is a representation function, that is, if $f : \mathbf{Z} \to \mathbf{N}_0 \cup \{\infty\}$ satisfies the condition card$(f^{-1}(0)) < \infty$, then for every $h \geqslant 2$ there exists a set A of integers such that $r_A(n,h) = f(n)$ for every integer n.

A special case of the representation function problem for nonnegative integers with $h = 2$ is the conjecture of Erdős and Turán [2] that the representation function $r_A(n,2)$ of an asymptotic basis A of order 2 must be unbounded, that is,

$$\liminf_{n \to \infty} r_A(n,2) > 0 \Rightarrow \limsup_{n \to \infty} r_A(n,2) = \infty.$$

This is an important unsolved problem in additive number theory.

Dowd [1] and Grekos et al. [3] have given various equivalent formulations of the Erdős–Turán conjecture. In particular, Dowd proved that there exists a set A of nonnegative integers and a number c such that $r_A(n,2) \in [1,c]$ for all nonnegative integers n if and only if for every N there exists a finite set A_N of nonnegative integers with $\max(A_N) \geqslant N$ and $r_A(n,2) \in [1,c]$ for all $n = 0, 1, \ldots, \max(A_N)$. In this paper we apply Dowd's method to obtain similar results for a new class of generalized additive bases.

2. Generalized additive bases

We extend the idea of an additive basis of order h as follows: Let $\mathscr{H} = \{H_n\}_{n=0}^{\infty}$ be a sequence of nonempty finite sets of positive integers. For any set A of nonnegative

integers, we define the representation function

$$r_A(n, H_n) = \sum_{h_n \in H_n} r_A(n, h_n).$$

The set A of nonnegative integers will be called a *basis of order* $\mathscr{H}$ if

$$r_A(n, H_n) \geqslant 1 \tag{1}$$

for all $n \geqslant 0$, and an *asymptotic basis of order* $\mathscr{H}$ if the representation function $r_A(n, H_n)$ satisfies (1) for all sufficiently large n.

Let $\mathscr{R} = \{R_n\}_{n=0}^{\infty}$ be a sequence of nonempty finite sets of positive integers. If

$$r_A(n, H_n) \in R_n \tag{2}$$

for every nonnegative integer n, then A will be called an *$\mathscr{R}$-basis of order $\mathscr{H}$*. Since each R_n is a nonempty set of positive integers, it follows that every $\mathscr{R}$-basis of order $\mathscr{H}$ is a basis of order $\mathscr{H}$. We shall call the set A an *asymptotic $\mathscr{R}$-basis of order $\mathscr{H}$* if (2) holds for all sufficiently large n.

For any sequences $\mathscr{H} = \{H_n\}_{n=0}^{\infty}$ and $\mathscr{R} = \{R_n\}_{n=0}^{\infty}$ of nonempty finite sets of positive integers, we can ask if there exists an $\mathscr{R}$-basis of order $\mathscr{H}$ or an asymptotic $\mathscr{R}$-basis of order $\mathscr{H}$. This is the *generalized representation function problem*. The original Erdős–Turán conjecture corresponds to the special case $H_n = \{2\}$ and $R_n = [1, c]$ for all $n \geqslant 0$. It is an open problem to determine the number of distinct $\mathscr{R}$-bases of order $\mathscr{H}$ and asymptotic $\mathscr{R}$-bases of order $\mathscr{H}$ for a given pair of sequences $\mathscr{H}$ and $\mathscr{R}$.

Let $h \geqslant 2$ and let f be a function such that $f(n)$ is a positive integer for every nonnegative integer n. We introduce the sets $H_n = \{h\}$ and $R_n = \{f(n)\}$ for all $n \in \mathbf{N}_0$, and the sequences $\mathscr{H} = \{H_n\}_{n=0}^{\infty}$ and $\mathscr{R} = \{R_n\}_{n=0}^{\infty}$. Then an $\mathscr{R}$-basis of order $\mathscr{H}$ is a basis A of order h whose representation function satisfies $r_A(n, h) = f(n)$ for all $n \in \mathbf{N}_0$, and so the representation function problem for bases of order h is a special case of the generalized representation function problem.

An $\mathscr{R}$-basis of order $\mathscr{H}$ is not necessarily infinite. For example, if $H_0 = R_0 = \{1\}$ and if $H_n = \{n\}$ and $1 \in R_n$ for all $n \geqslant 1$, then the set $\{0, 1\}$ is an $\mathscr{R}$-basis of order $\mathscr{H}$.

Theorem 1. *Let $\mathscr{H} = \{H_n\}_{n=0}^{\infty}$ be a sequence of nonempty finite sets of positive integers. There exists a finite set A that is a basis of order $\mathscr{H}$ or an asymptotic basis of order $\mathscr{H}$ if and only if*

$$\liminf_{n \to \infty} \frac{\max(H_n)}{n} > 0.$$

Proof. Let $h_n^* = \max(H_n)$. Let A be a finite set of nonnegative integers that is a basis of order $\mathscr{H}$. Then $0, 1 \in A$ and so $\max(A) \geqslant 1$. Every positive integer n can be

represented as the sum of h_n elements of A for some $h_n \in H_n$, and so

$$n \leqslant h_n \max(A) \leqslant h_n^* \max(A).$$

It follows that

$$\liminf_{n\to\infty} \frac{h_n^*}{n} \geqslant \frac{1}{\max(A)} > 0.$$

Conversely, if $\liminf_{n\to\infty} h_n^*/n > 0$, then there exists a positive integer m such that

$$h_n^* \geqslant \frac{n}{m}$$

for all $n \geqslant 0$. Consider the finite set $A = [0, m]$. By the division algorithm, every positive integer n can be written in the form $n = qm + r$, where q and r are nonnegative integers and $0 \leqslant r \leqslant m-1$. If $r = 0$, then $q = n/m \leqslant h_n^*$ and

$$n = q \cdot m + (h_n^* - q) \cdot 0 \in h_n^* A.$$

If $1 \leqslant r \leqslant m-1$, then $q = (n-r)/m < h_n^*$. Since h_n^* and q are integers, it follows that $h_n^* \geqslant q+1$ and

$$n = q \cdot m + 1 \cdot r + (h_n^* - q - 1) \cdot 0 \in h_n^* A.$$

In both cases, $r_A(n, H_n) \geqslant r_A(n, h_n^*) \geqslant 1$, and the finite set A is a basis of order $\mathscr{H}$.

If A is an asymptotic basis of order $\mathscr{H}$, then $r_A(n, H_n) = 0$ for only finitely many $n \in \mathbf{N}_0$, and so there is a finite set F of nonnegative integers such that $A \cup F$ is a basis of order $\mathscr{H}$. Therefore, there exists a finite set that is a basis of order $\mathscr{H}$ if and only if there exists a finite set that is an asymptotic basis of order $\mathscr{H}$. This completes the proof. □

Let $\mathscr{H} = \{H_n\}_{n=0}^{\infty}$ and $\mathscr{R} = \{R_n\}_{n=0}^{\infty}$ be sequences of nonempty finite sets of positive integers. A nonempty finite set A of nonnegative integers will be called a *finite basis of order* $\mathscr{H}$ if

$$r_A(n, H_n) \geqslant 1$$

for all $n \in [0, \max(A)]$, and a *finite* $\mathscr{R}$*-basis of order* $\mathscr{H}$ if

$$r_A(n, H_n) \in R_n$$

for all $n \in [0, \max(A)]$.

Theorem 2. *Let* $\mathscr{H} = \{H_n\}_{n=0}^{\infty}$ *and* $\mathscr{R} = \{R_n\}_{n=0}^{\infty}$ *be sequences of nonempty finite sets of positive integers.*

(i) *If* A *is a basis of order* $\mathscr{H}$*, or if* A *is a finite basis of order* $\mathscr{H}$ *with* $\max(A) \geqslant 1$*, then* $0, 1 \in A$.

(ii) *If A is an $\mathscr{R}$-basis of order $\mathscr{H}$ or if A is a finite $\mathscr{R}$-basis of order $\mathscr{H}$ with* $\max(A) \geqslant 1$, *then* $\operatorname{card}(H_0) \in R_0$ *and* $\operatorname{card}(H_1) \in R_1$.

(iii) *If A is an $\mathscr{R}$-basis of order $\mathscr{H}$, then $A_N = A \cap [0, N]$ is a finite $\mathscr{R}$-basis of order $\mathscr{H}$ for every $N \geqslant 0$.*

(iv) *If $A \neq \{0\}$ is a finite $\mathscr{R}$-basis of order $\mathscr{H}$, then $A' = A \backslash \{\max(A)\}$ is also a finite $\mathscr{R}$-basis of order $\mathscr{H}$.*

Proof. To prove (i) and (ii), we observe that if $r_A(0, H_0) \geqslant 1$, then $0 \in A$. If $r_A(1, H_1) \geqslant 1$, then $1 \in A$. Since, for every $h \geqslant 1$, both 0 and 1 have unique representations as sums of exactly h nonnegative integers, it follows that if $r_A(0, H_0) \in R_0$, then

$$r_A(0, H_0) = \sum_{h_0 \in H_0} r_A(0, h_0) = \sum_{h_0 \in H_0} 1 = \operatorname{card}(H_0) \in R_0.$$

Similarly, if $r_A(1, H_1) \in R_1$, then

$$r_A(1, H_1) = \operatorname{card}(H_1) \in R_1.$$

The statements (iii) and (iv) follow immediately from the definition of a finite basis. □

3. König's lemma

The principal tool in this paper is König's lemma on the existence of infinite paths in trees. For completeness, we include a short proof below.

A *graph* G consists of a nonempty set $\{v\}$, whose elements are called *vertices*, and a set $\{e\}$, whose elements are called *edges*. Each edge is a set $e = \{v, v'\}$, where v and v' are vertices and $v \neq v'$. Thus, we are considering only graphs without loops or multiple edges.

We use the following terminology. The vertices v and v' are called *adjacent* if $\{v, v'\}$ is an edge. The *degree* of a vertex v is the number of edges e with $v \in e$. A *path* in G from vertex v to vertex v' is a sequence of vertices $v_0, v_1, v_2, \ldots, v_n$ such that $v_0 = v$, $v_n = v'$, and v_{i-1} is adjacent to v_i for all $i = 1, \ldots, n$. We define the *length* of this path by n. The graph G is *connected* if for every two vertices v and v' with $v \neq v'$ there is a path from v to v'. A graph is connected if and only if, for some vertex v_0, there is a path from v_0 to v for every vertex $v \neq v_0$.

A *simple path* in G is a path whose vertices are pairwise distinct. A *simple circuit* is a sequence of vertices $v_0, v_1, v_2, \ldots, v_n$ such that $n \geqslant 3$, $\{v_{i-1}, v_i\}$ is an edge for $i = 1, \ldots, n$, $v_i \neq v_j$ for $0 \leqslant i < j \leqslant n-1$, and $v_0 = v_n$. A graph G has no simple circuits if and only, for every pair of distinct vertices v and v', there is at most one simple path from v to v'. An *infinite simple path* is an infinite sequence of pairwise distinct vertices $v_0, v_1, v_2, \ldots$ such that v_{i-1} is adjacent to v_i for all $i \geqslant 1$.

A *tree* is a connected graph with no simple circuits. A *rooted tree* is a tree with a distinguished vertex, called the *root* of the tree. In a rooted tree, for every vertex v different from the root, there is a unique simple path in the tree from the root to v.

Theorem 3 (König's lemma). *If T is a rooted tree with infinitely many vertices such that every vertex has finite degree, then T contains an infinite simple path beginning at the root.*

Proof. Let v_0 be the root of the tree. We use induction to prove that for every n there is a simple path $v_0, v_1, \ldots, v_n$ such that the tree T contains infinitely many vertices v for which the unique simple path from the root v_0 to v begins with the vertices $v_0, v_1, \ldots, v_n$. Since T has infinitely many vertices, the root v_0 satisfies this condition.

Let $n \geqslant 1$, and assume that we have constructed a simple path $v_0, v_1, \ldots, v_{n-1}$ of vertices of the tree T with the property that T contains an infinite set I_{n-1} of vertices such that, for every $v \in I_{n-1}$, the unique simple path from v_0 to v passes through vertex v_{n-1}. Since the degree of v_{n-1} is finite, the set F_n of vertices $v \neq v_{n-2}$ that are adjacent to v_{n-1} is a finite set. For every vertex $v \in I_{n-1}$, there is a unique simple path in T that begins at v_0, passes through v_{n-1} and exactly one of the vertices in F_n, and ends at v. By the pigeonhole principle, since I_{n-1} is infinite, there is a vertex $v_n \in F_n$ and an infinite set $I_n \subseteq I_{n-1}$ of vertices such that, for every $v \in I_n$, the unique path from v_0 to v passes through v_n. This completes the induction. The vertices $v_0, v_1, v_2, \ldots$ are pairwise distinct, and $v_0, v_1, v_2, \ldots$ is an infinite simple path in T. This completes the proof. □

4. The generalized representation function problem

In this section we prove that there exists an infinite $\mathscr{R}$-basis of order $\mathscr{H}$ if and only if there exist arbitrarily large finite $\mathscr{R}$-bases of order $\mathscr{H}$.

Theorem 4. *Let $\mathscr{R} = \{R_n\}_{n=0}^{\infty}$ and $\mathscr{H} = \{H_n\}_{n=0}^{\infty}$ be sequences of nonempty finite sets of positive integers such that*

$$\lim_{n \to \infty} \frac{\max(H_n)}{n} = 0. \tag{3}$$

There exists an $\mathscr{R}$-basis A of order $\mathscr{H}$ if and only if for every N there exists a finite $\mathscr{R}$-basis A_N of order $\mathscr{H}$ with $\max(A_N) \geqslant N$.

Proof. If A is a $\mathscr{R}$-basis of order $\mathscr{H}$, then, by (3) and Theorem 1, the set A is infinite, hence for every N there is an integer $a(N) \in A$ with $a(N) \geqslant N$. By Theorem 2, the set $A_N = A \cap [0, a(N)]$ is a finite $\mathscr{R}$-basis of order $\mathscr{H}$ with $\max(A_N) \geqslant N$.

Conversely, suppose that for every N there exists a finite $\mathscr{R}$-basis A_N of order $\mathscr{H}$ with $\max(A_N) \geqslant N$. If $N \geqslant 1$, then $0, 1 \in A_N$ and the sets $\{0\}$ and $\{0, 1\}$ are finite $\mathscr{R}$-bases of order $\mathscr{H}$.

We construct the graph T whose vertices are the finite $\mathscr{R}$-bases of order $\mathscr{H}$. This graph has infinitely many vertices, since there are finite $\mathscr{R}$-bases of order $\mathscr{H}$ with arbitrarily large maximum elements.

Vertices V and V' will be called adjacent in this graph if $V' \subseteq V$ and $V \backslash V' = \{\max(V)\}$. The sets $\{0\}$ and $\{0,1\}$ are adjacent vertices of this graph, and $\{0,1\}$ is the only vertex adjacent to $\{0\}$. If V is a vertex and $\operatorname{card}(V) \geqslant 2$, then it follows from Theorem 2 that $V' = V \backslash \{\max(V)\}$ is a vertex. Moreover, V' is the unique vertex adjacent to V in T such that $\operatorname{card}(V') = \operatorname{card}(V) - 1$. If V'' is adjacent to V and $V'' \neq V'$, then $V = V'' \backslash \{\max(V'')\}$ and $\operatorname{card}(V'') = \operatorname{card}(V) + 1$.

We shall prove that T is a rooted tree with root $V_0 = \{0\}$. Let $V = \{a_0, a_1, \ldots, a_n\}$ be a vertex, where $0 = a_0 < a_1 < \cdots < a_n$. For every $k = 0, 1, \ldots, n$, the set $V_k = \{a_0, a_1, \ldots, a_k\}$ is a finite $\mathscr{R}$-basis of order $\mathscr{H}$, hence is a vertex of T. Then $V_0 = \{0\}$, $V_n = V$, and $V_0, V_1, \ldots, V_n$ is a simple path in T from the root V_0 to V. It follows that the graph T is connected.

Suppose that $n \geqslant 3$ and $V_0, V_1, \ldots, V_{n-1}, V_n$ is a simple circuit in T, where $V_n = V_0$. Let $V_{n+1} = V_1$. Since each vertex is a finite set of integers, we can choose $k \in [1, n]$ such that V_k is a vertex in the circuit of maximum cardinality. Vertices V_{k-1} and V_{k+1} are adjacent to V_k, hence $\operatorname{card}(V_k) - \operatorname{card}(V_{k-1}) = \pm 1$ and $\operatorname{card}(V_k) - \operatorname{card}(V_{k+1}) = \pm 1$. The maximality of $\operatorname{card}(V_k)$ implies that $\operatorname{card}(V_k) - \operatorname{card}(V_{k-1}) = \operatorname{card}(V_k) - \operatorname{card}(V_{k+1}) = 1$, and so $V_{k-1} = V_k \backslash \{\max(V_k)\} = V_{k+1}$, which is impossible. Therefore, T contains no simple circuit, and so T is a tree.

To apply König's lemma, we must prove that every vertex of this tree has finite degree. The only vertex adjacent to the root $\{0\}$ is $\{0,1\}$, hence $\{0\}$ has finite degree. Let $V \neq \{0\}$ be a vertex of T. Then $1 \in V$ and so $\max(V) \geqslant 1$. Suppose that V is adjacent to infinitely many vertices V'. The only subset of V that is a vertex adjacent to V is $V \backslash \{\max(V)\}$. Every other vertex V' adjacent to V is a superset of V of the form $V' = V \cup \{\max(V')\}$. For each such V', the integer $n = \max(V') - 1$ must be an element of the sumset $h_n V$ for some $h_n \in H_n$, and so

$$n \leqslant h_n \max(V) \leqslant \max(H_n)\max(V).$$

Since $n \in h_n V$ for infinitely many integers n, it follows that

$$\limsup_{n \to \infty} \frac{\max(H_n)}{n} \geqslant \frac{1}{\max(V)} > 0,$$

which contradicts (3). Thus, every vertex of the infinite tree T has finite degree. By König's lemma, the tree must contain an infinite simple path $\{0\} = V_0, V_1, V_2, \ldots$. For each nonnegative integer n, let $a_n = \max(V_n)$. Then $V_n = \{a_0, a_1, \ldots, a_n\}$ for all $n = 0, 1, 2, \ldots$. Let

$$A = \{a_n\}_{n=0}^{\infty} = \bigcup_{n=0}^{\infty} V_n.$$

Since $a_n \geqslant n$, it follows that

$$r_A(n, H_n) = \sum_{h_n \in H_n} r_A(n, h_n) = \sum_{h_n \in H_n} r_{V_n}(n, h_n) = r_{V_n}(n, H_n) \in R_n,$$

and so A is a $\mathscr{R}$-basis of order $\mathscr{H}$. This completes the proof. □

Theorem 5. *Let $h \geqslant 2$ and let f be a function such that $f(n)$ is a positive integer for every nonnegative integer n. There exists a basis A of order h with representation function $r_A(n,h) = f(n)$ if and only if for every N there exists a finite set A_N of nonnegative integers with $\max(A_N) \geqslant N$ and $r_A(n,h) = f(n)$ for all $n = 0, 1, \ldots, \max(A_N)$.*

Proof. This follows immediately from Theorem 4 with $H_n = \{h\}$ and $R_n = \{f(n)\}$ for all nonnegative integers n. □

Applying Theorem 4 to the classical Erdős–Turán conjecture, we obtain the following result of Dowd [1, Theorem 2.1].

Theorem 6. *Let $c \geqslant 1$ and $h \geqslant 2$. There exists a basis A of order h such that*

$$r_A(n,h) \leqslant c \quad \text{for all } n \geqslant 0$$

if and only if, for every N, there exists a finite set A_N of nonnegative integers such that $\max(A_N) \geqslant N$ and

$$1 \leqslant r_{A_N}(n,h) \leqslant c$$

for all $n = 0, 1, \ldots, \max(A_N)$.

Proof. This follows immediately from Theorem 4 with $H_n = \{h\}$ and $R_n = [1, c]$ for all $n \geqslant 0$. □

5. Ordered representation functions

There are other important representation functions in additive number theory. For example, for any set A of integers, the *ordered representation function* $r'_A(n,h)$ counts the number of h-tuples $(a_1, \ldots, a_h) \in A^h$ such that $a_1 + \cdots + a_h = n$. Nathanson [4] proved the following uniqueness theorem for ordered representation functions: For any function $f : \mathbf{N}_0 \to \mathbf{N}_0$ and for any positive integer h, there exists at most one set A of nonnegative integers such that $r'_A(n,h) = f(n)$ for all $n \in \mathbf{N}_0$. He also showed that uniqueness does not hold if ordered representation functions only eventually coincide, and he described all pairs of sets A and B of nonnegative integers such that $r'_A(n,2) = r'_B(n,2)$ for all sufficiently large integers n.

We can define *basis of order* $\mathscr{H}$ and $\mathscr{R}$*-basis of order* $\mathscr{H}$ in terms of the ordered representation function. Theorem 4 is also true for ordered representation functions.

References

[1] M. Dowd, Questions related to the Erdős–Turán conjecture, SIAM J. Discrete Math. 1 (1988) 142–150.
[2] P. Erdős, P. Turán, On a problem of Sidon in additive number theory and some related questions, J. London Math. Soc. 16 (1941) 212–215.
[3] G. Grekos, L. Haddad, C. Helou, J. Pihko, On the Erdős–Turán conjecture, J. Number Theory 102 (2003) 339–352.
[4] M.B. Nathanson, Representation functions of sequences in additive number theory, Proc. Amer. Math. Soc. 72 (1978) 16–20.
[5] M.B. Nathanson, Every function is the representation function of an additive basis for the integers, arXiv: math.NT/0302091. Port. Math., to appear.