Journal of Number Theory 131 (2011) 1286–1295

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Self-approximation of Dirichlet L-functions

Ramūnas Garunkštis [1]

Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, 03225 Vilnius, Lithuania

ARTICLE INFO

Article history:
Received 9 June 2010
Accepted 26 January 2011
Available online 21 March 2011
Communicated by K. Soundararajan

Keywords:
Dirichlet L-function
Self-approximation
Strong recurrence

ABSTRACT

Let d be a real number, let s be in a fixed compact set of the strip $1/2 < \sigma < 1$, and let $L(s,\chi)$ be the Dirichlet L-function. The hypothesis is that for any real number d there exist 'many' real numbers τ such that the shifts $L(s+i\tau,\chi)$ and $L(s+id\tau,\chi)$ are 'near' each other. If d is an algebraic irrational number then this was obtained by T. Nakamura. Ł. Pańkowski solved the case then d is a transcendental number. We prove the case then $d \neq 0$ is a rational number. If $d=0$ then by B. Bagchi we know that the above hypothesis is equivalent to the Riemann hypothesis for the given Dirichlet L-function. We also consider a more general version of the above problem.

1. Introduction

Let, as usual, $s = \sigma + it$ denote a complex variable. For $\sigma > 1$, the Dirichlet L-function is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $\chi(n)$ is a Dirichlet character mod q. For $q=1$ we get $L(s,\chi) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function.

In [6] Bohr proved that if χ is a nonprincipal character, then the Riemann hypothesis for $L(s,\chi)$ is equivalent to the almost periodicity of $L(s,\chi)$ in the half plane $\sigma > 1/2$. A function $f(s)$ is *almost periodic* in a region $E \subset \mathbb{C}$ if for any positive ε and any compact subset K in E there exists a sequence of real numbers $\cdots < \tau_{-1} < 0 < \tau_1 < \tau_2 < \cdots$ such that

E-mail address: ramunas.garunkstis@mif.vu.lt.
URL: http://www.mif.vu.lt/~garunkstis.

[1] Supported by grant No. MIP-94 from the Research Council of Lithuania.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2011.01.013

$$\liminf_{m\to\pm\infty}(\tau_{m+1}-\tau_m)>0, \qquad \limsup_{m\to\pm\infty}\frac{\tau_m}{m}<\infty$$

and

$$\left|f(s+i\tau_m)-f(s)\right|<\varepsilon \quad \text{for all } s\in K \text{ and } m\in\mathbb{Z}$$

hold. Bohr [6] also obtained that every Dirichlet series is almost-periodic in its half-plane of absolute convergence. Effective upper bounds for the almost periodicity of Dirichlet series with Euler products in the half-plane of absolute convergence were considered by Girondo and Steuding [7]. Note that every Dirichlet L-function is almost periodic in the sense of Besicovitch on any vertical line of the strip $1/2<\sigma<1$. For this and related results see Besicovitch [5] and Mauclaire [13,14].

Bagchi [2] proved that the Riemann hypothesis for $L(s,\chi)$ (χ is an arbitrary Dirichlet character) is true if and only if for any compact subset $\mathcal{K}$ of the strip $1/2<\sigma<1$ and for any $\varepsilon>0$

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]\colon \max_{s\in\mathcal{K}}\left|L(s+i\tau,\chi)-L(s,\chi)\right|<\varepsilon\Big\}>0, \tag{1}$$

where meas A stands for the Lebesgue measure of a measurable set A. Bagchi says that the Dirichlet L-function $L(s,\chi)$ is *strongly recurrent* on the strip $\sigma_0<\sigma<\sigma_1$ if (1) is valid for any compact $\mathcal{K}$ of the strip $\sigma_0<\sigma<\sigma_1$. The strong recurrence is connected with the universality property of Dirichlet series. More about the universality and the strong recurrence see Bagchi [1–3], and Steuding [17].

There are several unconditional results concerning the self-approximation of Dirichlet L-functions in the critical strip. Let $\mathcal{K}$ be a compact subset of the strip $1/2<\sigma<1$ and let $\lambda\in\mathbb{R}$ be such that $\mathcal{K}$ and $\mathcal{K}+i\lambda:=\{s+i\lambda\colon s\in\mathcal{K}\}$ are disjoint. From Kaczorowski, Laurinčikas and Steuding [10] it follows that for any character χ and any $\varepsilon>0$

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]\colon \max_{s\in\mathcal{K}}\left|L(s+i\lambda+i\tau,\chi)-L(s+i\tau,\chi)\right|<\varepsilon\Big\}>0.$$

Nakamura [15] considered the joint universality of shifted Dirichlet L-functions. His Theorem 1.1 leads to the following statement. If $1=d_1,d_2,\dots,d_m$ are algebraic real numbers *linearly independent* over $\mathbb{Q}$, then for any Dirichlet character χ and any $\varepsilon>0$

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]\colon \max_{1\leqslant j,k\leqslant m}\max_{s\in\mathcal{K}}\left|L(s+id_j\tau,\chi)-L(s+id_k\tau,\chi)\right|<\varepsilon\Big\}>0. \tag{2}$$

If $m=2$ then Pańkowski [16] using Six Exponentials Theorem showed that (2) holds for d_1,d_2 are real numbers linearly independent over $\mathbb{Q}$.

We prove the following theorem.

Theorem 1. *Let $1=d_1,d_2,\dots,d_m$ be nonzero algebraic real numbers and let $\mathcal{K}$ be a compact subset of the strip $1/2<\sigma<1$. Then for any Dirichlet character χ and any $\varepsilon>0$ the inequality (2) is valid.*

Note that Theorem 1 remains true if $d_1,d_2,\dots,d_m$ are replaced by $dd_1,dd_2,\dots,dd_m$, where $d\in\mathbb{R}$. The next theorem shows that 'lim inf' in the inequality (2) often can be replaced by 'lim'.

Theorem 2. *Let $d_1,d_2,\dots,d_m$ be any real numbers, let $\chi_1,\chi_2,\dots,\chi_m$ be any Dirichlet characters, and let $\mathcal{K}$ be a compact subset of the strip $1/2<\sigma<1$. Then for any $\varepsilon>0$, except an at most countable set of ε, there exists a limit*

$$\lim_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]\colon \max_{1\leqslant j,k\leqslant m}\max_{s\in\mathcal{K}}\left|L(s+id_j\tau,\chi_j)-L(s+id_k\tau,\chi_k)\right|<\varepsilon\Big\}.$$

The mentioned results of Nakamura and Pańkowski together with Theorem 1 and Theorem 2 lead to the following corollary.

Corollary 3. *Let d be a nonzero real number and let $\mathcal{K}$ be a compact subset of the strip $1/2 < \sigma < 1$. Then for any Dirichlet character χ and any $\varepsilon > 0$, except an at most countable set of ε,*

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T]\colon \max_{s\in\mathcal{K}}\big|L(s+i\tau, \chi) - L(s+id\tau, \chi)\big| < \varepsilon\Big\} > 0. \tag{3}$$

From the proof of Theorem 2 we see that for any real numbers $d_1, \dots, d_m$ and for any Dirichlet characters $\chi_1, \dots, \chi_m$ the function

$$g(\tau) = \max_{1\leqslant j,k\leqslant m} \max_{s\in\mathcal{K}}\big|L(s+id_j\tau, \chi_j) - L(s+id_k\tau, \chi_k)\big|$$

is Besicovitch almost periodic function (for the definition see Section 3 above the proof of Theorem 2). Let $\varepsilon > 0$ be such that the limit (3) exists. For such ε we define a characteristic function $I_\varepsilon(\tau)$, $\tau \in \mathbb{R}$, by

$$I_\varepsilon(\tau) = \begin{cases} 1, & \text{if } g(\tau) < \varepsilon, \\ 0, & \text{if } g(\tau) \geqslant \varepsilon. \end{cases} \tag{4}$$

It is known (Jessen and Wintner [9, Section 12]) that $I_\varepsilon(\tau)$ is Besicovitch almost periodic function also. Thus we can say that self-approximations of Dirichlet L-functions, considered in this paper, usually appear in a regular way.

Theorem 1 and Theorem 2 are proved in Section 3. Next we state several lemmas.

2. Lemmas

We start from the following statement.

Lemma 4. *Let $\mathcal{K}$ be a compact subset of the rectangle U. Let*

$$d = \min_{z\in\partial U} \min_{s\in\mathcal{K}} |s - z|.$$

If $f(s)$ is analytic on U and

$$\int_U \big|f(s)\big|^2 \, d\sigma \, dt \leqslant \varepsilon,$$

then

$$\max_{s\in\mathcal{K}}\big|f(s)\big| \leqslant \frac{\sqrt{\varepsilon/\pi}}{d}.$$

Proof. The lemma can be found in Gonek [8, Lemma 2.5]. □

Lemma 5. *Let $a_1, \dots, a_N$ be real numbers linearly independent over the rational numbers. Let γ be a region of the N-dimensional unit cube with volume V (in the Jordan sense). Let $I_\gamma(T)$ be the sum of the intervals between $t = 0$ and $t = T$ for which the point $(a_1 t, \dots, a_N t)$ is* mod 1 *inside γ. Then*

$$\lim_{T\to\infty} \frac{I_\gamma(T)}{T} = V.$$

Proof. This is Theorem 1 in Appendix, Section 8, of Voronin and Karatsuba [11]. □

For a curve $\omega(t)$ in $\mathbb{R}^N$ we introduce the notation

$$\{\omega(t)\} = \left(\omega_1(t) - [\omega_1(t)], \dots, \omega_N(t) - [\omega_N(t)]\right),$$

where $[x]$ denotes the integral part of $x \in \mathbb{R}$.

Lemma 6. *Suppose that the curve $\omega(t)$ is uniformly distributed* mod 1 *in $\mathbb{R}^N$. Let D be a closed and Jordan measurable subregion of the unit cube in $\mathbb{R}^N$ and let Ω be a family of complex-valued continuous functions defined on D. If Ω is uniformly bounded and equicontinuous, then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\{\omega(t)\}\right) 1_D(t)\, dt = \int_D f(x_1, \dots, x_N)\, dx_1 \dots dx_N$$

uniformly with respect to $f \in \Omega$, where $1_D(t)$ is equal to 1 *if $\omega(t) \in D$* mod 1, *and* 0 *otherwise.*

Proof. The lemma is Theorem 3 in Appendix, Section 8, of Voronin and Karatsuba [11]. □

Lemma 7. *Let p_n be the nth prime number and $1 = d_1, d_2, \dots, d_l$ be algebraic real numbers which are linearly independent over $\mathbb{Q}$. Then the set $\{d_k \log p_n\}_{n\in\mathbb{N}}^{1\leqslant k\leqslant l}$ is linearly independent over $\mathbb{Q}$.*

Proof. This is Proposition 2.2 in Nakamura [15]. The proof is based on Baker's [4, Theorem 2.4] result. □

3. Proof of Theorem 1 and Theorem 2

Proof of Theorem 1. We define a truncated Dirichlet L-function

$$L_v(s, \chi) = \prod_{p\leqslant v} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Roughly speaking, we first prove Theorem 1 for the truncated Dirichlet L-function and later we show that the tail is small.

Let $\{d_1, d_2, \dots, d_l\}$ be a maximal linearly independent (over $\mathbb{Q}$) subset of the set $\{d_1, d_2, \dots, d_m\}$. Then there are integers $a \neq 0$ and $a_{k,1}, a_{k,2}, \dots, a_{k,l}$ such that

$$d_k = \frac{1}{a}(a_{k,1}d_1 + a_{k,2}d_2 + \dots + a_{k,l}d_l) \quad \text{for } l < k \leqslant m. \tag{5}$$

Let

$$A = \max_{l<k\leqslant m} \left\{|a_{k,1}| + |a_{k,2}| + \dots + |a_{k,l}|\right\}.$$

Denote by $\|x\|$ the minimal distance of $x \in \mathbb{R}$ to an integer. If

$$\left\| \tau \frac{d_n \log p}{2\pi a} \right\| < \delta \quad \text{for } p \leqslant v \text{ and } 1 \leqslant n \leqslant l, \tag{6}$$

then

$$\left\| \tau \frac{d_n \log p}{2\pi} \right\| < a\delta \quad \text{for } p \leqslant v \text{ and } 1 \leqslant n \leqslant l$$

and, by the relation (5),

$$\left\| \tau \frac{d_k \log p}{2\pi} \right\| < A\delta \quad \text{for } p \leqslant v \text{ and } l < k \leqslant m.$$

By this and by the continuity of the function $L_v(s, \chi)$ we have that for any $\varepsilon > 0$ there is $\delta > 0$ such that for τ satisfying (6)

$$\max_{1 \leqslant k,n \leqslant m} \max_{s \in \mathcal{K}} \bigl| \log L_v(s + id_k\tau, \chi) - \log L_v(s + id_n\tau, \chi) \bigr| < \varepsilon. \tag{7}$$

For positive numbers δ, v, and T we define the set

$$S_T = S_T(\delta, v) = \left\{ \tau\colon\ \tau \in [0, T],\ \left\| \tau \frac{d_n \log p}{2\pi a} \right\| < \delta,\ p \leqslant v,\ 1 \leqslant n \leqslant l \right\}. \tag{8}$$

Let U be an open bounded rectangle with vertices on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$, where $1/2 < \sigma_1 < \sigma_2 < 1$, such that the set $\mathcal{K}$ is in U. Let $y > v$. We have

$$\begin{aligned} &\frac{1}{T} \int\limits_{S_T} \int\limits_{U} \sum_{k=1}^{m} \bigl| \log L_y(s + id_k\tau, \chi) - \log L_v(s + id_k\tau, \chi) \bigr|^2 \, d\sigma \, dt \, d\tau \\ &\quad = \sum_{k=1}^{m} \int\limits_{U} \frac{1}{T} \int\limits_{S_T} \bigl| \log L_y(s + id_k\tau, \chi) - \log L_v(s + id_k\tau, \chi) \bigr|^2 \, d\tau \, d\sigma \, dt. \end{aligned}$$

For the inner integrals of the right-hand side of the last equality we will apply Lemma 6. Let p_n be the nth prime number. There are indexes M and N such that $p_M \leqslant v < p_{M+1}$ and $p_N \leqslant y < p_{N+1}$. By generalized Kronecker's theorem (Lemma 5) and by Lemma 7 the curve

$$\omega(\tau) = \left(\tau \frac{d_k \log p_n}{2\pi a} \right)_{1 \leqslant n \leqslant N}^{1 \leqslant k \leqslant l}$$

is uniformly distributed mod 1 in $\mathbb{R}^{lN}$. Let R' be a subregion of the lN-dimensional unit cube defined by inequalities

$$\| y_{k,n} \| \leqslant \delta \quad \text{for } 1 \leqslant k \leqslant l \text{ and } 1 \leqslant n \leqslant M$$

and

$$\left| y_{k,n} - \frac{1}{2} \right| \leqslant \frac{1}{2} \quad \text{for } 1 \leqslant k \leqslant l \text{ and } M + 1 \leqslant n \leqslant N.$$

Let R be a subregion of the lM-dimensional unit cube defined by inequalities

$$\| y_{k,n} \| \leqslant \delta \quad \text{for } 1 \leqslant k \leqslant l \text{ and } 1 \leqslant n \leqslant M.$$

Clearly

$$\operatorname{meas} R' = \operatorname{meas} R = (2\delta)^{lM}.$$

Note that

$$\begin{aligned}
\log L_y(s+id_k\tau,\chi) - \log L_v(s+id_k\tau,\chi) &= \log \frac{L_y}{L_v}(s+id_k\tau,\chi) \\
&= -\sum_{v<p\leqslant y} \log\left(1-\frac{\chi(p)}{p^{s+id_k\tau}}\right) = \sum_{v<p\leqslant y}\sum_{j=1}^{\infty}\frac{\chi^j(p)}{jp^{j(s+id_k\tau)}} \\
&= \sum_{M<n\leqslant N}\sum_{j=1}^{\infty}\frac{\chi^j(p_n)}{jp_n^{j(s+id_k\tau)}}. \qquad (9)
\end{aligned}$$

Thus in view of the linear dependence (5) we get

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_{S_T}\sum_{k=1}^{m}\left|\log\frac{L_y}{L_v}(s+id_k\tau,\chi)\right|^2 d\tau \\
&\quad= \lim_{T\to\infty}\frac{1}{T}\int_{S_T}\left(\sum_{k=1}^{l}\left|\log\frac{L_y}{L_v}(s+id_k\tau,\chi)\right|^2\right. \\
&\qquad\left. + \sum_{k=l+1}^{m}\left|\log\frac{L_y}{L_v}\left(s+\frac{i}{a}(a_{k,1}d_1+a_{k,2}d_2+\cdots+a_{k,l}d_l)\tau,\chi\right)\right|^2\right)d\tau.
\end{aligned}$$

By Lemma 6 and equality (9) we obtain that the last limit is equal to

$$\begin{aligned}
&\int_{R'}\left(\sum_{k=1}^{l}\left|\sum_{M<n\leqslant N}\sum_{j=1}^{\infty}\frac{\chi^j(p)e^{-2\pi ijay_{k,n}}}{jp_n^{js}}\right|^2\right. \\
&\qquad\left.+\sum_{k=l+1}^{m}\left|\sum_{M<n\leqslant N}\sum_{j=1}^{\infty}\frac{\chi^j(p)e^{2\pi ij(a_{k,1}y_{1,n}+a_{k,2}y_{2,n}+\cdots+a_{k,l}y_{l,n})}}{jp_n^{js}}\right|^2\right)dy_{1,1}\ldots dy_{l,N} \\
&\quad= \operatorname{meas} R\int_0^1\cdots\int_0^1\left(\sum_{k=1}^{l}\left|\sum_{M<n\leqslant N}\sum_{j=1}^{\infty}\frac{\chi^j(p)e^{-2\pi ijay_{k,n}}}{jp_n^{js}}\right|^2\right. \\
&\qquad\left.+\sum_{k=l+1}^{m}\left|\sum_{M<n\leqslant N}\sum_{j=1}^{\infty}\frac{\chi^j(p)e^{2\pi ij(a_{k,1}y_{1,n}+a_{k,2}y_{2,n}+\cdots+a_{k,l}y_{l,n})}}{jp_n^{js}}\right|^2\right)dy_{1,M+1}\ldots dy_{l,N} \\
&\quad= m\operatorname{meas} R\sum_{v<p\leqslant y}\sum_{j=1}^{\infty}\frac{1}{jp^{2j\sigma}} \ll \operatorname{meas} R\sum_{p>v}\frac{1}{p^{2\sigma}}.
\end{aligned}$$

Consequently

$$\frac{1}{T}\int_{S_T}\int_U \sum_{k=1}^{m} \left|\log L_y(s+id_k\tau,\chi) - \log L_v(s+id_k\tau,\chi)\right|^2 d\sigma\, dt\, d\tau \ll \operatorname{meas} R \sum_{p>v} \frac{1}{p^{2\sigma_1}}. \tag{10}$$

Again, by generalized Kronecker's theorem (Lemma 5),

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas} S_T = \operatorname{meas} R. \tag{11}$$

By (10) and (11), for large v, as $T \to \infty$, we have

$$\operatorname{meas}\left\{\tau\colon\ \tau \in S_T,\ \int_U \sum_{k=1}^{m} \left|\log \frac{L_y}{L_v}(s+id_k\tau,\chi)\right|^2 d\sigma\, dt < \sqrt{\sum_{p>v} \frac{1}{p^{2\sigma_1}}}\right\} > \frac{1}{2} T \operatorname{meas} R.$$

Then Lemma 4 gives

$$\operatorname{meas}\left\{\tau\colon\ \tau \in S_T,\ \max_{s\in\mathcal{K}} \sum_{k=1}^{m} \left|\log \frac{L_y}{L_v}(s+id_k\tau,\chi)\right|^2 d\tau \leqslant \frac{1}{d\sqrt{\pi}}\left(\sum_{p>v} \frac{1}{p^{2\sigma_1}}\right)^{\frac{1}{4}}\right\} > \frac{1}{2} T \operatorname{meas} R,$$

where $d = \min_{z\in\partial U} \min_{s\in\mathcal{K}} |s-z|$. By the continuity of the logarithm we obtain that for any $\varepsilon > 0$ there is $v = v(\varepsilon)$ such that for any $y > v$

$$\operatorname{meas}\left\{\tau\colon\ \tau \in S_T,\ \max_{s\in\mathcal{K}} \sum_{k=1}^{m} \left|L_y(s+id_k\tau,\chi) - L_v(s+id_k\tau,\chi)\right|^2 d\tau < \varepsilon\right\} > \frac{1}{2} T \operatorname{meas} R. \tag{12}$$

Now we will prove that for any $\delta > 0$ there is $y = y(\delta)$ such that

$$\operatorname{meas}\left\{\tau\colon\ \tau \in [0,T],\ \max_{s\in\mathcal{K}} \sum_{k=1}^{m} \left|L(s+id_k\tau,\chi) - L_y(s+id_k\tau,\chi)\right|^2 d\tau < \delta\right\} > (1-\delta)T. \tag{13}$$

The last formula together with (7), (8) and (12) yields Theorem 1. We return to the proof of (13). By the mean value theorem of the Dirichlet L-function (Steuding [17, Corollary 6.11]) and by Carlson's theorem (Titchmarsh [18, Chapter 9.51]) we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|L(s+ix\tau,\chi) - L_y(s+ix\tau,\chi)\right|^2 d\tau = \sum_{n>y} \frac{|\chi(n)|}{n^{2\sigma}},$$

where x is fixed. Thus (13) follows in view of

$$\int_0^T \int_U \sum_{k=1}^{m} \left|L(s+id_k\tau,\chi) - L_y(s+id_k\tau,\chi)\right|^2 d\sigma\, dt\, d\tau \ll \sum_{n>y} \frac{|\chi(n)|}{n^{2\sigma_1}}.$$

Theorem 1 is proved. $\square$

The proof of Theorem 2 is based on the ideas of Mauclaire [13,14]. It uses the theory of Besicovitch almost periodic functions. We recall related definitions.

Let

$$P(\tau) = \sum_{n \in F} a_n e^{i\lambda_n \tau},$$

where F is a finite set, λ_n are any real numbers, and the coefficients a_n are any complex numbers. For real τ we say that $P(\tau)$ is a *trigonometric polynomial.*

A function $f : \mathbb{R} \to \mathbb{C}$ is called *uniformly almost periodic* ($U.A.P.$) if given any $\varepsilon > 0$, there exists a trigonometric polynomial $P(\tau)$ such that

$$\sup_{\tau \in \mathbb{R}} \left| f(\tau) - P(\tau) \right| \leqslant \varepsilon.$$

A function $f : \mathbb{R} \to \mathbb{C}$ is called B^q *almost periodic* ($B^q.A.P.$), $q \geqslant 1$, if given any $\varepsilon > 0$, there exists a trigonometric polynomial $P(\tau)$ such that

$$\limsup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| f(\tau) - P(\tau) \right|^q d\tau \leqslant \varepsilon. \tag{14}$$

If $q = 1$ then we write $B.A.P.$ (Besikovitch almost periodic) instead of $B^1.A.P.$ For any $q \geqslant 1$ it is clear that every $U.A.P.$ function is $B^q.A.P.$ and that every $B^q.A.P.$ function is $B.A.P.$

Proof of Theorem 2. Let

$$g(\tau) = \max_{1 \leqslant j,k \leqslant m} \max_{s \in \mathcal{K}} \left| L(s + id_j\tau, \chi_j) - L(s + id_k\tau, \chi_k) \right|$$

and let

$$F_T(x) = \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0, T]:\ g(\tau) < x \right\}$$

be a distribution function of $g(\tau)$. If $g(\tau)$ is $B.A.P.$ then it is known (see Jessen and Wintner [9, Theorem 27] or Laurinčikas [12, Theorem 6.3, Chapter 2]) that there is a distribution function $F(x)$ such that $F_T(x)$ converges weakly to $F(x)$ for $T \to \infty$. It means that if $F(x)$ is continuous at $x = \varepsilon$ then

$$\lim_{T \to \infty} F_T(\varepsilon)$$

exists. Thus to obtain Theorem 2 we need to show that $g(\tau)$ is $B.A.P.$

We remark that if $a(t)$ and $b(t)$ are both non-negative $B.A.P.$ functions of t, then $t \mapsto \max(a(t), b(t))$ is also $B.A.P.$ since $\max(a(t), b(t))$ can be written as

$$\max\bigl(a(t), b(t)\bigr) = \frac{1}{2}\Bigl(\bigl|a(t) - b(t)\bigr| + \bigl(a(t) + b(t)\bigr)\Bigr),$$

and the modulus of $B.A.P.$ function is again $B.A.P.$ By this we have that $g(\tau)$ is $B.A.P.$ if the function

$$f(\tau) = \max_{s \in \mathcal{K}} \left| L(s + id_1\tau, \chi_1) - L(s + id_2\tau, \chi_2) \right|$$

is $B.A.P.$ In view of the note below the formula (14) the function $f(\tau)$ is $B.A.P.$ if there are $U.A.P$ functions $f_N(\tau)$ such that

$$\lim_{N\to+\infty}\left(\lim\sup_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\left|f(\tau)-f_N(\tau)\right|^2\,d\tau\right)=0. \tag{15}$$

Let

$$L_N(s,\chi)=\sum_{n\leqslant N}\frac{\chi(n)}{n^s}$$

be a partial sum of the Dirichlet series associated with $L(s,\chi)$. Next we show that the equality (15) is true with

$$f_N(\tau)=\max_{s\in\mathcal{K}}\left|L_N(s+id_1\tau,\chi_1)-L_N(s+id_2\tau,\chi_2)\right|.$$

By repeating the proof of Proposition 12 of Mauclaire [13] we get that $f_N(\tau)$ is $U.A.P.$ for any $d_1,d_2\in\mathbb{R}$. Note that the case when d_1 or d_2 is equal to zero is already included in Proposition 12 of Mauclaire [13].

Further we have that

$$\begin{aligned}
&L(s+id_1\tau,\chi_1)-L(s+id_2\tau,\chi_2)\\
&\quad=\bigl(L(s+id_1\tau,\chi_1)-L_N(s+id_1\tau,\chi_1)+L_N(s+id_2\tau,\chi_2)-L(s+id_2\tau,\chi_2)\bigr)\\
&\qquad+\bigl(L_N(s+id_1\tau,\chi_1)-L_N(s+id_2\tau,\chi_2)\bigr),
\end{aligned}$$

and as a consequence, we get that

$$\begin{aligned}
\left|f(\tau)-f_N(\tau)\right|&\leqslant\sup_{s\in K}\left|L(s+id_1\tau,\chi_1)-L_N(s+id_1\tau,\chi_1)+L_N(s+id_2\tau,\chi_2)-L(s+id_2\tau,\chi_2)\right|\\
&\leqslant\sup_{s\in K}\left|L(s+id_1\tau,\chi_1)-L_N(s+id_1\tau,\chi_1)\right|\\
&\quad+\sup_{s\in K}\left|L_N(s+id_2\tau,\chi_2)-L(s+id_2\tau,\chi_2)\right|.
\end{aligned}$$

Then, in view of the inequality $(a+b)^2\leqslant 2a^2+2b^2$, we obtain that

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\left|f(\tau)-f_N(\tau)\right|^2dt&\leqslant\frac{1}{T}\int_{-T}^{T}\Bigl(\sup_{s\in K}\left|L(s+id_1\tau,\chi_1)-L_N(s+id_1\tau,\chi_1)\right|\Bigr)^2dt\\
&\quad+\frac{1}{T}\int_{-T}^{T}\Bigl(\sup_{s\in K}\left|L(s+id_2\tau,\chi_2)-L_N(s+id_2\tau,\chi_2)\right|\Bigr)^2dt.
\end{aligned}$$

By Mauclaire [14, Theorem 5.1] we have that, for any real d,

$$\lim_{N\to+\infty}\left(\lim\sup_{T\to+\infty}\frac{1}{2T'}\int_{-T}^{T}\Bigl(\sup_{s\in K}\left|f(s+idt)-f_N(s+idt)\right|\Bigr)^2dt\right)=0.$$

This proves the equality (15) and Theorem 2 $\square$

From the proof we see that Theorem 2 remains true with Dirichlet L-functions $L(s, \chi_j)$, $j = 1, \dots, m$, replaced by any general Dirichlet series satisfying conditions of Theorem 5.1 of Mauclaire [14].

Remark. The 'lim inf' version of Corollary 3 is independently obtained by Takashi Nakamura in "The generalized strong recurrence for nonzero rational parameters", Arch. Math. 95 (2010) 549–555.

Acknowledgments

We thank Jean-Loup Mauclaire for suggesting Theorem 2 and for other useful comments which helped to improve the paper.

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