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## Hessenberg matrices and the Pell and Perrin numbers

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## ABSTRACT

In this paper, we investigate the Pell sequence and the Perrin sequence and we derive some relationships between these sequences and permanents and determinants of one type of Hessenberg matrices.

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## 1. Introduction

The Pell and Perrin sequences [1] are defined by the following recurrence relations, respectively:

$$P_n = 2P_{n-1} + P_{n-2}, \quad \text{where } P_1 = 1, P_2 = 2,$$

$$R_n = R_{n-2} + R_{n-3}, \quad \text{where } R_0 = 3, R_1 = 0, R_2 = 2$$

for  $n > 2$ . The first few values of the sequences are

| $n$   | 1 | 2 | 3 | 4  | 5  | 6  | 7   | 8   | 9   |
|-------|---|---|---|----|----|----|-----|-----|-----|
| $P_n$ | 1 | 2 | 5 | 12 | 29 | 70 | 169 | 408 | 985 |
| $R_n$ | 0 | 2 | 3 | 1  | 2  | 5  | 5   | 7   | 10  |

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The permanent of a matrix is similar to the determinant but all the signs used in the Laplace expansion of minors are positive. The permanent of an  $n$ -square matrix is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$$

where the summation extends over all permutations  $\sigma$  of the symmetric group  $S_n$  [6].

Let  $A = [a_{ij}]$  be an  $m \times n$  matrix with row vectors  $r_1, r_2, \dots, r_m$ . We call  $A$  contractible on column  $k$ , if column  $k$  contains exactly two nonzero elements. Suppose that  $A$  is contractible on column  $k$  with  $a_{ik} \neq 0 \neq a_{jk}$  and  $i \neq j$ . Then the  $(m - 1) \times (n - 1)$  matrix  $A_{ij:k}$  obtained from  $A$  replacing row  $i$  with  $a_{jk}r_i + a_{ik}r_j$  and deleting row  $j$  and column  $k$  is called the contraction of  $A$  on column  $k$  relative to rows  $i$  and  $j$ . If  $A$  is contractible on row  $k$  with  $a_{ki} \neq 0 \neq a_{kj}$  and  $i \neq j$ , then the matrix  $A_{k:ij} = [A_{ij:k}^T]^T$  is called the contraction of  $A$  on row  $k$  relative to columns  $i$  and  $j$ . We know that if  $A$  is a nonnegative matrix and  $B$  is a contraction of  $A$  [2], then

$$perA = perB. \tag{1}$$

It is known that there are a lot of relations between determinants or permanents of matrices and well-known number sequences. For example, in [2], the authors consider the relationships between the sums of Fibonacci and Lucas numbers by Hessenberg matrices.

In [4], Lee defined the matrix

$$\mathcal{L}_n = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{bmatrix}$$

and showed that

$$per(\mathcal{L}_n) = L_{n-1}$$

where  $L_n$  is the  $n$ th Lucas number.

In [5], the author investigated general tridiagonal matrix determinants and permanents. Also he showed that the permanent of the tridiagonal matrix based on  $\{a_i\}$ ,  $\{b_i\}$ ,  $\{c_i\}$  is equal to the determinant of the matrix based on  $\{-a_i\}$ ,  $\{b_i\}$ ,  $\{c_i\}$ .

In [3], the authors found  $(0, 1, -1)$  tridiagonal matrices whose determinants and permanents are negatively subscripted Fibonacci and Lucas numbers. Also, they give an  $n \times n$   $(1, -1)$  matrix  $S$ , such that  $perA = \det(A \circ S)$ , where  $A \circ S$  denotes Hadamard product of  $A$  and  $S$ . Let  $S$  be a  $(1, -1)$  matrix of order  $n$ , defined with

$$S = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & \dots & 1 & 1 \\ 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & -1 & 1 \end{bmatrix}. \tag{2}$$

In the present paper, we consider the Pell and Perrin numbers as determinants and permanents of upper Hessenberg matrices,  $A = (a_{ij})$  is an upper Hessenberg matrix if  $a_{ij} = 0$  for  $i > j + 1$ .

**2. Determinantal representations of the Pell and Perrin numbers**

In this section, we define one type of upper Hessenberg matrix of odd order and show that the permanents of these type of matrices are the Pell numbers. Let  $H_n = [h_{ij}]_{n \times n}$  be an  $n$ -square matrix with  $h_{t,t+2} = 1$ ,  $h_{s,s+2} = -1$  for  $t = 2, 4, \dots, \frac{n-3}{2}$  and  $s = 1, 3, \dots, \frac{n-1}{2}$  and  $h_{i,j} = 1$  for  $|i - j| \leq 1$  and otherwise 0. Namely:

$$H_n = \begin{bmatrix} 1 & 1 & -1 & & & & \\ 1 & 1 & 1 & 1 & & & 0 \\ & 1 & 1 & 1 & -1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 1 & 1 \\ & & & & 1 & 1 & 1 & -1 \\ 0 & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}. \tag{3}$$

**Theorem 1.** Let  $H_n$  be an  $n$ -square matrix as in (3), then

$$perH_n = perH_n^{(n-2)} = P_n$$

where  $P_n$  is the  $n$ th Pell number.

**Proof.** By definition of the matrix  $H_n$ , it can be contracted on column 1. Let  $H_n^r$  be the  $r$ th contraction of  $H_n$ . If  $r = 1$ , then

$$H_n^1 = \begin{bmatrix} 2 & 0 & 1 & & & & 0 \\ 1 & 1 & 1 & -1 & & & \\ 0 & 1 & 1 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 1 & -1 \\ & & & & 1 & 1 & 1 \\ 0 & & & & & 1 & 1 \end{bmatrix}.$$

Since  $H_n^1$  also can be contracted according to the first column,

$$H_n^2 = \begin{bmatrix} 2 & 3 & -2 & 0 & 0 & & \\ 1 & 1 & 1 & 1 & 0 & & 0 \\ & 1 & 1 & 1 & -1 & & \\ & & 1 & 1 & 1 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & 1 & 1 & 1 & -1 \\ 0 & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}.$$

Going with this process, we have

$$H_n^3 = \begin{bmatrix} 5 & 0 & 2 & 0 & 0 & & \\ 1 & 1 & 1 & 1 & 0 & & 0 \\ & 1 & 1 & 1 & -1 & & \\ & & 1 & 1 & 1 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & 1 & 1 & 1 & -1 \\ 0 & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}$$

and contracting  $H_n^3$  according to the first column

$$H_n^4 = \begin{bmatrix} 5 & 7 & -5 & 0 & 0 & & \\ 1 & 1 & 1 & 1 & 0 & & 0 \\ & 1 & 1 & 1 & -1 & & \\ & & 1 & 1 & 1 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & 1 & 1 & 1 & -1 \\ 0 & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}.$$

Continuing this method, we obtain the  $r$ th contraction

$$H_n^r = \begin{bmatrix} P_{r+1} & 0 & P_r & 0 & 0 & & \\ 1 & 1 & 1 & -1 & 0 & & 0 \\ & 1 & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & -1 & \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & 1 & 1 & 1 & -1 \\ 0 & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}, \text{ if } r \text{ is odd}$$

$$H_n^r = \begin{bmatrix} P_r & P_{r-1} + P_r & -P_r & 0 & 0 & & \\ 1 & 1 & 1 & 1 & 0 & & 0 \\ & 1 & 1 & 1 & -1 & & \\ & & 1 & 1 & 1 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots \\ & & & & 1 & 1 & 1 & -1 \\ 0 & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}, \text{ if } r \text{ is even}$$

where  $2 \leq r \leq n - 4$ . Hence

$$H_n^{n-3} = \begin{bmatrix} P_k & P_{k-1} + P_k & -P_k \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

which, by contraction of  $H_n^{n-3}$  on column 1,

$$H_n^{n-2} = \begin{bmatrix} P_n & 0 \\ 1 & 1 \end{bmatrix}.$$

By (1), we have  $perH_n = perH_n^{(n-2)} = P_n$ .  $\square$

Let  $K(n) = [k_{ij}]$  be  $n \times n$  matrix with  $k_{11} = 1, k_{12} = 2, k_{13} = 3, k_{21} = 1, k_{23} = 1$  and  $k_{m,m+1} = k_{m+1,m} = 1$  for  $m = 3, 4, 5, \dots, n - 1$  and  $k_{p,p+2} = 1$  for  $p = 3, 4, \dots, n - 2$ . Clearly:

$$K_n = \begin{bmatrix} 1 & 2 & 3 & 0 & & & \\ 1 & 0 & 0 & 0 & & & 0 \\ & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 & 1 \\ 0 & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}. \tag{4}$$

**Theorem 2.** Let  $K_n$  be an  $n$ -square matrix as in (3), then

$$perK_n = perK_n^{(n-2)} = R_n$$

where  $R_n$  is the  $n$ th Perrin number.

**Proof.** By definition of the matrix  $K_n$ , it can be contracted on column 1. Namely,

$$K_n^1 = \begin{bmatrix} 2 & 3 & 0 & 0 & & & 0 \\ 1 & 0 & 1 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 0 & 1 \\ 0 & & & & & 1 & 0 \end{bmatrix}.$$

$K_n^1$  also can be contracted on the first column,

$$K_n^2 = \begin{bmatrix} 3 & 2 & 2 & 0 & 0 & & 0 \\ 1 & 0 & 1 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 0 & 1 \\ 0 & & & & & 1 & 0 \end{bmatrix}.$$



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