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Poitou–Tate duality over extensions of global fields

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ABSTRACT

In this paper, we are interested in the Poitou–Tate duality in Galois cohomology. We will formulate and prove a theorem for a nice class of modules (with a continuous Galois action) over a pro-p ring. The theorem will comprise of the Tate local duality, Poitou–Tate duality and the Poitou–Tate's exact sequence.

1. Introduction

The classical Poitou–Tate duality is a duality principle for a local–global statement, namely it relates the kernels of the localization maps. Using compactly supported cohomology groups, one can give a cleaner formulation of the statement which we now do. Let F be a global field with characteristic not equal to p, and let S be a finite set of primes of F containing all primes above p and all Archimedean primes of F. We let $G_{F,S}$ denote the Galois group $\mathrm{Gal}(F_S/F)$ of the maximal unramified outside S extension F_S of F inside a fixed separable closure of F. In its usual formulation, Poitou–Tate duality relates the kernels of the localization maps on the $G_{F,S}$-cohomology of a module and the Tate twist of its Pontryagin dual. For simplicity, we assume in this introduction that p is odd if F has any real places. The general result without this assumption can be found in Theorem 4.2.6.

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The nth compactly supported $G_{F,S}$-cohomology group $H^n_{c,\mathrm{cts}}(G_{F,S}, M)$ with coefficients in a topological $G_{F,S}$-module M is defined as the nth cohomology group of the complex

$$\mathrm{Cone}\left(\dot{C}_{\mathrm{cts}}(G_{F,S}, M) \xrightarrow{\mathrm{res}_S} \bigoplus_{v\in S_f} \dot{C}_{\mathrm{cts}}(G_{F_v}, M)\right)[-1],$$

where G_{F_v} is the absolute Galois group of the completion of F at v, and res_S is the sum of restriction maps on the continuous cochain complexes. It therefore fits in a long exact sequence

$$\cdots \to H^n_{c,\mathrm{cts}}(G_{F,S}, M) \to H^n_{\mathrm{cts}}(G_{F,S}, M) \to \bigoplus_{v\in S} H^n_{\mathrm{cts}}(G_{F_v}, M) \to H^{n+1}_{c,\mathrm{cts}}(G_{F,S}, M) \to \cdots.$$

We now let R denote a commutative complete Noetherian local ring with finite residue field of characteristic p. Then we have the following formulation of Poitou–Tate duality due to Nekovář [9, Prop. 5.4.3(i)].

Theorem. *Let T be a finitely generated R-module with a continuous (R-linear) $G_{F,S}$-action. Then there are isomorphisms*

$$H^n_{\mathrm{cts}}(G_{F,S}, T) \xrightarrow{\sim} H^{3-n}_{c,\mathrm{cts}}\big(G_{F,S}, T^\vee(1)\big)^\vee,$$
$$H^n_{c,\mathrm{cts}}(G_{F,S}, T) \xrightarrow{\sim} H^{3-n}_{\mathrm{cts}}\big(G_{F,S}, T^\vee(1)\big)^\vee$$

of R-modules for all n, where $T^\vee = \mathrm{Hom}_{\mathrm{cts}}(T, \mathbb{Q}_p/\mathbb{Z}_p)$.

We now recall some notation from the language of derived categories. We denote by $\mathbf{D}(Mod_R)$ the derived category of R-modules which is obtained from the category $\mathrm{Ch}(Mod_R)$ of chain complexes of R-modules by inverting the quasi-isomorphisms, i.e., the maps of complexes that induce isomorphisms on cohomology. We have the derived functors $\mathbf{R}\mathrm{Hom}_R(-,-)$, $\mathbf{R}\Gamma_{\mathrm{cts}}(G_{F,S}, -)$ and $\mathbf{R}\Gamma_{c,\mathrm{cts}}(G_{F,S}, -)$ that are obtained from $\mathrm{Hom}_R(-,-)$, $C_{\mathrm{cts}}(G_{F,S}, -)$ and $C_{c,\mathrm{cts}}(G_{F,S}, -)$. Then Poitou–Tate duality can be reformulated as the following isomorphisms

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G_{F,S}, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbf{R}\Gamma_{c,\mathrm{cts}}\big(G_{F,S}, T^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\big)[-3],$$
$$\mathbf{R}\Gamma_{c,\mathrm{cts}}(G_{F,S}, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbf{R}\Gamma_{\mathrm{cts}}\big(G_{F,S}, T^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\big)[-3]$$

in $\mathbf{D}(Mod_R)$.

Now suppose that F_∞ is a p-adic Lie extension of F contained in F_S. We denote by Γ the Galois group of the extension F_∞/F, and we let $\Lambda = R[\![\Gamma]\!]$ denote the resulting Iwasawa algebra over R. Let T be a finitely generated R-module with a continuous (R-linear) $G_{F,S}$-action, and let A be a cofinitely generated R-module with a continuous (R-linear) $G_{F,S}$-action. The Λ-modules of interest are the following direct and inverse limits of cohomology groups (and their counterparts with compact support)

$$\varinjlim_{F_\alpha} H^n_{\mathrm{cts}}\big(\mathrm{Gal}(F_S/F_\alpha), A\big) \quad \text{and} \quad \varprojlim_{F_\alpha} H^n_{\mathrm{cts}}\big(\mathrm{Gal}(F_S/F_\alpha), T\big),$$

where the limits are taken over all finite Galois extensions F_α of F which are contained in F_∞. By an application of Shapiro's lemma, one can show that they are respectively isomorphic to

$$H^n_{\mathrm{cts}}\big(G_{F,S}, F_\Gamma(A)\big) \quad \text{and} \quad H^n_{\mathrm{cts}}\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big),$$

where the Λ-modules $F_\Gamma(A)$ and $\mathscr{F}_\Gamma(T)$ are defined by

$$\varinjlim_{F_\alpha} \mathrm{Hom}_R\big(R\big[\mathrm{Gal}(F_\alpha/F)\big], A\big) \quad \text{and} \quad \varprojlim_{F_\alpha} R\big[\mathrm{Gal}(F_\alpha/F)\big] \otimes_R T$$

respectively. Therefore, we can reduce the question of finding dualities on the Iwasawa modules of interest to that of obtaining dualities over $G_{F,S}$, but with R replaced by Λ.

In his monograph [9], Nekovář considers the above situation over a commutative p-adic Lie extension (e.g., a $\mathbb{Z}_p^r$-extension) and develops an extension of Poitou–Tate global duality for the above cohomology groups. In view of the vast activity in the study of noncommutative generalizations of the main conjecture of Iwasawa theory [1,3,6,11], one would like to extend the above theory to the noncommutative setting.

In fact, in this paper, we study generalizations of the above duality of Poitou–Tate over a general pro-p ring Λ (not necessarily commutative). Together with the module theory, we carefully develop the theory of continuous group cohomology in our setting. From there, we are able to state and prove our duality theorem (cf. Theorem 4.2.6).

Theorem. *Let M be a bounded complex of objects that are profinite Λ-modules with a continuous (Λ-linear) $G_{F,S}$-action. Then we have the following isomorphism*

$$\begin{array}{ccc}
\bigoplus_{v\in S} \mathbf{R}\Gamma(G_v, M)[-1] & \longrightarrow & \bigoplus_{v\in S} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_v, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma_c(G_{F,S}, M) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_{F,S}, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_{F,S}, M) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma_c(G_{F,S}, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-3]
\end{array}$$

of exact triangles in $\mathbf{D}(Mod_\Lambda)$.

We now give a brief description of the contents of each section of the paper. In Section 2, we introduce notations and results from homological algebra required for the paper. Section 3 is about the discussion of profinite rings and their topological modules. We also introduce continuous cohomology groups with coefficients in compact modules and discrete modules. In Section 4, we will formulate and prove our duality theorems. In Section 5, we will apply the duality theorems proved in Section 4 to extensions of global fields.

2. Preliminaries

We begin by reviewing certain objects and notation that will be used in this write-up. Most of the material presented in this section can be found in [4,9,13]. Throughout the paper, every ring is associative and has a unit.

Fix an abelian category $\mathcal{A}$ and denote the category of (cochain) complexes of objects in $\mathcal{A}$ by $\mathrm{Ch}(\mathcal{A})$. We also denote the category of bounded below complexes, bounded above complexes and bounded complexes by $\mathrm{Ch}^+(\mathcal{A})$, $\mathrm{Ch}^-(\mathcal{A})$ and $\mathrm{Ch}^b(\mathcal{A})$ respectively. For each $n \in \mathbb{Z}$, the translation by n of a complex X is given by

$$X[n]^i = X^{n+i}, \qquad d^i_{X[n]} = (-1)^n d^{n+i}_X.$$

If $f : X \longrightarrow Y$ is a morphism of complexes, then $f[n] : X[n] \longrightarrow Y[n]$ is given by $f[n]^i = f^{n+i}$.

If X is a complex, we have the following truncations of X:

$$\sigma_{\leqslant i}X = \big[\cdots \longrightarrow X^{i-2} \longrightarrow X^{i-1} \longrightarrow X^i \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\big],$$
$$\tau_{\leqslant i}X = \big[\cdots \longrightarrow X^{i-2} \longrightarrow X^{i-1} \longrightarrow \ker\big(d_X^i\big) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\big],$$
$$\sigma_{\geqslant i}X = \big[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^i \longrightarrow X^{i+1} \longrightarrow X^{i+2} \longrightarrow \cdots\big],$$
$$\tau_{\geqslant i}X = \big[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{coker}\big(d_X^{i-1}\big) \longrightarrow X^{i+1} \longrightarrow X^{i+2} \longrightarrow \cdots\big].$$

The cone of a morphism $f : X \longrightarrow Y$ is defined by $\operatorname{Cone}(f) = Y \oplus X[1]$ with differential

$$d^i_{\operatorname{Cone}(f)} = \begin{pmatrix} d_Y^i & f^{i+1} \\ 0 & -d_X^{i+1} \end{pmatrix} : Y^i \oplus X^{i+1} \longrightarrow Y^{i+1} \oplus X^{i+2}.$$

There is an exact sequence of complexes

$$0 \longrightarrow Y \xrightarrow{j} \operatorname{Cone}(f) \xrightarrow{p} X[1] \longrightarrow 0,$$

where j and p are the canonical inclusion and projection respectively. The corresponding boundary map

$$\delta : H^i\big(X[1]\big) = H^{i+1}(X) \longrightarrow H^{i+1}(Y)$$

is induced by f^{i+1}.

If X is a complex and $x \in X^i$, we write $\bar{x} = i$ for the degree.

Let Λ, S and T be rings. Let M (resp., N) be a left Λ-S-bimodule (resp., a Λ-T-bimodule). Then $\operatorname{Hom}_\Lambda(M, N)$ is taken to be the S-T-bimodule of all left Λ-module homomorphisms from M to N, where the left S-action is given by $(s \cdot f)(m) = f(ms)$ and the right T-action is given by $(f \cdot t)(m) = f(m)t$ for $f \in \operatorname{Hom}_\Lambda(M, N)$, $m \in M$, $s \in S$ and $t \in T$. If $M^\bullet$ is a complex of Λ-S-bimodules and $N^\bullet$ a complex of Λ-T-bimodules, we define a complex $\operatorname{Hom}^\bullet_\Lambda(M^\bullet, N^\bullet)$ of S-T-bimodules by

$$\operatorname{Hom}^n_\Lambda\big(M^\bullet, N^\bullet\big) = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_\Lambda\big(M^i, N^{i+n}\big)$$

with differentials defined as follows: for $f \in \operatorname{Hom}_\Lambda(M^i, N^{i+n})$, we have

$$df = d_N^{i+n} \circ f + (-1)^n f \circ d_M^{i-1}.$$

In the case when $S = T$, we have a similar definition for the complexes $\operatorname{Hom}^\bullet_{\Lambda\text{-}S}(M^\bullet, N^\bullet)$ of abelian groups, where $\operatorname{Hom}_{\Lambda\text{-}S}(M, N)$ is the group of all of Λ-S-bimodule homomorphisms from M to N. It follows immediately from the definition that for an element $f \in \operatorname{Hom}^0_{\Lambda\text{-}S}(M^\bullet, N^\bullet)$, we have $f \in \operatorname{Hom}_{\operatorname{Ch}(\Lambda\text{-}S)}(M^\bullet, N^\bullet)$ if and only if $df = 0$. Here $\operatorname{Ch}(\Lambda\text{-}S)$ denotes the category of complexes of Λ-S-bimodules.

Suppose that $M^\bullet$ is a complex of Λ-S-bimodules and $L^\bullet$ a complex of S-T-bimodules. We define the complex $M^\bullet \otimes_S L^\bullet$ of Λ-T-bimodules by

$$\big(M^\bullet \otimes_S L^\bullet\big)^n = \bigoplus_{i\in\mathbb{Z}} M^i \otimes_S L^{n-i}$$

with differentials

$$d(m \otimes l) = dm \otimes l + (-1)^{\bar{m}} m \otimes dl.$$

We end the section by collecting some technical results which will be used in the paper.

Lemma 2.1. *The following formulas define isomorphisms of complexes*:

$$\begin{gathered}
\operatorname{Hom}_{\Lambda}^{\bullet}\big(M^{\bullet}, N^{\bullet}\big)[n] \cong \operatorname{Hom}_{\Lambda}^{\bullet}\big(M^{\bullet}, N^{\bullet}[n]\big), \\
f \mapsto f, \\
\big(M^{\bullet}[n]\big) \otimes_S L^{\bullet} \cong \big(M^{\bullet} \otimes_S L^{\bullet}\big)[n], \\
m \otimes l \mapsto m \otimes l, \\
M^{\bullet} \otimes_S \big(L^{\bullet}[n]\big) \cong \big(M^{\bullet} \otimes_S L^{\bullet}\big)[n], \\
m \otimes l \mapsto (-1)^{n\bar{m}} m \otimes l.
\end{gathered}$$

Proof. This follows from a straightforward verification of the definition of translation and the sign conventions. □

Lemma 2.2. *The adjunction morphisms define morphisms*

$$\begin{gathered}
\operatorname{Hom}_{\Lambda\text{-}T}^{\bullet}\big(M^{\bullet} \otimes_S L^{\bullet}, N^{\bullet}\big) \longrightarrow \operatorname{Hom}_{\Lambda\text{-}S}^{\bullet}\big(M^{\bullet}, \operatorname{Hom}_{T^o}^{\bullet}\big(L^{\bullet}, N^{\bullet}\big)\big), \\
f \mapsto \big(m \mapsto \big(l \mapsto f(m \otimes l)\big)\big), \\
\operatorname{Hom}_{\Lambda\text{-}T}^{\bullet}\big(M^{\bullet} \otimes_S L^{\bullet}, N^{\bullet}\big) \longrightarrow \operatorname{Hom}_{S\text{-}T}^{\bullet}\big(L^{\bullet}, \operatorname{Hom}_{\Lambda}^{\bullet}\big(M^{\bullet}, N^{\bullet}\big)\big), \\
f \mapsto \big(l \mapsto \big(m \mapsto (-1)^{\bar{m}\bar{l}} f(m \otimes l)\big)\big)
\end{gathered}$$

of complexes and morphisms

$$\begin{gathered}
\operatorname{Hom}_{\operatorname{Ch}(\Lambda\text{-}T)}\big(M^{\bullet} \otimes_S L^{\bullet}, N^{\bullet}\big) \longrightarrow \operatorname{Hom}_{\operatorname{Ch}(\Lambda\text{-}S)}\big(M^{\bullet}, \operatorname{Hom}_{T^o}^{\bullet}\big(L^{\bullet}, N^{\bullet}\big)\big), \\
\operatorname{Hom}_{\operatorname{Ch}(\Lambda\text{-}T)}\big(M^{\bullet} \otimes_S L^{\bullet}, N^{\bullet}\big) \longrightarrow \operatorname{Hom}_{\operatorname{Ch}(S\text{-}T)}\big(L^{\bullet}, \operatorname{Hom}_{\Lambda}^{\bullet}\big(M^{\bullet}, N^{\bullet}\big)\big)
\end{gathered}$$

of abelian groups. All of these maps are monomorphisms; *they are isomorphisms if $M^{\bullet}$ and $L^{\bullet}$ are bounded above and $N^{\bullet}$ is bounded below.*

Lemma 2.3. *Given the following data*:

(1) *Complexes A_1, B_1 of Λ-S-bimodules, complexes A_2, B_2 of S-T-bimodules, and complexes A_3, B_3 of Λ-T-bimodules.*
(2) *Morphisms of complexes $f_j : A_j \longrightarrow B_j$ preserving the respective bimodule structures.*
(3) *Morphisms of complexes of Λ-T-bimodules*

$$\begin{gathered}
\cup_A : A_1 \otimes_S A_2 \longrightarrow A_3, \\
\cup_B : B_1 \otimes_S B_2 \longrightarrow B_3
\end{gathered}$$

such that $f_3 \circ \cup_A = \cup_B \circ (f_1 \otimes f_2)$. *For* $j = 1, 2, 3$, *define* E_j *to be the complex*

$$\mathrm{Cone}(A_j \xrightarrow{f_j} B_j)[-1].$$

Then we have morphisms of complexes

$$\cup_0, \cup_1 : E_1 \otimes_S E_2 \longrightarrow E_3$$

given by the formulas

$$(a_1, b_1) \cup_0 (a_2, b_2) = \big(a_1 \cup_A a_2, (-1)^{\bar{a}_1} f_1(a_1) \cup_B b_2\big),$$
$$(a_1, b_1) \cup_1 (a_2, b_2) = \big(a_1 \cup_A a_2, b_1 \cup_B f_2(a_2)\big),$$

and the formula

$$s\big((a_1, b_1) \otimes (a_2, b_2)\big) = \big(0, (-1)^{\bar{a}_1} b_1 \cup_B b_2\big)$$

defines a homotopy $s : \cup_1 \rightsquigarrow \cup_0$.

Proof. This is a special case of [9, Prop. 1.3.2]. □

3. Profinite rings

Completed group algebras of profinite groups arise naturally in the study of Iwasawa theory, and such rings are profinite rings. In this section, we shall study the properties of profinite rings and their (topological) modules. We will also develop a cohomological theory over such rings.

Throughout the section, Λ will always denote a profinite ring, and $\mathcal{I}$ is a directed fundamental system of open neighborhoods of zero consisting of two-sided ideals of Λ. We use Λ° to denote the opposite ring to Λ.

3.1. Topological Λ-modules

In this subsection, we will study the topological modules over a profinite ring Λ. These are Hausdorff topological abelian groups with a continuous Λ-action. In particular, we are interested in the following two classes of topological Λ-modules.

Definition 3.1.1. We say that a topological Λ-module M is a compact (resp., discrete) Λ-module if its underlying topology is compact (resp., discrete). The category of compact Λ-modules (resp., discrete Λ-modules) is denoted by $\mathcal{C}_\Lambda$ (resp., $\mathcal{D}_\Lambda$).

The following proposition records some of the properties of the above two categories, whose proofs can be found in [12, Chap. 5].

Proposition 3.1.2. (i) *Every compact* Λ*-module is a projective limit of finite modules and has a fundamental system of neighborhoods of zero consisting of open submodules. In particular, it is an abelian profinite group.*

(ii) *Every discrete* Λ*-module is the direct limit of finite* Λ*-modules. In particular, it is an abelian torsion group.*

(iii) *Pontryagin duality induces a duality between the category* $\mathcal{C}_\Lambda$ *of compact* Λ*-modules and the category* $\mathcal{D}_{\Lambda^\circ}$ *of discrete* Λ°*-modules.*

(iv) *The category* $\mathcal{C}_\Lambda$ *is abelian and has enough projectives and exact inverse limits. The category* $\mathcal{D}_\Lambda$ *is abelian and has enough injectives and exact direct limits.*

We give another description of discrete Λ-modules. If M is a Λ-module and $\mathfrak{a}$ is a two-sided ideal of Λ, we define

$$M[\mathfrak{a}] = \{x \in M \mid \mathfrak{a} \subseteq \mathrm{Ann}(x)\}.$$

With this, we have the following lemma.

Lemma 3.1.3. *Let M be an abstract Λ-module. Then M is a discrete Λ-module (i.e., the Λ-action is continuous with respect to the discrete topology on M) if and only if*

$$M = \bigcup_{\mathfrak{a} \in \mathcal{I}} M[\mathfrak{a}].$$

Proof. Suppose that M is a discrete Λ-module. Let $x \in M$. Then by the continuity of the Λ-action, there exists $\mathfrak{a} \in \mathcal{I}$ such that $\mathfrak{a} \cdot x = 0$. This implies that $x \in M[\mathfrak{a}]$.

Conversely, suppose that

$$M = \bigcup_{\mathfrak{a} \in \mathcal{I}} M[\mathfrak{a}].$$

We shall show that the action

$$\theta : \Lambda \times M \longrightarrow M$$

is continuous, where M is given the discrete topology. In other words, for each $x \in M$, we need to show that $\theta^{-1}(x)$ is open in $\Lambda \times M$. Let $(\lambda, y) \in \theta^{-1}(x)$. Then $y \in M[\mathfrak{a}]$ for some $\mathfrak{a} \in \mathcal{I}$. Therefore, we have $(\lambda, y) \in (\lambda + \mathfrak{a}) \times \{y\}$, and the latter set is an open set contained in $\theta^{-1}(x)$. □

When working with topological Λ-modules, one will have to consider continuous homomorphisms between the modules. In general, an abstract homomorphism of modules may not be continuous. In the next lemma, we record a few situations where every abstract homomorphism is continuous. We say that a topological Λ-module M is endowed with the $\mathcal{I}$-adic topology if the collection $\{\mathfrak{a}M\}_{\mathfrak{a} \in \mathcal{I}}$ forms a fundamental system of neighborhoods of zero.

Lemma 3.1.4. *Let M and N be two topological Λ-modules. Suppose one of the following cases holds.*

(1) *Both M and N have the $\mathcal{I}$-adic topology.*
(2) *Both M and N have the discrete topology.*
(3) *M is a finitely generated Λ-module endowed with the $\mathcal{I}$-adic topology, and N is a compact Λ-module.*
(4) *M is a finitely generated Λ-module endowed with the $\mathcal{I}$-adic topology, and N is a discrete Λ-module.*

Then every abstract Λ-homomorphism is continuous. In other words, we have

$$\mathrm{Hom}_{\Lambda,\mathrm{cts}}(M, N) = \mathrm{Hom}_{\Lambda}(M, N).$$

Proof. (1) and (2) are straightforward.

(3) Suppose M is generated by $e_1, \ldots, e_r$. Let $f : M \longrightarrow N$ be an abstract Λ-homomorphism, and for each i, set $x_i = f(e_i)$. Let V be an open Λ-submodule of N. By continuity of the Λ-action on N, for each i, there exists $\mathfrak{a}_i \in \mathcal{I}$ such that $\mathfrak{a}_i \cdot x_i \subseteq V$. Since $\mathcal{I}$ is directed, we can find $\mathfrak{a} \in \mathcal{I}$ such that $\mathfrak{a} \subseteq \mathfrak{a}_i$ for all i. It follows that $f(\mathfrak{a}M) \subseteq V$, establishing the continuity of f.

(4) We retain the notation in (3). By Lemma 3.1.3, for each i, there exists $\mathfrak{a}_i \in \mathcal{I}$ such that $x_i \in N[\mathfrak{a}_i]$. Since $\mathcal{I}$ is directed, we can find $\mathfrak{a} \in \mathcal{I}$ such that $\mathfrak{a} \subseteq \mathfrak{a}_i$ for all i, and $f(\mathfrak{a}M) = 0$. □

Corollary 3.1.5. *Let M be a compact Λ-module. Then every finitely generated abstract Λ-submodule of M is a closed subset of M. In particular, every finitely generated left (or right) ideal of Λ is closed in Λ.*

Proof. Let N be a Λ-submodule of M generated by $x_1, \ldots, x_r$. By Lemma 3.1.4(3), the following Λ-homomorphism

$$\begin{aligned} \phi : \bigoplus_{i=1}^{r} \Lambda &\longrightarrow M, \\ e_i &\mapsto x_i \end{aligned}$$

is continuous. Since $\bigoplus_{i=1}^{r} \Lambda$ is compact, so is its image N. □

Corollary 3.1.6. *Let M be a finitely presented abstract Λ-module. Then M is a compact Λ-module.*

Proof. Since M is finitely presented, we have an exact sequence $\Lambda^r \xrightarrow{f} \Lambda^s \longrightarrow M \longrightarrow 0$ for some integers r and s. By Lemma 3.1.4(1), the map f is a continuous Λ-homomorphism of compact Λ-modules. Since the category $\mathcal{C}_\Lambda$ is abelian by Lemma 3.1.2(iv), it follows that M is an object in $\mathcal{C}_\Lambda$. □

In view of Corollary 3.1.6, one may ask the following two questions. The first is if one can say anything about the $\mathcal{I}$-adic topology on an abstract Λ-module M. In general, it is not even clear whether this topology is Hausdorff. The second question that one may ask is if there are other ways to endow a finitely presented Λ-module with a topology such that it becomes a compact Λ-module. In response to these two questions, we have the following proposition. In fact, as we shall see, if M is already a compact Λ-module, the $\mathcal{I}$-adic topology is Hausdorff, and it is the only one with which one can endow a finitely presented Λ-module in order to make it into a compact Λ-module. One may compare the following proposition with [10, Prop. 5.2.17].

Proposition 3.1.7. *Let M be a compact Λ-module. Then the $\mathcal{I}$-adic topology is finer than the original topology of M, and the canonical homomorphism*

$$\alpha : M \longrightarrow \varprojlim_{\mathfrak{a} \in \mathcal{I}} M/\mathfrak{a}M$$

of Λ-modules is injective. Furthermore, if M is a finitely generated Λ-module, then the topologies coincide, and the above homomorphism is a continuous isomorphism of compact Λ-modules.

Proof. Let N be an open submodule of M. Then by continuity, for each $x \in N$, there exist a neighborhood V_x of x and $\mathfrak{a}_x \in \mathcal{I}$ such that $\mathfrak{a}_x V_x \subseteq N$. Since M is compact, it is covered by finitely many such sets, say $V_{x_1}, V_{x_2}, \ldots, V_{x_r}$. Choose $\mathfrak{a} \in \mathcal{I}$ such that $\mathfrak{a} \subseteq \mathfrak{a}_i$ for all $i = 1, \ldots, r$. Then we have $\mathfrak{a}M \subseteq N$, and this shows the first assertion. Since M is Hausdorff under its original topology, it follows that M is Hausdorff under the $\mathcal{I}$-adic topology and so

$$\ker \alpha = \bigcap_{\mathfrak{a} \in \mathcal{I}} \mathfrak{a}M = 0.$$

Now if M is finitely generated, we have a surjection

$$\Lambda^m \twoheadrightarrow (M \text{ with } \mathcal{I}\text{-adic topology}),$$

which is continuous by Lemma 3.1.4(1). This implies that M with the $\mathcal{I}$-adic topology is compact. By the first assertion, the identity map

$$(M \text{ with } \mathcal{I}\text{-adic topology}) \longrightarrow M$$

is continuous. This in turn gives a continuous bijection between compact spaces and is therefore a homeomorphism. If M is given the $\mathcal{I}$-adic topology, then the image of α is dense in $\varprojlim_{\mathfrak{a}\in\mathcal{I}} M/\mathfrak{a}M$, and so is surjective since M is compact. □

We conclude with a description of projective objects in $\mathcal{C}_\Lambda$ that are finitely generated over Λ.

Proposition 3.1.8. *Let P be a projective object in $\mathcal{C}_\Lambda$ that is finitely generated over Λ. Then P is a projective Λ-module. Conversely, let P be a finitely generated projective Λ-module. Then P, endowed with the $\mathcal{I}$-adic topology, is a compact Λ-module and is a projective object in $\mathcal{C}_\Lambda$.*

Proof. Let P be a projective object in $\mathcal{C}_\Lambda$ that is finitely generated over Λ. Then there is a surjection $f : \Lambda^r \twoheadrightarrow P$ of Λ-modules. By Proposition 3.1.7, the topology on P is precisely the $\mathcal{I}$-adic topology, and it follows from Lemma 3.1.4(1) that f is a continuous homomorphism of compact Λ-modules. Now since P is a projective object in $\mathcal{C}_\Lambda$, the map f has a continuous Λ-linear section. In particular, this implies that we have an isomorphism $\Lambda^r \cong P \oplus (\ker f)$ of Λ-modules. Hence P is a projective Λ-module.

Conversely, suppose that P is a finitely generated projective Λ-module. Then there exists a finitely generated projective Λ-module Q such that $P \oplus Q$ is a free Λ-module of finite rank. We then have a surjection $\pi : \Lambda^n \twoheadrightarrow Q$, and this gives a finite presentation

$$\Lambda^n \longrightarrow P \oplus Q \longrightarrow P \longrightarrow 0$$

of P where the first map sends an element x of Λ^n to $(0, \pi(x))$ and the second map is the canonical projection. It then follows from Proposition 3.1.6 that P is a compact Λ-module under the $\mathcal{I}$-adic topology. Now suppose we are given the following diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle \alpha} \\ M & \xrightarrow{\varepsilon} & N \end{array}$$

of compact Λ-modules and continuous Λ-homomorphisms. Since P is a projective Λ-module, there is an abstract Λ-homomorphism $\beta : P \to M$ such that $\varepsilon\beta = \alpha$. On the other hand, it follows from Lemma 3.1.4(3) that β is also continuous. Therefore, this shows that P is a projective object of $\mathcal{C}_\Lambda$. □

3.2. Continuous cochains

Definition 3.2.1. Let G be a profinite group. We define $\mathcal{C}_{\Lambda,G}$ to be the category where the objects are compact Λ-modules with a continuous Λ-linear G-action and the morphisms are continuous $\Lambda[G]$-homomorphisms. Similarly, we define $\mathcal{D}_{\Lambda,G}$ to be the category where the objects are discrete Λ-modules with a continuous Λ-linear G-action and the morphisms are (continuous) $\Lambda[G]$-homomorphisms.

Proposition 3.2.2. (i) *The category $\mathcal{C}_{\Lambda,G}$ is abelian, has enough projectives and exact inverse limits.*

(ii) *The category $\mathcal{D}_{\Lambda,G}$ is abelian, has enough injectives and exact direct limits.*

(iii) *The Pontryagin duality induces a contravariant equivalence between $\mathcal{C}_{\Lambda,G}$ and $\mathcal{D}_{\Lambda^\circ,G}$ (resp. $\mathcal{C}_{\Lambda^\circ,G}$ and $\mathcal{D}_{\Lambda,G}$).*

Proof. We shall prove (iii) first. For a topological group A, we shall denote $A^\vee$ to be its Pontryagin dual. By Proposition 3.1.2, it suffices to show that if M (resp., N) is an object of $\mathcal{C}_{\Lambda,G}$ (resp., $\mathcal{D}_{\Lambda^\circ,G}$), then $M^\vee$ (resp., $N^\vee$) is an object of $\mathcal{D}_{\Lambda^\circ,G}$ (resp., $\mathcal{C}_{\Lambda,G}$). We define a G-action on $M^\vee$ by $\sigma\cdot f(m) = f(\sigma^{-1}m)$ for $f\in M^\vee, \sigma\in G$ and $m\in M$. This is clearly Λ°-linear, and since G is profinite, we may apply [2, Prop. 3] to conclude that the G-action is continuous. The same argument works for N. Hence we have proven (iii). It remains to prove (ii), since (i) will follow from (ii) and (iii).

To prove (ii), we note that it is clear that $\mathcal{D}_{\Lambda,G}$ is abelian and has exact direct limits. It remains to show that it has enough injectives. By the lemma to follow, we see that the functor

$$M\mapsto \bigcup_{\mathfrak{a}\in\mathcal{I}}\bigcup_{U}\big(M[\mathfrak{a}]\big)^U : Mod_{\Lambda[G]}\longrightarrow \mathcal{D}_{\Lambda,G}$$

is right adjoint to an exact functor, and so preserves injectives by [13, Prop. 2.3.10]. Since $Mod_{\Lambda[G]}$ has enough injectives, it follows that $\mathcal{D}_{\Lambda,G}$ also has enough injectives. □

Lemma 3.2.3. *An abstract $\Lambda[G]$-module N is an object in $\mathcal{D}_{\Lambda,G}$ if and only if*

$$N=\bigcup_{\mathfrak{a}\in\mathcal{I}}\bigcup_{U}\big(N[\mathfrak{a}]\big)^U,$$

where U runs through all the open subgroups of G. Moreover, if M is an abstract $\Lambda[G]$-module, then

$$\bigcup_{\mathfrak{a}\in\mathcal{I}}\bigcup_{U}\big(M[\mathfrak{a}]\big)^U,$$

is an object of $\mathcal{D}_{\Lambda,G}$, and there is a canonical isomorphism

$$\mathrm{Hom}_{\Lambda[G],\mathrm{cts}}\Big(N,\bigcup_{\mathfrak{a}\in\mathcal{I}}\bigcup_{U}\big(M[\mathfrak{a}]\big)^U\Big)\cong \mathrm{Hom}_{\Lambda[G]}(N,M)$$

for every $N\in\mathcal{D}_{\Lambda,G}$.

Proof. Suppose N is an object in $\mathcal{D}_{\Lambda,G}$. Then, in particular, it is a discrete Λ-module. By Lemma 3.1.3, we have $N=\bigcup_{\mathfrak{a}\in\mathcal{I}}N[\mathfrak{a}]$. Let $x\in N[\mathfrak{a}]$. Then by continuity of the G-action, there exists an open subgroup U of G such that $U\cdot x = x$.

Conversely, suppose that

$$N=\bigcup_{\mathfrak{a}\in\mathcal{I}}\bigcup_{U}\big(N[\mathfrak{a}]\big)^U.$$

Clearly this implies that $N=\bigcup_{\mathfrak{a}\in\mathcal{I}}N[\mathfrak{a}]$, and so N is a discrete Λ-module. It remains to show that the G-action

$$\theta : G\times N\longrightarrow N$$

is continuous. Let $x\in N$, and let $(\sigma, y)\in\theta^{-1}(x)$. Then $y\in N[\mathfrak{a}]^U$ for some $\mathfrak{a}\in\mathcal{I}$ and open subgroup U. In particular, we have $(\sigma, y)\in\sigma U\times\{y\}\subseteq\theta^{-1}(x)$. Therefore, this proves the first assertion. The second assertion is an immediate consequence of the first. □

Lemma 3.2.4. *Let M be an object of $\mathcal{C}_{\Lambda,G}$. Then M has a fundamental system of neighborhoods of zero consisting of open $\Lambda[G]$-submodules.*

Proof. Let N be an open Λ-submodule of M. Then for each $g \in G$, there exist an open Λ-submodule N_g of M and an open subgroup U_g of G such that $gU_g \cdot N_g \subseteq N$. Since G is compact, it is covered by finite number of such cosets, say $g_1U_{g_1}, \ldots, g_rU_{g_r}$. Set $N_0 = \bigcap_{i=1}^{r} N_{g_i}$. This is an open Λ-submodule of M. Then $\Lambda[G] \cdot N_0$ is a $\Lambda[G]$-submodule of M which contains N_0 and is contained in N. □

For the remainder of the subsection, we will be studying the continuous cochain complex (and its cohomology) of G with coefficients in certain classes of topological Λ-modules.

Definition 3.2.5. Let M be a topological Λ-module with a continuous Λ-linear G-action. The (inhomogeneous) continuous cochains $C^i_{\mathrm{cts}}(G, M)$ of degree $i \geqslant 0$ on G with values in M are defined to be the left Λ-module of continuous maps $G^i \to M$ with the usual differential

$$\begin{aligned}\big(\delta^i c\big)(g_1, \ldots, g_{i+1}) = g_1 c(g_2, \ldots, g_{i+1}) + \sum_{j=1}^{i} (-1)^j c(g_1, \ldots, g_j g_{j+1}, \ldots, g_{i+1}) \\ + (-1)^{i-1} c(g_1, \ldots, g_i),\end{aligned}$$

which maps $C^i_{\mathrm{cts}}(G, M_\alpha)$ to $C^{i+1}_{\mathrm{cts}}(G, M_\alpha)$. It then follows that

$$\cdots \longrightarrow C^i_{\mathrm{cts}}(G, M) \xrightarrow{\delta^i_M} C^{i+1}_{\mathrm{cts}}(G, M) \longrightarrow \cdots$$

is a complex of Λ-modules and its ith cohomology group is denoted by $H^i_{\mathrm{cts}}(G, M)$. The following lemma is a standard result (cf. [10, Lemma 2.7.2]).

Lemma 3.2.6. *Let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of topological Λ-modules with a continuous Λ-linear G-action such that the topology of M' is induced by that of M and such that β has a continuous (not necessarily Λ-linear) section. Then

$$0 \to C^\bullet_{\mathrm{cts}}\big(G, M'\big) \xrightarrow{\alpha_*} C^\bullet_{\mathrm{cts}}(G, M) \xrightarrow{\beta_*} C^\bullet_{\mathrm{cts}}\big(G, M''\big) \to 0$$

is an exact sequence of complexes of Λ-modules.

We are particularly interested in the case when M is an object of $\mathcal{C}_{\Lambda,G}$ or $\mathcal{D}_{\Lambda,G}$. We now discuss cohomology and limits.

Proposition 3.2.7. *Let $N = \varinjlim_\alpha N_\alpha$ be an object of $\mathcal{D}_{\Lambda,G}$, where $N_\alpha \in \mathcal{D}_{\Lambda,G}$. Then we have an isomorphism*

$$C^i_{\mathrm{cts}}(G, N) \cong \varinjlim_\alpha C^i_{\mathrm{cts}}(G, N_\alpha)$$

of continuous cochain groups which induces an isomorphism

$$H^i_{\mathrm{cts}}(G, N) \cong \varinjlim_\alpha H^i_{\mathrm{cts}}(G, N_\alpha)$$

of cohomology groups.

Proof. The first isomorphism is immediate and the second follows from the first since direct limit is exact. □

In the next proposition, we shall examine the relationship between cohomology and inverse limit. We shall denote $\varprojlim^{(i)}$ to be the ith derived functor of $\varprojlim$.

Proposition 3.2.8. *Let $M = \varprojlim_\alpha M_\alpha$ be an object in $\mathcal{C}_{\Lambda,G}$, where each M_α is finite. Then we have an isomorphism*

$$C_{\mathrm{cts}}(G, M) \cong \varprojlim_\alpha C_{\mathrm{cts}}(G, M_\alpha)$$

of complexes of Λ-modules and a spectral sequence

$$\varprojlim_\alpha{}^{(i)} H^j_{\mathrm{cts}}(G, M_\alpha) \Longrightarrow H^{i+j}_{\mathrm{cts}}(G, M).$$

Suppose further that G has the property that $H^m_{\mathrm{cts}}(G, N)$ is finite for all finite discrete Λ-modules N with a continuous commuting G-action and for all $m \geqslant 0$. Then

$$H^i_{\mathrm{cts}}(G, M) \cong \varprojlim_\alpha H^i_{\mathrm{cts}}(G, M_\alpha).$$

Proof. The first assertion is immediate from the definition. The second assertion follows from a similar argument as in [9, Prop. 8.3.5]. We consider the two hypercohomology spectral sequences for the functor $\varprojlim$ and the inverse system $C^i_{\mathrm{cts}}(G, M_\alpha)$:

$$\varprojlim_\alpha{}^{(j)} C^i_{\mathrm{cts}}(G, M_\alpha) \Longrightarrow H^{i+j},$$
$$\varprojlim_\alpha{}^{(i)} H^j_{\mathrm{cts}}(G, M_\alpha) \Longrightarrow H^{i+j}.$$

For each i, it is clear that

$$\varprojlim_\alpha C^i_{\mathrm{cts}}(G, M_\alpha) \longrightarrow C^i_{\mathrm{cts}}(G, M_\alpha)$$

is surjective for every α, and so the inverse system $C^i_{\mathrm{cts}}(G, M_\alpha)$ is "weakly flabby" in the sense of [5, Lemma 1.3]. Therefore, by [5, Thm. 1.8], we have that $\varprojlim_\alpha{}^{(j)} C^i_{\mathrm{cts}}(G, M_\alpha) = 0$ for $j > 0$. Hence, the first spectral sequence degenerates and we obtain

$$H^i = H^i\Big(\varprojlim_\alpha C_{\mathrm{cts}}(G, M_\alpha)\Big) \cong H^i\big(C_{\mathrm{cts}}(G, M)\big) = H^i_{\mathrm{cts}}(G, M).$$

For the last assertion, the additional assumption allows one to invoke [5, Cor. 7.2] to conclude that $\varprojlim_\alpha{}^{(i)} H^j_{\mathrm{cts}}(G, M_\alpha) = 0$ for $i > 0$. □

For the remainder of the subsection, we let $\mathcal{A}$ denote either $\mathcal{C}_{\Lambda,G}$ or $\mathcal{D}_{\Lambda,G}$. Let $M^\bullet$ be a complex of objects in $\mathcal{A}$ with differentials denoted by d^i_M. We define $C^\bullet_{\mathrm{cts}}(G, M^\bullet)$ by

$$C^n_{\mathrm{cts}}\big(G, M^\bullet\big) = \bigoplus_{i+j=n} C^j_{\mathrm{cts}}\big(G, M^i\big).$$

Its differential $\delta_{M^\bullet}^{i+j}$ is determined as follows: restriction of $\delta_{M^\bullet}^{i+j}$ to $C_{\mathrm{cts}}^j(G, M^i)$ is the sum of

$$\left(d_M^i\right)_* : C_{\mathrm{cts}}^j\left(G, M^i\right) \longrightarrow C_{\mathrm{cts}}^j\left(G, M^{i+1}\right)$$

and

$$(-1)^i \delta_{M^i}^j : C_{\mathrm{cts}}^j\left(G, M^i\right) \longrightarrow C_{\mathrm{cts}}^{j+1}\left(G, M^i\right).$$

We denote its ith cohomology group by $H^i(G, M^\bullet)$.

Proposition 3.2.9. *Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of objects in $\mathcal{A}$. Then*

$$0 \to C_{\mathrm{cts}}^\bullet\left(G, M'\right) \xrightarrow{\alpha_*} C_{\mathrm{cts}}^\bullet(G, M) \xrightarrow{\beta_*} C_{\mathrm{cts}}^\bullet\left(G, M''\right) \to 0$$

is an exact sequence of complexes of Λ-modules. The statement also holds true if we replace M', M, M'' by complexes of objects in $\mathcal{A}$.

Proof. By Lemma 3.2.6, it suffices to show that β has a continuous section. If $\mathcal{A} = \mathcal{D}_{\Lambda,G}$, this is obvious. In the case when $\mathcal{A} = \mathcal{C}_{\Lambda,G}$, since every compact Λ-module is profinite by Proposition 3.1.2, every continuous surjection has a continuous section. □

Let $M^\bullet$ be a complex of objects in $\mathcal{A}$. The filtration $\tau_{\leqslant j} M^\bullet$ induces a filtration

$$\tau_{\leqslant j} C_{\mathrm{cts}}^\bullet\left(G, M^\bullet\right) = C_{\mathrm{cts}}^\bullet\left(G, \tau_{\leqslant j} M^\bullet\right)$$

on the cochain groups which fit into the following exact sequence of complexes

$$0 \longrightarrow C_{\mathrm{cts}}^\bullet\left(G, \tau_{\leqslant j} M^\bullet\right) \longrightarrow C_{\mathrm{cts}}^\bullet\left(G, \tau_{\leqslant j+1} M^\bullet\right) \longrightarrow \tau_{\leqslant j+1} C_{\mathrm{cts}}^\bullet\left(G, M^\bullet\right) / \tau_{\leqslant j} C_{\mathrm{cts}}^\bullet\left(G, M^\bullet\right) \longrightarrow 0$$

by Proposition 3.2.9. This filtration gives rise to the following hypercohomology spectral sequence

$$H_{\mathrm{cts}}^i\left(G, H^j\left(M^\bullet\right)\right) \Longrightarrow H_{\mathrm{cts}}^{i+j}\left(G, M^\bullet\right),$$

which is convergent if $M^\bullet$ is cohomologically bounded below.

Lemma 3.2.10. *Let $f : M^\bullet \longrightarrow N^\bullet$ be a quasi-isomorphism of cohomologically bounded below complexes of objects in $\mathcal{A}$. Then the induced map*

$$f_* : C_{\mathrm{cts}}^\bullet\left(G, M^\bullet\right) \longrightarrow C_{\mathrm{cts}}^\bullet\left(G, N^\bullet\right)$$

is also a quasi-isomorphism.

Proof. The map f induces isomorphisms

$$H_{\mathrm{cts}}^i\left(G, H^j\left(M^\bullet\right)\right) \xrightarrow{\sim} H_{\mathrm{cts}}^i\left(G, H^j\left(N^\bullet\right)\right).$$

By convergence of the above spectral sequence, this implies that the induced maps

$$H_{\mathrm{cts}}^i\left(G, M^\bullet\right) \longrightarrow H_{\mathrm{cts}}^i\left(G, N^\bullet\right)$$

are isomorphisms. □

Hence we can conclude the following.

Proposition 3.2.11. *The functor*

$$C^{\bullet}_{\mathrm{cts}}(G,-):\mathrm{Ch}^{+}(\mathcal{A})\longrightarrow \mathrm{Ch}^{+}(Mod_{\Lambda})$$

preserves homotopy, exact sequences and quasi-isomorphisms, hence induces the following exact derived functors

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G,-):\mathbf{D}^{b}(\mathcal{C}_{\Lambda,G})\longrightarrow \mathbf{D}^{+}(Mod_{\Lambda}),$$
$$\mathbf{R}\Gamma_{\mathrm{cts}}(G,-):\mathbf{D}^{+}(\mathcal{D}_{\Lambda,G})\longrightarrow \mathbf{D}^{+}(Mod_{\Lambda}).$$

Proof. This proposition follows from what we have done so far. The only subtlety lies in the fact that $\mathcal{C}_{\Lambda,G}$ does not necessarily have enough injectives and therefore we do not know if $\mathbf{D}^{+}(\mathcal{C}_{\Lambda,G})$ exists. However, we know that $\mathcal{C}_{\Lambda,G}$ has enough projectives. Therefore, $\mathbf{D}^{-}(\mathcal{C}_{\Lambda,G})$ exists, and we may apply Lemma 3.2.10 to $\mathbf{D}^{b}(\mathcal{C}_{\Lambda,G})$. □

We now like to extend Proposition 3.2.8 to the case of complexes. Before that, we first prove a lemma which will be required in our discussion.

Lemma 3.2.12. *Let $f: M\longrightarrow N$ be a morphism of objects in $\mathcal{C}_{\Lambda,G}$. Then there exists a directed indexing set I with the following properties*:

(1) *There exists a fundamental system $\{U_i\}$ (resp., $\{V_i\}$) of neighborhoods of zero consisting of open $\Lambda[G]$-submodules of M (resp., N).*
(2) *For each $i\in I$, there is a $\Lambda[G]$-homomorphism $f_i: M/U_i\longrightarrow N/V_i$ which fits into the following commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M/U_i & \xrightarrow{f_i} & N/V_i \end{array}$$

where the vertical morphisms are the canonical quotient map.
(3) *One has $f=\varprojlim_i f_i$.*

Proof. Let $\{U_\alpha\}_{\alpha\in I_M}$ (resp., $\{V_\beta\}_{\beta\in I_N}$) be a system of neighborhoods of zero consisting of open $\Lambda[G]$-submodules of M (resp., N). Then we set $I=I_M\times I_N, U_{\alpha,\beta}=U_\alpha\cap f^{-1}(V_\beta)$ and $V_{\alpha,\beta}=V_\beta$. It is then straightforward to verify that f factors through $M/U_{\alpha,\beta}$ to give a $\Lambda[G]$-homomorphism $f_{\alpha,\beta}: M/U_{\alpha,\beta}\longrightarrow N/V_{\alpha,\beta}$ and $f=\varprojlim_{\alpha,\beta} f_{\alpha,\beta}$. □

In view of the above lemma, we say that a morphism $f: M\longrightarrow N$ in $\mathcal{C}_{\Lambda,G}$ is compatible with a directed indexing set I if the conclusion in the lemma holds. By the lemma, we have that for every morphism $f: M\longrightarrow N$ in $\mathcal{C}_{\Lambda,G}$, there exists a directed indexing set I such that f is compatible with I. In particular, if M is a bounded complex in $\mathcal{C}_{\Lambda,G}$, we can find a directed indexing set I such that the differentials are compatible with I.

Proposition 3.2.13. *Suppose that G has the property that $H^m_{\mathrm{cts}}(G, N)$ is finite for all finite discrete Λ-modules N with a continuous commuting G-action and for all $m \geqslant 0$. Let $M^\bullet = \varprojlim_{i\in I} M_i^\bullet$ be a bounded complex of objects in $\mathcal{C}_{\Lambda,G}$ with I-compatible differentials. Then we have the following isomorphism*

$$H^n_{\mathrm{cts}}\big(G, M^\bullet\big) \cong \varprojlim_i H^n_{\mathrm{cts}}\big(G, M_i^\bullet\big)$$

of hypercohomology groups for each n.

Proof. The canonical chain map $M^\bullet \longrightarrow M_i^\bullet$ induces the following morphism of (convergent) spectral sequences

$$\begin{array}{ccc} H^r_{\mathrm{cts}}(G, H^s(M^\bullet)) & \Longrightarrow & H^{r+s}_{\mathrm{cts}}(G, M^\bullet) \\ \downarrow & & \\ H^r_{\mathrm{cts}}(G, H^s(M_i^\bullet)) & \Longrightarrow & H^{r+s}_{\mathrm{cts}}(G, M_i^\bullet) \end{array}$$

which is compatible with i. By the hypothesis, the bottom spectral sequence is a spectral sequence of finite Λ-modules. Therefore, the inverse limit is compatible with the inverse system of the spectral sequences, and we have the following morphism

$$\begin{array}{ccc} H^r_{\mathrm{cts}}(G, H^s(M^\bullet)) & \Longrightarrow & H^{r+s}_{\mathrm{cts}}(G, M^\bullet) \\ \downarrow & & \\ \varprojlim_i H^r_{\mathrm{cts}}(G, H^s(M_i^\bullet)) & \Longrightarrow & \varprojlim_i H^{r+s}_{\mathrm{cts}}(G, M_i^\bullet) \end{array}$$

of (convergent) spectral sequences. By Proposition 3.2.8, we have the isomorphisms

$$H^r_{\mathrm{cts}}\big(G, H^s\big(M^\bullet\big)\big) \cong \varprojlim_i H^r_{\mathrm{cts}}\big(G, H^s\big(M_i^\bullet\big)\big).$$

Hence, by the convergence of the spectral sequences, we obtain the required isomorphism. □

For ease of notation, we will drop the '$\bullet$' for complexes. We also drop the notation 'cts'. Therefore, we write $C(G, M)$ as the complex of continuous cochains and $\mathbf{R}\Gamma(G, M)$ for its derived functor. Its ith cohomology group is then written as $H^i(G, M)$.

3.3. Total cup products

We first recall the definition for topological G-modules (in other words, abelian Hausdorff topological groups with a continuous G-action).

Definition 3.3.1 *(Cup products).* Let A, B and C be topological G-modules. Suppose

$$\langle\ ,\ \rangle : A \times B \longrightarrow C$$

is a continuous map satisfying $\sigma\langle a, b\rangle = \langle \sigma a, \sigma b\rangle$ for $a \in A$, $b \in B$ and $\sigma \in G$. Then we define the cup product on the cochain groups

$$C^i(G, A) \times C^j(G, B) \longrightarrow C^{i+j}(G, C)$$

as follows: for $\alpha \in C^i(G, A)$, $\beta \in C^j(G, B)$ and $\sigma_1, \ldots, \sigma_{i+j} \in G$, we have

$$(\alpha \cup \beta)(\sigma_1, \ldots, \sigma_{i+j}) = \big\langle \alpha(\sigma_1, \ldots, \sigma_i), \sigma_1 \cdots \sigma_i \beta(\sigma_{i+1}, \ldots, \sigma_{i+j}) \big\rangle.$$

The cup product satisfies the following relation

$$\delta_C(\alpha \cup \beta) = (\delta_A \alpha) \cup \beta + (-1)^i \alpha \cup (\delta_B \beta)$$

and induces a pairing

$$H^i(G, A) \times H^j(G, B) \longrightarrow H^{i+j}(G, C)$$

on the cohomology groups.

Now fix a prime p. For the remainder of the paper, we shall assume that our profinite ring Λ is pro-p. In other words, for each $\mathfrak{a} \in \mathcal{I}$, the ring $\Lambda/\mathfrak{a}$ is finite of a p-power cardinality. Let M and N be objects in $\mathcal{C}_{\Lambda,G}$ and $\mathcal{D}_{\Lambda^\circ,G}$ respectively, and let A be a topological G-module. Suppose there is a continuous pairing

$$\langle\ ,\ \rangle : N \times M \longrightarrow A$$

such that

(1) $\sigma\langle y, x\rangle = \langle \sigma y, \sigma x\rangle$ for $x \in M$, $y \in N$ and $\sigma \in G$, and
(2) $\langle y\lambda, x\rangle = \langle y, \lambda x\rangle$ for $x \in M$, $y \in N$ and $\lambda \in \Lambda$.

As before, condition (1) will give rise to the cup product

$$C^i(G, N) \times C^j(G, M) \longrightarrow C^{i+j}(G, A),$$

which is Λ-balanced by condition (2). The cup product induces a group homomorphism

$$C^i(G, N) \otimes_\Lambda C^j(G, M) \longrightarrow C^{i+j}(G, A)$$

which gives rise to the following morphism

$$C(G, N) \otimes_\Lambda C(G, M) \longrightarrow C(G, A)$$

of complexes of abelian groups. Taking the adjoint, we have a morphism

$$C(G, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}\big(C(G, N), C(G, A)\big)$$

of complexes of Λ-modules.

Lemma 3.3.2. *Suppose we are given another continuous pairing*

$$(\ ,\ ) : N' \times M' \longrightarrow A$$

such that

(1) $\sigma(y', x') = (\sigma y', \sigma x')$ *for* $x' \in M'$, $y' \in N'$ *and* $\sigma \in G$;
(2) $(y'\lambda, x') = (y', \lambda x')$ *for* $x' \in M'$, $y' \in N'$ *and* $\lambda \in \Lambda$; *and*
(3) *there are morphisms* $f : N' \longrightarrow N$ *in* $\mathcal{D}_{\Lambda^\circ, G}$ *and* $g : M \longrightarrow M'$ *in* $\mathcal{C}_{\Lambda, G}$ *such that the following diagram*

$$\begin{array}{ccc} N' \otimes_\Lambda M & \xrightarrow{\mathrm{id} \otimes g} & N' \otimes_\Lambda M' \\ {\scriptstyle f \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle (\ ,\ )} \\ N \otimes_\Lambda M & \xrightarrow{\langle\ ,\ \rangle} & A \end{array}$$

commutes. Then we have the following commutative diagram

$$\begin{array}{ccc} C(G, M) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(C(G, N), C(G, A)) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ C(G, M') & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(C(G, N'), C(G, A)) \end{array}$$

of complexes of Λ*-modules.*

Proof. It follows from a direct calculation that following diagram

$$\begin{array}{ccc} C(G, N') \otimes_\Lambda C(G, M) & \xrightarrow{\mathrm{id} \otimes g} & C(G, N') \otimes_\Lambda C(G, M') \\ {\scriptstyle f \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \cup_{(\ ,\ )}} \\ C(G, N) \otimes_\Lambda C(G, M) & \xrightarrow{\cup_{\langle\ ,\ \rangle}} & C(G, A) \end{array}$$

is commutative, where $\cup_{(\ ,\ )}$ and $\cup_{\langle\ ,\ \rangle}$ are the cup products induced by the pairings $(\ ,\ )$ and $\langle\ ,\ \rangle$ respectively. By taking the adjoint and another straightforward calculation, we have the commutative diagram in the lemma. □

Now let M and N be bounded complexes of objects in $\mathcal{C}_{\Lambda, G}$ and $\mathcal{D}_{\Lambda^\circ, G}$ respectively, and let A be a bounded complex of topological G-modules. Suppose there is a collection of continuous pairings

$$\langle\ ,\ \rangle_{a,b} : N^a \times M^b \longrightarrow A^{a+b}$$

where each pairing satisfies conditions (1) and (2), and the following hold:

(a) $\langle d_N^a y, x\rangle_{a+1,b} = d_A^{a+b}(\langle y, x\rangle_{a,b})$ for $y \in N^a$ and $x \in M^b$, and
(b) $(-1)^a \langle y, d_M^b x\rangle_{a,b+1} = d_A^{a+b}(\langle y, x\rangle_{a,b})$ for $y \in N^a$ and $x \in M^b$.

For each pair (a, b), we have a morphism

$$\cup_{ij}^{ab} : C^i\big(G, N^a\big) \otimes_\Lambda C^j\big(G, M^b\big) \longrightarrow C^{i+j}\big(G, A^{a+b}\big)$$

of abelian groups induced by the cup product. Then the total cup product

$$\cup : C(G, N) \otimes_\Lambda C(G, M) \longrightarrow C(G, A)$$

is a morphism of complexes of $\mathbb{Z}_p$-modules given by the collection $\cup = ((-1)^{ib}\cup_{ij}^{ab})$. The definition given for the total cup products follows that in [9, 3.4.5.2]. We also have an analogous result to Lemma 3.3.2 for complexes.

Lemma 3.3.3. *Suppose we are given another collection of continuous pairings*

$$\langle\ ,\ \rangle_{a,b} : N'^a \times M'^b \longrightarrow A'^{a+b}$$

as above. Then we have the following commutative diagram

$$\begin{array}{ccc} C(G, M) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(C(G, N), C(G, A)) \\ \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle f_*} \\ C(G, M') & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(C(G, N'), C(G, A)) \end{array}$$

of complexes of Λ-modules.

3.4. Tate cohomology groups

We shall now describe the Tate cochain complexes of a finite group G. We begin by giving an alternative description of the (inhomogeneous) cochain complexes. Throughout this subsection, G will always denote a finite group. Consider the standard $\mathbb{Z}[G]$-resolution (in inhomogeneous form) of $\mathbb{Z}$ (cf. [13, Sect. 6.5]),[1]

$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots$$

where $X_0 = \mathbb{Z}[G]$ and, for $n \geqslant 1$, X_n is the free $\mathbb{Z}[G]$-module generated by the set of all symbols $(g_1, \ldots, g_n)$ with $g_i \in G$, and the differentials are given by the formula

$$\partial^n(g_1, \ldots, g_n) = g_1(g_2, \ldots, g_n) + \sum_{j=1}^{n-1}(-1)^j(g_1, \ldots, g_j g_{j+1}, \ldots, g_n) + (-1)^n(g_1, \ldots, g_{n-1}).$$

For any $\mathbb{Z}[G]$-module M, there is a natural isomorphism $C^i(G, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(X_i, M)$ which is compatible with the differentials, thus giving an identification of complexes. Furthermore, if M is a $\Lambda[G]$-module, the above identification is an isomorphism of complexes of Λ-modules.

We now construct the complete cochain groups. For a $\mathbb{Z}[G]$-module A, we write $A^* = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$. Note that this is a $\mathbb{Z}[G]$-module in a natural way. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the long exact sequence

[1] Weibel calls this the unnormalized bar resolution.

$$0 \longleftarrow \mathbb{Z} \longleftarrow X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots,$$

we obtain the following long exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow X_0^* \longrightarrow X_1^* \longrightarrow X_2^* \longrightarrow \cdots,$$

since each X_i is a free $\mathbb{Z}$-module. Splicing the two long exact sequence and applying $\mathrm{Hom}_{\mathbb{Z}[G]}(-, M)$ to the resulting long exact sequence, we obtain the following complex

$$\cdots \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}\big(X_1^*, M\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}\big(X_0^*, M\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(X_0, M) \longrightarrow \cdots.$$

The completed cochain complexes $\hat{C}^i(G, M)$ are defined by

$$\hat{C}^i(G, M) = \begin{cases} C^i(G, M) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(X_i, M) & \text{if } i \geqslant 0, \\ \mathrm{Hom}_{\mathbb{Z}[G]}(X^*_{-1-i}, M) & \text{if } i \leqslant -1. \end{cases}$$

Following [9, 5.7.2], we may extend the above definition to a complex $M^\bullet$ of G-modules by setting

$$\hat{C}^n\big(G, M^\bullet\big) = \bigoplus_{i+j=n} \hat{C}^i\big(G, M^j\big)$$

with differential defined using the sign conventions of the previous sections. As before, for ease of notation, we will drop the '$\bullet$' for complexes. The usual cup product for Tate cohomology groups (cf. [10, Prop. 1.4.6]) extends to a total cup product with the same sign convention as in the preceding section.

4. Duality over pro-p rings

Let p be a fixed prime. Throughout the section, our profinite ring Λ will always be pro-p. In this section, we will formulate and prove Tate's (and Poitou's) local and global duality theorems.

4.1. Tate's local duality

Let F be a non-Archimedean local field with characteristic not equal to p. Fix a separable closure F^{sep} of F. Set $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$.

Lemma 4.1.1. *We have*

$$H^j\big(G_F, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \cong \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p & \text{if } j = 2, \\ 0 & \text{if } j > 2. \end{cases}$$

Proof. For $j > 2$, the conclusion follows from the fact that G_F has p-cohomological dimension 2 (see [10, Thm. 7.1.8(i)]). By [10, Thm. 7.1.8(ii)], we have $H^2(G_v, \mathbb{Z}/p^r(1)) \cong \mathbb{Z}/p^r$. The assertion now follows by taking direct limits. □

By the preceding lemma, we have a quasi-isomorphism $\mathbb{Q}_p/\mathbb{Z}_p[-2] \xrightarrow{i} \tau_{\geqslant 2} C(G_F, \mathbb{Q}_p/\mathbb{Z}_p(1))$ of complexes of $\mathbb{Z}_p$-modules. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is an injective $\mathbb{Z}_p$-module, the map i has a homotopy inverse. We shall fix one such map

$$r : \tau_{\geqslant 2} C\big(G_F, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p[-2].$$

This gives a morphism

$$\theta : C\big(G_F, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \longrightarrow \tau_{\geqslant 2} C\big(G_F, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \xrightarrow{r} \mathbb{Q}_p/\mathbb{Z}_p[-2]$$

of complexes of $\mathbb{Z}_p$-modules.

Let M be a bounded complex of objects in $\mathcal{C}_{\Lambda,G_F}$. We shall write $M^\vee$ to be the complex $\mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$. The obvious pairing

$$M^\vee(1) \otimes_\Lambda M \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

induces the total cup product

$$C\big(G_F, M^\vee(1)\big) \otimes_\Lambda C(G_F, M) \longrightarrow C\big(G_F, \mathbb{Q}_p/\mathbb{Z}_p(1)\big).$$

Suppose that N is another bounded complex of objects in $\mathcal{C}_{\Lambda,G_F}$, and there is a morphism $f : M \longrightarrow N$ of complexes in $\mathcal{C}_{\Lambda,G_F}$. Then we have the following commutative diagram

$$\begin{array}{ccc} N^\vee(1) \otimes_\Lambda M & \xrightarrow{\mathrm{id}\otimes f} & N^\vee(1) \otimes_\Lambda N \\ \downarrow{\scriptstyle f^\vee \otimes \mathrm{id}} & & \downarrow \\ M^\vee(1) \otimes_\Lambda M & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p(1) \end{array}$$

with the obvious pairings. Applying cochains and θ, we obtain the following commutative diagram

$$\begin{array}{ccc} C(G_F, N^\vee(1)) \otimes_\Lambda C(G_F, M) & \xrightarrow{\mathrm{id}\otimes f} & C(G_F, N^\vee(1)) \otimes_\Lambda C(G_F, N) \\ \downarrow{\scriptstyle f^\vee \otimes \mathrm{id}} & & \downarrow \\ C(G_F, M^\vee(1)) \otimes_\Lambda C(G_F, M) & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p[-2] \end{array}$$

which induces the following commutative diagram

$$\begin{array}{ccc} C(G_F, M) & \xrightarrow{\alpha_M} & \mathrm{Hom}_{\mathbb{Z}_p}(C(G_F, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-2] \\ \downarrow & & \downarrow \\ C(G_F, N) & \xrightarrow{\alpha_N} & \mathrm{Hom}_{\mathbb{Z}_p}(C(G_F, N^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-2] \end{array}$$

of complexes of Λ-modules by Lemma 3.3.2. We are now able to prove the following formulation of Tate's local duality.

Theorem 4.1.2. *Let M be a bounded complex of objects in $\mathcal{C}_{\Lambda,G_F}$. Then we have the following isomorphism*

$$\mathbf{R}\Gamma(G_F, M) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbf{R}\Gamma\big(G_F, M^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\big)[-2]$$

in $\mathbf{D}(Mod_\Lambda)$.

Proof. We shall show that α_M (in the above diagram) is a quasi-isomorphism. Now if $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is an exact triangle[2] in $\mathbf{D}^b(\mathcal{C}_{\Lambda,G_F})$, we then have a morphism

$$\begin{array}{ccc}
\mathbf{R}\Gamma(G_F, A) & \xrightarrow{\alpha_A} & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_F, A^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-2] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_F, B) & \xrightarrow{\alpha_B} & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_F, B^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-2] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_F, C) & \xrightarrow{\alpha_C} & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_F, C^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-2]
\end{array}$$

of exact triangles. Therefore, if any two of the morphisms α_A, α_B and α_C are isomorphisms, so is the third. For a bounded complex M in $\mathcal{C}_{\Lambda,G_F}$, we have the following exact triangle

$$\sigma_{\leqslant i-1}M \longrightarrow \sigma_{\leqslant i}M \longrightarrow M^i[-i].$$

Therefore, by induction, we are reduced to showing that α_M is a quasi-isomorphism in the case when M is a single module. Write $M = \varprojlim_\beta M_\beta$, where each M_β is a finite module. By the functoriality of α, we have the following commutative diagram

$$\begin{array}{ccc}
C(G_F, M) & \xrightarrow{\alpha_M} & \mathrm{Hom}_{\mathbb{Z}_p}(C(G_F, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-2] \\
\downarrow u & & \downarrow v \\
\varprojlim_\beta C(G_F, M_\beta) & \xrightarrow{\varprojlim \alpha_{M_\beta}} & \varprojlim_\beta \mathrm{Hom}_{\mathbb{Z}_p}(C(G_F, M_\beta^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-2]
\end{array}$$

of complexes of Λ-modules. By Proposition 3.2.7 and Proposition 3.2.8, we have that u and v in the above diagram are isomorphisms of complexes, and the vertical maps in the following commutative diagram

$$\begin{array}{ccc}
H^i(G_F, M) & \xrightarrow{(\alpha_M)_*} & \mathrm{Hom}_{\mathbb{Z}_p}(H^{2-i}(G_F, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p) \\
\downarrow u_* & & \downarrow v_* \\
\varprojlim_\beta H^i(G_F, M_\beta) & \xrightarrow{\varprojlim (\alpha_{M_\beta})_*} & \varprojlim_\beta \mathrm{Hom}_{\mathbb{Z}_p}(H^{2-i}(G_F, M_\beta^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)
\end{array}$$

are isomorphisms. Since each $(\alpha_{M_\beta})_*$ is an isomorphism by Tate local duality [10, Thm. 7.2.6], we have the required conclusion. □

[2] We write an exact triangle $A \to B \to C \to A[1]$ more compactly as $A \to B \to C$ throughout.

4.2. Global duality over pro-p rings

Let F be a global field with characteristic not equal to p, and let S be a finite set of primes of F containing all primes above p and all Archimedean primes of F (if F is a number field). Let S_f (resp., $S_{\mathbb{R}}$) denote the collection of non-Archimedean primes (resp., real primes) of F in S.

Fix a separable closure F^{sep} of F. Set $G_{F,S} = \text{Gal}(F_S/F)$, where F_S is the maximal subextension of F^{sep}/F unramified outside S. For each $v \in S_f$, we fix a separable closure F_v^{sep} of F_v and an embedding $F^{\text{sep}} \hookrightarrow F_v^{\text{sep}}$. This induces a continuous group homomorphism $G_v := \text{Gal}(F_v^{\text{sep}}/F_v) \to G_{F,S}$. If v is a real prime, we also write G_v for $\text{Gal}(\mathbb{C}/\mathbb{R})$.

If M is a complex in $\mathcal{C}_{\Lambda,G_{F,S}}$ (resp., $\mathcal{D}_{\Lambda,G_{F,S}}$), then we can view M as a complex in $\mathcal{C}_{\Lambda,G_v}$ (resp., $\mathcal{D}_{\Lambda,G_v}$) via the continuous homomorphism $G_v \longrightarrow G_{F,S}$. Therefore, the cochain complexes $C(G_{F,S}, M)$ and $C(G_v, M)$ can be defined. Recall that for $v \in S_f$, we have the restriction map

$$\text{res}_v : C(G_{F,S}, M) \longrightarrow C(G_v, M)$$

induced by the group homomorphism $G_v \longrightarrow G_{F,S}$. For a real prime v, we have the following

$$\text{res}_v : C(G_{F,S}, M) \longrightarrow C(G_v, M) \hookrightarrow \hat{C}(G_v, M).$$

To shorten notation in what follows, for $v \in S_{\mathbb{R}}$, we will abuse notation and use $C(G_v, M)$, $H^i(G_v, M)$, and $\mathbf{R}\Gamma(G_v, M)$ to denote the Tate cochains $\hat{C}(G_v, M)$, its cohomology groups, and its derived object. We now make the following definition.

Definition 4.2.1. Let M be a complex in $\mathcal{C}_{\Lambda,G_{F,S}}$ or $\mathcal{D}_{\Lambda,G_{F,S}}$. The complex of continuous cochains of M with compact support is defined as

$$C_c(G_{F,S}, M) = \text{Cone}\left(C(G_{F,S}, M) \xrightarrow{\text{res}_S} \bigoplus_{v \in S} C(G_v, M) \right)[-1],$$

where the elements of

$$C_c^i(G_{F,S}, M) = C^i(G_{F,S}, M) \oplus \left(\bigoplus_{v \in S} C^{i-1}(G_v, M) \right)$$

have the form (a, a_S) with $a \in C^i(G_{F,S}, M)$, $a_S = (a_v)_{v \in S}$, $a_v \in C^{i-1}(G_v, M)$, and the differential is given by

$$d(a, a_S) = \big(da, -\text{res}_S(a) - da_S\big).$$

The ith cohomology group of $C_c(G_{F,S}, M)$ is denoted by $H_c^i(G_{F,S}, M)$.

Remark. If F is a function field in one variable over a finite field or F is a totally imaginary number field, then $S_{\mathbb{R}}$ is empty, and the cone is given by

$$\text{Cone}\left(C(G_{F,S}, M) \xrightarrow{\text{res}_{S_f}} \bigoplus_{v \in S_f} C(G_v, M) \right)[-1].$$

Now suppose that p is odd and F is a number field with at least one real prime. Let $v \in S_{\mathbb{R}}$. Then $\hat{H}^i(G_v, M) = 0$ for every M in $\mathcal{C}_{\Lambda,G_{F,S}}$ (resp., $\mathcal{D}_{\Lambda,G_{F,S}}$) and for all i, since G_v is a finite group of

order 2 and M is an inverse limit of finite p-groups (resp., direct limit of finite p-groups). Therefore, it follows that the canonical map

$$\mathrm{Cone}\left(C(G_{F,S},M)\xrightarrow{\mathrm{res}_{S_f}}\bigoplus_{v\in S_f}C(G_v,M)\right)[-1]\longrightarrow C_c(G_{F,S},M)$$

is a quasi-isomorphism. Therefore, one may take the above cone as a definition of the complex of continuous cochains with compact support in this case.

Proposition 4.2.2. *The functor*

$$C_c(G_{F,S},-):\mathrm{Ch}^+(\mathcal{C}_{\Lambda,G_{F,S}})\longrightarrow\mathrm{Ch}(Mod_\Lambda)$$
$$\big(\text{resp., } C_c(G_{F,S},-):\mathrm{Ch}^+(\mathcal{D}_{\Lambda,G_{F,S}})\longrightarrow\mathrm{Ch}(Mod_\Lambda)\big)$$

preserves homotopy, exact sequences and quasi-isomorphisms, hence induces the following exact derived functors

$$\mathbf{R}\Gamma_c(G_{F,S},-):\mathbf{D}^b(\mathcal{C}_{\Lambda,G_{F,S}})\longrightarrow\mathbf{D}(Mod_\Lambda)$$
$$\big(\text{resp., } \mathbf{R}\Gamma_c(G_{F,S},-):\mathbf{D}^+(\mathcal{D}_{\Lambda,G_{F,S}})\longrightarrow\mathbf{D}(Mod_\Lambda)\big)$$

such that for M in $\mathbf{D}^b(\mathcal{C}_{\Lambda,G_{F,S}})$ or $\mathbf{D}^+(\mathcal{D}_{\Lambda,G_{F,S}})$, we have the following exact triangle

$$\mathbf{R}\Gamma_c(G_{F,S},M)\longrightarrow\mathbf{R}\Gamma(G_{F,S},M)\longrightarrow\bigoplus_{v\in S}\mathbf{R}\Gamma(G_v,M)$$

in $\mathbf{D}(Mod_\Lambda)$ and the following long exact sequence

$$\cdots\longrightarrow H^i_c(G_{F,S},M)\longrightarrow H^i(G_{F,S},M)\longrightarrow\bigoplus_{v\in S}H^i(G_v,M)\longrightarrow H^{i+1}_c(G_{F,S},M)\longrightarrow\cdots.$$

Proof. This is immediate from the definition of the cone. □

By [10, Thm. 7.1.8(iii), Thm. 8.3.19], Proposition 3.2.8 can be applied to $G_{F,S}$ and G_v, where $v\in S_f$. For $v\in S_{\mathbb{R}}$, G_v is a finite group of order 2, and so the finiteness hypothesis in Proposition 3.2.8 is satisfied, so the conclusion also holds in this case. The following analogous statement to Proposition 3.2.8 for cohomology groups with compact support will now follow from the definition of the cone and the long exact sequence of cohomology groups in the preceding proposition.

Proposition 4.2.3. *The functor $C_c(G_{F,S},-)$ preserves direct limits in $\mathcal{D}_{\Lambda,G_{F,S}}$. Moreover, if $M=\varinjlim_\alpha M_\alpha$ is an object in $\mathcal{C}_{\Lambda,G_{F,S}}$, where each M_α is finite, then we have the following isomorphism*

$$C_c(G_{F,S},M)\cong\varinjlim_\alpha C_c(G_{F,S},M_\alpha)$$

of complexes and isomorphisms

$$H^i_c(G_{F,S},M)\cong\varinjlim_\alpha H^i_c(G_{F,S},M_\alpha)$$

of cohomology groups.

Lemma 4.2.4. *We have*

$$H_c^j\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \cong \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p & \text{if } j = 3, \\ 0 & \text{if } j > 3. \end{cases}$$

Proof. By the long exact sequence of Poitou–Tate [10, 8.6.13], we have the following exact sequence

$$H^2\big(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(1)\big) \longrightarrow \bigoplus_{v\in S} H^2\big(G_v, \mathbb{Z}/p^n\mathbb{Z}(1)\big) \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0$$

and an isomorphism

$$H^3\big(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(1)\big) \xrightarrow{\text{res}} \bigoplus_{v\in S_{\mathbb{R}}} \hat{H}^3\big(G_v, \mathbb{Z}/p^n\mathbb{Z}(1)\big).$$

By the definition of continuous cochains with compact support and the fact that $\mathrm{cd}_p(G_v) = 2$ for $v \in S_f$, we have $H_c^3(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(1)) \cong \mathbb{Z}/p^n\mathbb{Z}$. The remainder of the lemma will then follow from a similar argument to that in Lemma 4.1.1. □

Let M be a bounded complex in $\mathcal{C}_{\Lambda,G_{F,S}}$. For each $v \in S$, we define a morphism $\cup_v$ of complex of $\mathbb{Z}_p$-modules to be

$$C\big(G_v, M^\vee(1)\big) \otimes_\Lambda C(G_v, M) \longrightarrow C\big(G_v, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \longrightarrow C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big)[1],$$

where the first map is the total cup product, and the second is the natural morphism arising from the definition of the cone. By Lemma 4.2.4, we have a quasi-isomorphism $\mathbb{Q}_p/\mathbb{Z}_p[-3] \xrightarrow{i} \tau_{\geqslant 3}C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1))$ of complexes of $\mathbb{Z}_p$-modules. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is an injective $\mathbb{Z}_p$-module, the map i has a homotopy inverse. We shall fix one such map

$$r : \tau_{\geqslant 3}C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p[-3],$$

and this induces the following morphism

$$\vartheta : C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \longrightarrow \tau_{\geqslant 3}C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \xrightarrow{r} \mathbb{Q}_p/\mathbb{Z}_p[-3]$$

of complexes of $\mathbb{Z}_p$-modules. Combining this with $\cup_v$, we obtain a morphism

$$C\big(G_v, M^\vee(1)\big) \otimes_\Lambda C(G_v, M) \xrightarrow{\cup_v} C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big)[1] \xrightarrow{\vartheta[1]} \mathbb{Q}_p/\mathbb{Z}_p[-2]$$

of complexes of $\mathbb{Z}_p$-modules. For $v \in S_f$, this is essentially the morphism constructed in Section 3.1, which will give the Tate local duality as in Theorem 4.1.2. We also have the following.

Theorem 4.2.5. *Let $p = 2$, and let $v \in S_{\mathbb{R}}$. For a bounded complex M of objects in $\mathcal{C}_{\Lambda,G_v}$, we have the following isomorphism*

$$\widehat{\mathbf{R}\Gamma}(G_v, M) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\big(\widehat{\mathbf{R}\Gamma}\big(G_v, M^\vee(1)\big), \mathbb{Q}_2/\mathbb{Z}_2\big)[-2]$$

in $\mathbf{D}(Mod_\Lambda)$.

Proof. By a similar argument to that in Theorem 4.1.2, it suffices to consider a finite module M, and the conclusion then follows from [10, Thm. 7.2.17]. □

For ease of notation, we shall write $P(G_{F,S}, -)$ for $\bigoplus_{v\in S} C(G_v, -)$. We now define a morphism $\cup_S$ by

$$P\big(G_{F,S}, M^\vee(1)\big) \otimes_\Lambda P(G_{F,S}, M) \xrightarrow{\nu} \bigoplus_{v\in S} C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \xrightarrow{\sum} C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big),$$

where $\nu(a_S, b_S) = (a_v \cup_v b_v)_{v\in S}$.

We now construct the total cup products for the compactly supported cochain groups. Since these are defined as cones, it follows from Lemma 2.3 that there are two morphisms

$$\cup_0, \cup_1 : C_c\big(G_{F,S}, M^\vee(1)\big) \otimes_\Lambda C_c(G_{F,S}, M) \longrightarrow C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big)$$

of complexes of $\mathbb{Z}_p$-modules given by

$$(a, a_S) \cup_0 (b, b_S) = \big(a \cup b, (-1)^{\bar{a}} \operatorname{res}_S(a) \cup_S b_S\big),$$
$$(a, a_S) \cup_1 (b, b_S) = \big(a \cup b, a_S \cup_S \operatorname{res}_S(b)\big)$$

where $\cup$ is the total cup product

$$C\big(G_{F,S}, M^\vee(1)\big) \otimes_\Lambda C(G_{F,S}, M) \longrightarrow C\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big).$$

The morphisms $\cup_0$ and $\cup_1$ induce the following morphisms

$$\cup_c : C\big(G_{F,S}, M^\vee(1)\big) \otimes_\Lambda C_c(G_{F,S}, M) \longrightarrow C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big),$$
$${}_c\cup : C_c\big(G_{F,S}, M^\vee(1)\big) \otimes_\Lambda C(G_{F,S}, M) \longrightarrow C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big)$$

of complexes of abelian groups which are given by the following respective formulas (see also [9, 5.3.3.2, 5.3.3.3])

$$a \cup_c (b, b_S) = \big(a \cup b, (-1)^{\bar{a}} \operatorname{res}_S(a) \cup_S b_S\big),$$
$$(a, a_S) \,{}_c\cup b = \big(a \cup b, a_S \cup_S \operatorname{res}_S(b)\big).$$

All of these fit into the following diagram

$$\begin{array}{ccc} C_c(G_{F,S}, M^\vee(1)) \otimes_\Lambda C_c(G_{F,S}, M) & \longrightarrow & C(G_{F,S}, M^\vee(1)) \otimes_\Lambda C_c(G_{F,S}, M) \\ \downarrow & & \downarrow \cup_c \\ C_c(G_{F,S}, M^\vee(1)) \otimes_\Lambda C(G_{F,S}, M) & \xrightarrow{{}_c\cup} & C(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \end{array}$$

which is commutative up to homotopy by Lemma 2.3. Also, the following diagrams

$$\begin{array}{ccc}
C(G_{F,S}, M^\vee(1)) \otimes_\Lambda (P(G_v, M)[-1]) & \longrightarrow & P(G_{F,S}, M^\vee(1)) \otimes_\Lambda (P(G_{F,S}, M)[-1]) \\
\downarrow & & \downarrow t \\
 & & (P(G_{F,S}, M^\vee(1)) \otimes_\Lambda P(G_{F,S}, M))[-1] \\
 & & \downarrow \cup_S[-1] \\
C(G_{F,S}, M^\vee(1)) \otimes_\Lambda C_c(G_{F,S}, M) & \xrightarrow{\cup_c} & C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1))
\end{array}$$

$$\begin{array}{ccc}
P(G_{F,S}, M^\vee(1))[-1] \otimes_\Lambda C(G_{F,S}, M) & \longrightarrow & P(G_{F,S}, M^\vee(1))[-1] \otimes_\Lambda P(G_{F,S}, M) \\
\downarrow & & \downarrow t' \\
 & & (P(G_{F,S}, M^\vee(1)) \otimes_\Lambda P(G_{F,S}, M))[-1] \\
 & & \downarrow \cup_S[-1] \\
C_c(G_{F,S}, M^\vee(1)) \otimes_\Lambda C(G_{F,S}, M) & \xrightarrow{{}_c\cup} & C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1))
\end{array}$$

are commutative, where t and t' are the morphisms defined as in Lemma 2.1. These in turn induce the following morphism of exact triangles in $\mathbf{K}(Mod_\Lambda)$.

$$\begin{array}{ccc}
P(G_{F,S}, M)[-1] & \longrightarrow & \bigoplus_{v\in S} \mathrm{Hom}_{\mathbb{Z}_p}(C(G_v, M^\vee(1)), C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)))[-3] \\
\downarrow & & \downarrow \\
C_c(G_{F,S}, M) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(C(G_{F,S}, M^\vee(1)), C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)))[-3] \\
\downarrow & & \downarrow \\
C(G_{F,S}, M) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(C_c(G_{F,S}, M^\vee(1)), C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)))[-3]
\end{array}$$

Combining this with the morphism

$$\vartheta : C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \longrightarrow \tau_{\geqslant 3} C_c\big(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)\big) \xrightarrow{r} \mathbb{Q}_p/\mathbb{Z}_p[-3],$$

we obtain the following morphism of exact triangles

$$\begin{array}{ccc}
\bigoplus_{v\in S} \mathbf{R}\Gamma(G_v, M)[-1] & \longrightarrow & \bigoplus_{v\in S} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_v, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma_c(G_{F,S}, M) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_{F,S}, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-3] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma(G_{F,S}, M) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma_c(G_{F,S}, M^\vee(1)), \mathbb{Q}_p/\mathbb{Z}_p)[-3]
\end{array}$$

in $\mathbf{D}(Mod_\Lambda)$.

Theorem 4.2.6. *For any bounded complex M in $\mathcal{C}_{\Lambda,G_{F,S}}$, the above morphism of exact triangles is an isomorphism.*

Proof. The top morphism is an isomorphism by Theorem 4.1.2. It remains to show that the middle morphism is an isomorphism. By a similar argument (using Proposition 4.2.3 for the limiting argument for the compactly supported cohomology) to that of Theorem 4.1.2, we can reduce to the case that M is a single finite module. The conclusion then follows from the usual Poitou–Tate duality (cf. [10, 8.6.13]). □

Remark. Theorem 4.1.2 and Theorem 4.2.6 are stated in [3] for the case that Λ is a profinite ring with a basis of neighborhoods consisting of powers of the Jacobson radical of Λ, and M is a finitely generated projective Λ-module.

5. Iwasawa modules

In this section, we will introduce certain modules over an Iwasawa algebra. The next two paragraphs will introduce some notations which will be adhered to throughout this section.

Fix a prime p. Let R be a commutative pro-p ring with a directed fundamental system $\mathcal{I}$ of neighborhoods of zero consisting of open ideals. Let G and Γ be two profinite groups such that there is continuous homomorphism $\pi : G \longrightarrow \Gamma$ of profinite groups. Set $\Lambda = R[\![\Gamma]\!]$. We now describe the natural profinite topology on Λ (see [12, Sect. 5.3]). Let $\mathcal{U}$ be the collection of open normal subgroups of Γ, and consider the following family of two-sided ideals:

$$\mathfrak{a}\Lambda + I(U), \quad \mathfrak{a} \in \mathcal{I},\ U \in \mathcal{U}.$$

Here $I(U)$ denotes the kernel of the map $\Lambda \twoheadrightarrow R[\Gamma/U]$. We take these ideals as a fundamental system of neighborhoods of zero.

We have a map $\iota : \Lambda \to \Lambda$ which sends γ to γ^{-1}. Note that this is only a homomorphism of R-modules. It is a ring homomorphism if and only if Γ is abelian. Denote by

$$\rho = \rho_\Gamma : G \xrightarrow{\pi} \Gamma \subseteq \Lambda^\times$$

the tautological one-dimensional representation of G over Λ.

Remark. In most situations, the ring R is usually a commutative complete Noetherian local ring with finite residue field of characteristic p, and the group Γ is a compact p-adic Lie group. However, despite motivated by the above situation, we shall consider the theory in more generality.

5.1. Induced modules

For a given Λ-module M, we define a Λ°-module M^ι by the formula $m \cdot_\iota \lambda := \iota(\lambda)m$ for $\lambda \in \Lambda$, $m \in M$. Similarly, if N is a Λ°-module, we define a Λ-module, which is also denoted as N^ι, by $\lambda \cdot_\iota m := m\iota(\lambda)$.

We shall prove the following lemma. Let A and B be two rings, and suppose that M has a left A-action and right B-action. We say that the actions of A and B are balanced if for every $a \in A$, $b \in B$ and $x \in M$, we have $a(xb) = (ax)b$.

Lemma 5.1.1. (a) *If M is a $\Lambda[G]$-module, then M^ι is a $\Lambda^\circ[G]$-module.*

(b) *If M is a $\Lambda[G]$-Λ-module (not necessarily balanced), then M^ι is a $\Lambda^\circ[G]$-Λ-module (not necessarily balanced).*

Proof. (a) Let $g \in G, \lambda \in \Lambda$ and $m \in M^\iota$. Then we have

$$(gm) \cdot_\iota \lambda = \iota(\lambda) gm = g\big(\iota(\lambda)m\big) = g(m \cdot_\iota \lambda).$$

(b) Similar argument as above. □

For a given $U \in \mathscr{U}$ and a given $R[G]$-module M, we define two $\Lambda[G]$-Λ-modules as follows:

$$ {}_U M = \mathrm{Hom}_R\big(R[\Gamma/U], M\big),$$
$$ M_U = R[\Gamma/U]^\iota \otimes_R M,$$

where G acts on $R[\Gamma/U]$ via $\rho_{\Gamma/U}$ and Λ acts on $R[\Gamma/U]$ via the canonical projection $\Lambda \twoheadrightarrow R[\Gamma/U]$. Note that the $\Lambda[G]$-Λ-modules defined above are balanced as Λ-Λ-modules. They are balanced as $\Lambda[G]$-Λ-modules if Γ/U is abelian.

Let $V \in \mathscr{U}$ with $U \subseteq V$. Then there is a canonical surjection $\mathrm{pr} : R[\Gamma/U] \twoheadrightarrow R[\Gamma/V]$ and a map $\mathrm{Tr} : R[\Gamma/V] \longrightarrow R[\Gamma/U]$ given by

$$gU \mapsto \sum_{v \in V/U} gvU.$$

These in turn induce the following maps.

$$\mathrm{pr}^* : {}_V M \longrightarrow {}_U M,$$
$$\mathrm{pr}_* : M_U \longrightarrow M_V,$$
$$\mathrm{Tr}^* : {}_U M \longrightarrow {}_V M,$$
$$\mathrm{Tr}_* : M_V \longrightarrow M_U.$$

Denote by $\delta_\beta : G/U \to \mathbb{Z}$ the Kronecker delta-function

$$\delta_\beta\big(\beta'\big) \cong \begin{cases} 1 & \text{if } \beta = \beta', \\ 0 & \text{if } \beta \neq \beta'. \end{cases}$$

The next two lemmas then follow from a straightforward calculation.

Lemma 5.1.2. *We have the following isomorphism of $R[G]$-modules*

$$M_U \xrightarrow{\sim} {}_U M,$$
$$\sum_{\beta \in G/U} \beta \otimes x_\beta \mapsto \sum_{\beta \in G/U} x_\beta \delta_\beta$$

which is functorial in M. Moreover, if V is another open normal subgroup of G such that $U \subseteq V$, then the isomorphism fits into the following commutative diagrams.

$$\begin{array}{ccc} M_U & \xrightarrow{\sim} & {}_U M \\ {\scriptstyle \mathrm{pr}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}^*} \\ M_V & \xrightarrow{\sim} & {}_V M \end{array} \qquad \begin{array}{ccc} M_V & \xrightarrow{\sim} & {}_V M \\ {\scriptstyle \mathrm{Tr}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}^*} \\ M_U & \xrightarrow{\sim} & {}_U M \end{array}$$

Lemma 5.1.3. *We have the following equalities of $\Lambda^\circ[G]$-modules*:

$$({}_U M)^\iota = \mathrm{Hom}_R\big(R[\Gamma/U]^\iota, M\big),$$
$$(M_U)^\iota = R[\Gamma/U] \otimes_R M.$$

Let M be an $R[G]$-module. We define two $\Lambda[G]$-Λ-modules as follows:

$$F_\Gamma(M) = \varinjlim_{U\in\mathscr{U}} {}_U M,$$
$$\mathscr{F}_\Gamma(M) = \varprojlim_{U\in\mathscr{U}} M_U,$$

where the transition maps are induced by the surjections $R[\Gamma/U] \twoheadrightarrow R[\Gamma/V]$ for $U \subseteq V$. Note that the $\Lambda[G]$-Λ-modules defined above are balanced as Λ-Λ-modules. They are balanced as $\Lambda[G]$-Λ-modules if and only if Γ is abelian. One easily sees from Lemma 3.1.3 that

$$F_\Gamma(M)^\iota = \varinjlim_{U\in\mathscr{U}} \mathrm{Hom}_R\big(R[\Gamma/U]^\iota, M\big) \quad \text{and}$$
$$\mathscr{F}_\Gamma(M)^\iota = \varprojlim_{U\in\mathscr{U}} \big(R[\Gamma/U] \otimes_R M\big).$$

We also have the following description of $F_\Gamma(A)$, when A is an object of $\mathcal{D}_{R,G}$.

Lemma 5.1.4. *If A is an object of $\mathcal{D}_{R,G}$, then $F_\Gamma(A)$ is an object of $\mathcal{D}_{\Lambda,G}$ and*

$$F_\Gamma(A) \cong \mathrm{Hom}_{R,\mathrm{cts}}(\Lambda, A).$$

Similarly, we have

$$F_\Gamma(A)^\iota \cong \mathrm{Hom}_{R,\mathrm{cts}}\big(\Lambda^\iota, A\big).$$

If $\{A_\alpha\}$ is a direct system of objects in $\mathcal{D}_{R,G}$, then we have isomorphisms

$$F_\Gamma(A) \cong \varinjlim_\alpha F_\Gamma(A_\alpha) \quad \Big(\textit{resp.}, F_\Gamma(A)^\iota \cong \varinjlim_\alpha F_\Gamma(A_\alpha)^\iota\Big)$$

in $\mathcal{D}_{\Lambda,G}$ (resp., in $\mathcal{D}_{\Lambda^\circ,G}$).

Proof. By Lemma 3.1.4(3), for each $U \in \mathscr{U}$, we have

$$\mathrm{Hom}_R\big(R[\Gamma/U], A\big) = \mathrm{Hom}_{R,\mathrm{cts}}\big(R[\Gamma/U], A\big).$$

Therefore, the lemma will now follow from [12, Prop. 5.1.4]. □

We would like to have a description of $\mathscr{F}_\Gamma(T)$, when T is an object in $\mathcal{C}_{R,G}$. Before we can do this, we shall recall the notion of a complete tensor product from [12]. Let M be an object in $\mathcal{C}_{\Lambda,G}$, and let N be an object in $\mathcal{C}_{R,G}$. The completed tensor product of M and N is taken to be

$$M \hat{\otimes}_R N = \varprojlim_{U,V} M/U \otimes_R N/V,$$

where U (resp., V) runs through the open $\Lambda[G]$-submodules of M (resp., open $R[G]$-submodules of N).

Lemma 5.1.5. *Let M be an object in $\mathcal{C}_{\Lambda,G}$ and N be an object in $\mathcal{C}_{R,G}$. Then the completed tensor product $M \hat{\otimes}_R N$ is an object of $\mathcal{C}_{\Lambda,G}$, and coincides with the usual tensor product if N is a finitely generated R-module. Moreover, as a functor, the completed tensor product is right exact (in both variables) and preserves inverse limits.*

Proof. It follows from [14, Lemma 7.7.2] that $M \hat{\otimes}_R N$ is a compact Λ-module. By a similar argument to that used in the proof of that lemma, we have that the G-action is continuous. □

We are now in position to describe $\mathscr{F}_\Gamma(T)$.

Lemma 5.1.6. *If T is an object of $\mathcal{C}_{R,G}$, then $\mathscr{F}_\Gamma(T)$ is isomorphic to $\Lambda^\iota \hat{\otimes}_R T$ and $\mathscr{F}_\Gamma(T)^\iota$ is isomorphic to $\Lambda \hat{\otimes}_R T$. If $\{T_\alpha\}$ is an inverse system of objects in $\mathcal{C}_{R,G}$ such that $T \cong \varprojlim_\alpha T_\alpha$, then we have isomorphisms*

$$\mathscr{F}_\Gamma(T) \cong \varprojlim_\alpha \mathscr{F}_\Gamma(T_\alpha) \quad \Big(\text{resp., } \mathscr{F}_\Gamma(T)^\iota \cong \varprojlim_\alpha \mathscr{F}_\Gamma(T_\alpha)^\iota\Big)$$

in $\mathcal{C}_{\Lambda,G}$ (resp., in $\mathcal{C}_{\Lambda^\circ,G}$).

Proof. We have

$$\mathscr{F}_\Gamma(T) = \varprojlim_U \big(R[\Gamma/U]^\iota \otimes_R T\big) = \varprojlim_U \big(R[\Gamma/U]^\iota \hat{\otimes}_R T\big) \cong \Big(\varprojlim_U R[\Gamma/U]^\iota\Big) \hat{\otimes}_R T \cong \Lambda^\iota \hat{\otimes}_R T.$$

Suppose $T \cong \varprojlim_\alpha T_\alpha$ in $\mathcal{C}_{R,G}$. Then

$$\mathscr{F}_\Gamma(T) \cong \Lambda^\iota \hat{\otimes}_R T \cong \varprojlim_\alpha \Lambda^\iota \hat{\otimes}_R T_\alpha \cong \varprojlim_\alpha \mathscr{F}_\Gamma(T_\alpha). \qquad \square$$

As a conclusion to the subsection, we record the following duality relation between the modules we have defined.

Proposition 5.1.7. *Let T be an object in $\mathcal{C}_{R,G}$. Then we have isomorphisms*

$$\mathscr{F}_\Gamma(T)^\vee \cong F_\Gamma\big(T^\vee\big)^\iota \quad \big(\text{resp., } \big(\mathscr{F}_\Gamma(T)^\iota\big)^\vee \cong F_\Gamma\big(T^\vee\big)\big)$$

in $\mathcal{D}_{\Lambda^\circ,G}$ (resp., in $\mathcal{D}_{\Lambda,G}$).

Proof. We will prove the first isomorphism, the second will follow from a similar argument. This follows by the following calculations:

$$\begin{aligned}
\mathscr{F}_\Gamma(T)^\vee &\cong \mathrm{Hom}_{\mathbb{Z}_p}\big(\Lambda^\iota \hat{\otimes}_R T, \mathbb{Q}_p/\mathbb{Z}_p\big) \quad \text{(by Lemma 5.1.6)} \\
&\cong \mathrm{Hom}_{R,\mathrm{cts}}\big(\Lambda^\iota, \mathrm{Hom}_{\mathbb{Z}_p,\mathrm{cts}}(T, \mathbb{Q}_p/\mathbb{Z}_p)\big) \quad \big(\text{by [12, Prop. 5.5.4(c)]}\big) \\
&\cong F_\Gamma\big(T^\vee\big)^\iota \quad \text{(by Lemma 5.1.4).} \qquad \square
\end{aligned}$$

5.2. Shapiro's lemma

As before, R denotes a commutative pro-p ring. Let G be a profinite group. Fix a closed normal subgroup H of G and write $\Gamma = G/H$. Let $\pi : G \longrightarrow \Gamma$ be the canonical quotient map. We identify $\mathscr{U}$ as the collection of open normal subgroups of G containing H. Therefore, in this context, for each $U \in \mathscr{U}$, and an $R[G]$-module M, we have

$$ {}_U M = \mathrm{Hom}_R\big(R[G/U], M\big),$$
$$ M_U = R[G/U]^{\iota} \otimes_R M.$$

We will apply Shapiro's lemma to see that the direct limits and inverse limits of cohomology groups over every intermediate field F_α can be viewed as cohomology groups of certain Λ-modules. The results in this section can be found in [9, 8.2.2, 8.3.3-5, 8.4.4.2].

Lemma 5.2.1. *Let U be an open normal subgroup of G, and let N be a bounded below complex of objects of $\mathcal{D}_{R,G}$. Then we have a quasi-isomorphism*

$$C(G, {}_U N) \xrightarrow{\sim} C(U, N)$$

of complexes of Λ-modules.

Proof. We first prove the lemma in the case that N is an object of $\mathcal{D}_{R,G}$. Then we may write $N = \varinjlim_\alpha N_\alpha$, where N_α is a finite object of $\mathcal{D}_{R,G}$. The usual Shapiro's lemma holds for such modules. Also, we note that ${}_U N \cong \varinjlim_\alpha {}_U(N_\alpha)$. Hence, we have

$$C(G, {}_U N) = C\Big(G, \varinjlim_\alpha {}_U(N_\alpha)\Big) \cong \varinjlim_\alpha C\big(G, {}_U(N_\alpha)\big) \xrightarrow{\mathrm{sh}} \varinjlim_\alpha C(U, N_\alpha) = C(U, N)$$

which gives the required conclusion for the case that N is an object of $\mathcal{D}_{R,G}$. For the case that N is a bounded below complex of objects of $\mathcal{D}_{R,G}$, one can prove this by a spectral sequence argument as used in Lemma 3.2.10. □

Recall that if A is a complex in $\mathcal{D}_{R,G}$, then $F_\Gamma(A) = \varinjlim_{U\in\mathscr{U}} {}_U A$ is a complex in $\mathcal{D}_{\Lambda,G}$ by Lemma 5.1.4. We then have the following proposition.

Proposition 5.2.2. *Let A be a bounded below complex of objects of $\mathcal{D}_{R,G}$. Then the composite morphism*

$$C\big(G, F_\Gamma(A)\big) \xrightarrow{\sim} \varinjlim_{U\in\mathscr{U}} C(G, {}_U A) \xrightarrow{\mathrm{sh}} \varinjlim_{U\in\mathscr{U}} C(U, A) \xrightarrow{\mathrm{res}} C(H, A)$$

is a quasi-isomorphism of complexes of Λ-modules. In other words, we have an isomorphism

$$\mathbf{R}\Gamma\big(G, F_\Gamma(A)\big) \xrightarrow{\sim} \mathbf{R}\Gamma(H, A)$$

in $\mathbf{D}(Mod_\Lambda)$.

The next result will give a Shapiro-type relation for cohomology groups of objects (and complexes of objects) in $\mathcal{C}_{R,G}$.

Lemma 5.2.3. *Let U be an open normal subgroup of G. Then for any bounded complex M in $\mathcal{C}_{R,G}$, we have a quasi-isomorphism*

$$C(G, M_U) \xrightarrow{\sim} C(U, M)$$

of complexes of Λ-modules.

Proof. By the same argument as that in Lemma 5.2.1, it suffices to consider the case when M is an object of $\mathcal{C}_{R,G}$. Then we have $M = \varprojlim_\alpha M_\alpha$, where M_α is a finite object in $\mathcal{C}_{R,G}$. Note that $M_U \cong \varprojlim_\alpha (M_\alpha)_U$. Then we have morphisms

$$C(G, M_U) \cong \varprojlim_\alpha C\big(G, (M_\alpha)_U\big) \xrightarrow{\mathrm{sh}} \varprojlim_\alpha C(U, M_\alpha) \cong C(U, M)$$

which induce a morphism

$$\begin{array}{ccc} \varprojlim_\alpha{}^{(i)} H^j(G, (M_\alpha)_U) & \Longrightarrow & H^{i+j}(G, M_U) \\ \Big\downarrow & & \\ \varprojlim_\alpha{}^{(i)} H^j(U, M_\alpha) & \Longrightarrow & H^{i+j}(U, M) \end{array}$$

of convergent spectral sequences. Since M_α is finite, the usual Shapiro's lemma implies that

$$H^j\big(G, (M_\alpha)_U\big) \cong H^j(U, M_\alpha)$$

is an isomorphism. This in turn implies that

$$\varprojlim_\alpha{}^{(i)} H^j\big(G, (M_\alpha)_U\big) \cong \varprojlim_\alpha{}^{(i)} H^j(U, M_\alpha).$$

By the convergence of the spectral sequences, we have isomorphisms

$$H^n(G, M_U) \cong H^n(U, M),$$

as required. □

Since inverse limits are not necessarily exact, we cannot always view inverse limits of cohomology groups over every intermediate field F_α as cohomology groups of certain Λ-modules in general. However, we can say something if we impose an extra assumption on G.

Proposition 5.2.4. *Let M be a bounded complex of objects in $\mathcal{C}_{R,G}$. Then we have the following isomorphism*

$$C\big(G, \mathscr{F}_\Gamma(M)\big) \xrightarrow{\sim} \varprojlim_{U \in \mathscr{U}} C(G, M_U)$$

of complexes of Λ-modules. Furthermore, if $H^m(G, N)$ is finite for all finite discrete Λ-modules N with a Λ-linear continuous G-action and all $m \geqslant 0$, then we have

$$H^j\big(G, \mathscr{F}_\Gamma(M)\big) \cong \varprojlim_{U \in \mathscr{U}} H^j(U, M).$$

Proof. As before, it suffices to consider the case when M is an object of $\mathcal{C}_{R,G}$. Write $M = \varprojlim_\alpha M_\alpha$, where each M_α is a finite object in $\mathcal{C}_{R,G}$. By Lemma 5.1.6, we have a continuous isomorphism

$$\mathscr{F}_\Gamma(M) \cong \varprojlim_\alpha \mathscr{F}_\Gamma(M_\alpha) \cong \varprojlim_{\alpha,U} (M_\alpha)_U.$$

The second assertion now follows from Proposition 3.2.8 and Lemma 5.2.3. □

5.3. Iwasawa setting

We now apply the discussion in Section 5.2 to the arithmetic situation. Let F_∞ be a Galois extension of F which is contained in F_S. Write $H = \mathrm{Gal}(F_S/F_\infty)$, and write $\Gamma = \mathrm{Gal}(F_\infty/F)$. Let $\mathscr{U}$ denote the collection of open normal subgroups of $G_{F,S}$ containing H. For each $U \in \mathscr{U}$, we let $F_U = (F_S)^U$, and define S_U to be the set of primes in F_U above S. As before, we write $\Lambda = R[\![\Gamma]\!]$, where R is a commutative pro-p ring. The following lemma is immediate from the discussion in the preceding subsection.

Lemma 5.3.1. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_{F,S}}$. Then we have the following isomorphisms*

$$H^j\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big) \cong \varprojlim_U H^j(G_{F,S}, T_U) \cong \varprojlim_U H^j(G_{F_U,S_U}, T),$$

$$H^j\big(G_{F,S}, F_\Gamma(T^\vee)\big) \cong \varinjlim_U H^j\big(G_{F,S}, {}_UT^\vee\big) \cong \varinjlim_U H^j\big(G_{F_U,S_U}, T^\vee\big) \cong H^j\big(\mathrm{Gal}(F_S/F_\infty), T^\vee\big).$$

Let $v \in S_f$. Fix an embedding $F^{\mathrm{sep}} \hookrightarrow F_v^{\mathrm{sep}}$, which induces a continuous group monomorphism

$$\alpha = \alpha_v : G_v \hookrightarrow G_F,$$

where $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let X be a finite discrete $R[G_F]$-module. For a finite Galois extension F' of F, write $U = \mathrm{Gal}(F^{\mathrm{sep}}/F')$ and $X_U = R[G_F/U] \otimes_R X$. The embedding $F^{\mathrm{sep}} \hookrightarrow F_v^{\mathrm{sep}}$ determines a prime v' of F' above v such that $F'_{v'}$ is a finite Galois extension of F_v and $G_{v'} := \mathrm{Gal}(F_v^{\mathrm{sep}}/F'_{v'}) = \alpha^{-1}(U)$.

Fix coset representatives $\sigma_i \in G_F$ of

$$G_F/U = \bigcup_i \sigma_i \alpha(G_v/G_{v'}).$$

Then the set of distinct primes in F' above v is given by the (finite) collection $\{\sigma_i(v')\}$. Then by [9, 8.1.7.6, 8.5.3.1], we have a quasi-isomorphism

$$C(G_v, X_U) \xrightarrow{\sim} \bigoplus_i C(G_{\sigma_i(v')}, X)$$

and isomorphisms

$$H^n(G_v, X_U) \cong \bigoplus_i H^n(G_{\sigma_i(v')}, X)$$

of cohomology groups for $n \geqslant 0$.

Now suppose that $p = 2$ and F is a number field with at least one real prime. Let $v \in S_{\mathbb{R}}$. Let F' be a finite Galois extension of F, and retain the above notations. Then the primes in F' above v are either all real or all complex. We first consider the case when all the primes above F' are all real. Then G_v acts trivially on X_U, and we have

$$C(G_v, X_U) = C(G_v, X)^{|G_F/U|}$$

which is precisely $\bigoplus_\tau C(G_{\tau(v')}, X)$, where v' is a prime above v and τ runs through a set of coset representatives for G_F/U. If all the primes above F' are complex, it remains to show that $\hat{H}^i(G_v, X_U) = 0$ for all i. Since G_v is cyclic (of order 2), we are reduced to showing this for $i = 1, 2$, which follows from Shapiro's lemma in the usual sense (since these are usual cohomology groups).

We shall apply the above discussion to finite discrete $R[G_{F,S}]$-modules, which we view as $R[G_F]$-modules via the canonical quotient map $G_F \twoheadrightarrow G_{F,S}$. By the compatibility of limits and the groups of continuous cochains, we can apply the above results to objects in $\mathcal{D}_{R,G_{F,S}}$ or $\mathcal{C}_{R,G_{F,S}}$.

Lemma 5.3.2. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_{F,S}}$. Then for $v \in S_f$, we have the following isomorphisms*

$$H^j\big(G_v, \mathscr{F}_\Gamma(T)\big) \cong \varprojlim_U H^j(G_v, T_U) \cong \varprojlim_U \bigoplus_{w|v} H^j(G_w, T),$$

$$H^j\big(G_v, F_\Gamma(T^\vee)\big) \cong \varinjlim_U H^j\big(G_v, {}_U T^\vee\big) \cong \varinjlim_U \bigoplus_{w|v} H^j\big(G_w, T^\vee\big).$$

The same conclusion holds for the case when $p = 2$ and $v \in S_{\mathbb{R}}$, if we replace the cohomology groups by the completed cohomology groups as defined in Section 3.4.

We would like to derive an analogue of Shapiro's lemma for compactly supported cohomology. Let F' be a finite Galois extension of F which is contained in F_S. Denote the set of primes of F' above S by S'. Let X be a discrete $R[G_{F,S}]$-module. We write $U = \mathrm{Gal}(F_S/F')$ and $X_U = R[G_{F,S}/U] \otimes_R X$. By the discussion in the previous subsection and the above, we have the following diagram

$$\begin{array}{ccccc} C(G_{F,S}, X_U) & \longrightarrow & \bigoplus_{v\in S} C(G_v, X_U) & \xrightarrow{\sim} & \bigoplus_{v\in S}\bigoplus_{v'|v} C(G_v, R[G_v/G_{v'}] \otimes_R X) \\ \downarrow \mathrm{sh} & & & & \downarrow \mathrm{sh} \\ C(G_{F',S'}, X) & & \longrightarrow & & \bigoplus_{v'\in S'} C(G_{v'}, X) \end{array}$$

which commutes up to homotopy. By a similar argument to that in [9, 8.1.7.2.1, 8.5.3.2], this in turn induces a quasi-isomorphism (functorial in X)

$$\mathrm{sh}_c : C_c(G_{F,S}, X_U) \longrightarrow C_c(G_{F',S'}, X)$$

which fits into the following commutative (up to homotopy) diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v\in S} C(G_v, X_U)[-1] & \longrightarrow & C_c(G_{F,S}, X_U) & \longrightarrow & C(G_{F,S}, X_U) & \longrightarrow & 0 \\
 & & \downarrow \wr & & \| & & \| & & \\
0 & \longrightarrow & \bigoplus_{v\in S}\bigoplus_{v'|v} C(G_v, R[G_v/G_{v'}]\otimes_R X)[-1] & \longrightarrow & C_c(G_{F,S}, X_U) & \longrightarrow & C(G_{F,S}, X_U) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{sh}[-1]} & & \downarrow{\scriptstyle \mathrm{sh}_c} & & \downarrow{\scriptstyle \mathrm{sh}} & & \\
0 & \longrightarrow & \bigoplus_{v'\in S'} C(G_v, X)[-1] & \longrightarrow & C_c(G_{F',S'}, X) & \longrightarrow & C(G_{F',S'}, X) & \longrightarrow & 0
\end{array}$$

Suppose that $F'' \subseteq F_S$ is another finite Galois extension of F containing F', and write S'' for the set of primes of F'' above S and $V = \mathrm{Gal}(F_S/F'')$. Again by similar arguments to that in [9, 8.5.3.4], we have the following morphisms

$$\mathrm{res}_c : C_c(G_{F',S'}, X) \longrightarrow C_c(G_{F'',S''}, X),$$
$$\mathrm{cor}_c : C_c(G_{F'',S''}, X) \longrightarrow C_c(G_{F',S'}, X),$$

which are functorial in X and fit in the following diagrams, which are commutative up to homotopy:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v'\in S'} C(G_{v'}, X)[-1] & \longrightarrow & C_c(G_{F',S'}, X) & \longrightarrow & C(G_{F',S'}, X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{res}[-1]} & & \downarrow{\scriptstyle \mathrm{res}_c} & & \downarrow{\scriptstyle \mathrm{res}} & & \\
0 & \longrightarrow & \bigoplus_{v''\in S''} C(G_{v''}, X)[-1] & \longrightarrow & C_c(G_{F'',S''}, X) & \longrightarrow & C(G_{F'',S''}, X) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v''\in S''} C(G_{v''}, X)[-1] & \longrightarrow & C_c(G_{F'',S''}, X) & \longrightarrow & C(G_{F'',S''}, X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{cor}[-1]} & & \downarrow{\scriptstyle \mathrm{cor}_c} & & \downarrow{\scriptstyle \mathrm{cor}} & & \\
0 & \longrightarrow & \bigoplus_{v'\in S'} C(G_{v'}, X)[-1] & \longrightarrow & C_c(G_{F',S'}, X) & \longrightarrow & C(G_{F',S'}, X) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccc}
C_c(G_{F,S}, X_U) & \xrightarrow{\mathrm{sh}_c} & C_c(G_{F',S'}, X) \\
\downarrow{\scriptstyle \mathrm{Tr}_*} & & \downarrow{\scriptstyle \mathrm{res}_c} \\
C_c(G_{F,S}, X_V) & \xrightarrow{\mathrm{sh}_c} & C_c(G_{F'',S''}, X)
\end{array}
\qquad
\begin{array}{ccc}
C_c(G_{F,S}, X_V) & \xrightarrow{\mathrm{sh}_c} & C_c(G_{F'',S''}, X) \\
\downarrow{\scriptstyle \mathrm{pr}_*} & & \downarrow{\scriptstyle \mathrm{cor}_c} \\
C_c(G_{F,S}, X_U) & \xrightarrow{\mathrm{sh}_c} & C_c(G_{F',S'}, X)
\end{array}$$

Since all the morphisms constructed above are functorial, they can be extended to complexes. Hence, we may conclude the following.

Proposition 5.3.3. (a) *For a bounded below complex A of objects in $\mathcal{D}_{R,G_{F,S}}$, the canonical morphism of complexes*

$$C_c\big(G_{F,S}, F_\Gamma(A)\big) \xrightarrow{\sim} \varinjlim_{U,\mathrm{Tr}} C_c(G_{F,S}, {}_U A)$$

is an isomorphism.

(b) *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_{F,S}}$. Then the canonical morphism of complexes*

$$C_c\big(G_{F,S},\mathscr{F}_\Gamma(T)\big)\longrightarrow \varprojlim_{U,\mathrm{pr}} C_c(G_{F,S},T_U)$$

is an isomorphism and induces isomorphisms

$$H^j_c\big(G_{F,S},\mathscr{F}_\Gamma(T)\big)\cong \varprojlim_{U,\mathrm{pr}} H^j_c(G_{F,S},T_U)\cong \varprojlim_{U,\mathrm{cor}_c} H^j_c(G_{F_U,S_U},T)$$

of cohomology groups for $j\geqslant 0$.

5.4. Duality over extensions of a global/local field

We retain the notation introduced in the previous subsection. Let F_∞ be a Galois extension of F which is contained in F_S. Write $H=\mathrm{Gal}(F_S/F_\infty)$, and write $\Gamma=\mathrm{Gal}(F_\infty/F)$. As before, we write $\Lambda=R[\![\Gamma]\!]$, where R is a commutative pro-p ring.

Applying Theorem 4.2.6 and Proposition 5.1.7, we obtain the following theorem. In the theorem, we abuse notation and use $\mathbf{R}\Gamma(G_v,M)$ to denote $\widehat{\mathbf{R}\Gamma}(G_v,M)$ for $v\in S_{\mathbb{R}}$.

Theorem 5.4.1. *Then, for a bounded complex T in $\mathcal{C}_{R,G_{F,S}}$, we have the following isomorphism of exact triangles*

$$\begin{array}{ccc}
\bigoplus_{v\in S}\mathbf{R}\Gamma(G_v,\mathscr{F}_\Gamma(T))[-1] & \xrightarrow{\sim} & \bigoplus_{v\in S}\mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_v,F_\Gamma(T^\vee)^\iota(1)),\mathbb{Q}_p/\mathbb{Z}_p)[-3]\\
\downarrow & & \downarrow\\
\mathbf{R}\Gamma_c(G_{F,S},\mathscr{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_{F,S},F_\Gamma(T^\vee)^\iota(1)),\mathbb{Q}_p/\mathbb{Z}_p)[-3]\\
\downarrow & & \downarrow\\
\mathbf{R}\Gamma(G_{F,S},\mathscr{F}_\Gamma(T)) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma_c(G_{F,S},F_\Gamma(T^\vee)^\iota(1)),\mathbb{Q}_p/\mathbb{Z}_p)[-3]
\end{array}$$

in $\mathbf{D}(\mathrm{Mod}_\Lambda)$.

We end by saying something about the situation over non-Archimedean local fields. Let F be a non-Archimedean local field of characteristic not equal to p. Let F_∞ be a Galois extension of F with Galois group Γ. Write $G_E=\mathrm{Gal}(F^{\mathrm{sep}}/E)$ for every Galois extension E/F. Recall that by [10, Thm. 7.1.8(i)], we have $\mathrm{cd}_p(G_F)=2$.

Let T be a bounded complex of objects in $\mathcal{C}_{R,G_F}$. By Proposition 5.2.2 and Proposition 5.2.4, we have

$$\begin{aligned}
C\big(G_F,F_\Gamma(T^\vee)\big)&\longrightarrow \varinjlim C\big(G_{F_\alpha},T^\vee\big),\\
H^i\big(G_F,F_\Gamma(T^\vee)\big)&\cong \varinjlim H^i\big(G_{F_\alpha},T^\vee\big)\cong H^i\big(G_{F_\infty},T^\vee\big),\\
C\big(G_F,\mathscr{F}_\Gamma(T)\big)&\longrightarrow \varprojlim C(G_{F_\alpha},T),\\
H^i\big(G_F,\mathscr{F}_\Gamma(T)\big)&\cong \varprojlim H^i(G_{F_\alpha},T),
\end{aligned}$$

where F_α runs through all finite Galois extension of F_∞/F. Applying Theorem 4.1.2 and Proposition 5.1.7, we obtain the following.

Theorem 5.4.2. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_F}$. Then we have the following isomorphism*

$$\mathbf{R}\Gamma\big(G_F, \mathscr{F}_\Gamma(T)\big) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbf{R}\Gamma\big(G_F, F_\Gamma\big(T^\vee\big)^\iota(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\big)[-2]$$

in $\mathbf{D}(Mod_\Lambda)$.

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