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General Section

# Independence between coefficients of two modular forms ☆

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Let $k$ be an even integer and $S_k$ be the space of cusp forms of weight $k$ on $\mathrm{SL}_2(\mathbb{Z})$. Let $S = \oplus_{k\in 2\mathbb{Z}} S_k$. For $f, g \in S$, we let $R(f, g)$ be the set of ratios of the Fourier coefficients of $f$ and $g$ defined by $R(f, g) := \{x \in \mathbb{P}^1(\mathbb{C}) \mid x = [a_f(p) : a_g(p)]$ for some prime $p\}$, where $a_f(n)$ (resp. $a_g(n)$) denotes the $n$th Fourier coefficient of $f$ (resp. $g$). In this paper, we prove that if $f$ and $g$ are nonzero and $R(f, g)$ is finite, then $f = cg$ for some constant $c$. This result is extended to the space of weakly holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$. We apply it to study the number of representations of a positive integer by a quadratic form.

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## 1. Introduction

The Fourier coefficients of a modular form play crucial roles in studying the theory of modular forms. In particular, the $q$-expansion principle (for example, see [1] or [4])

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shows that a modular form is determined by its Fourier coefficients. A natural question is to find relations between two modular forms when a connection between their Fourier coefficients is given. It was proved by Ramakrishnan [3, Appendix] that if $f$ and $g$ are normalized Hecke eigenforms of the same weight such that for all primes $p$ outside a set $T$ of density $\delta(T) < \frac{1}{18}$

$$a_f(p)^2 = a_g(p)^2,$$

then there exists a quadratic character $\chi$ such that

$$f = g \otimes \chi.$$

Here, $a_f(n)$ (resp. $a_g(n)$) denotes the $n$th Fourier coefficient of $f$ (resp. $g$). This result was also known to Blasius and Serre.

Let $k$ be an even integer, and $M_k^!$ be the space of weakly holomorphic modular forms of weight $k$ on $\mathrm{SL}_2(\mathbb{Z})$. Let

$$M^! := \oplus_{k \in 2\mathbb{Z}} M_k^!.$$

Suppose that $f$ and $g$ are weakly holomorphic modular forms in $M^!$. We define a subset $R(f,g)$ of $\mathbb{P}^1(\mathbb{C})$ by

$$R(f,g) := \left\{ x \in \mathbb{P}^1(\mathbb{C}) \mid x = [a_f(p) : a_g(p)] \text{ for some prime } p \right\}. \tag{1.1}$$

This is a set of ratios of the Fourier coefficients of $f$ and $g$. For example, if $a_f(p)^2 = a_g(p)^2$ for every prime $p$, then $R(f,g) = \{[1 : 1], [1 : -1]\}$. Therefore, if $f$ and $g$ are Hecke eigenforms and $R(f,g) = \{[1 : 1], [1 : -1]\}$, then $f = cg$ for some constant $c$. In this vein, the objective of this paper is to classify $f$ and $g$ in $M^!$ such that $R(f,g)$ is a finite set. Our main result is as follows.

**Theorem 1.1.** *Suppose that $f$ and $g$ are nonzero weakly holomorphic modular forms in $M^!$. If $R(f,g)$ is a finite set, then $f = cg$ for some constant $c$.*

This applies to study the number of representations of an integer by a quadratic form. Let $Q(x_1, \ldots, x_d)$ be a positive definite quadratic form over $\mathbb{Z}$ in $d$ variables with level one. For a positive integer $n$, let

$$r_Q(n) := \left| \{(x_1, \ldots, x_d) \in \mathbb{Z}^d \mid Q(x_1, \ldots, x_d) = n\} \right|$$

be the number of representations of the integer $n$ by a quadratic form $Q$. Note that

$$1 + \sum_{n=1}^{\infty} r_Q(n) q^n$$

is a modular form of weight $d/2$ on $\mathrm{SL}_2(\mathbb{Z})$. Here, $q$ denotes $e^{2\pi iz}$, where $z$ is a complex number whose imaginary part is positive. Therefore, Theorem 1.1 gives the following corollary.

**Corollary 1.2.** *Assume that $Q_1(x_1, \ldots, x_d)$ and $Q_2(x_1, \ldots, x_d)$ are positive definite quadratic forms over $\mathbb{Z}$ in $d$ variables with level one. If there is a positive integer $n$ such that the numbers of representations of $n$ by $Q_1$ and $Q_2$ are different, then the number of elements of the set*

$$R(Q_1, Q_2) := \left\{x \in \mathbb{P}^1(\mathbb{C}) \mid x = [r_{Q_1}(p) : r_{Q_2}(p)] \text{ for some prime } p\right\}$$

*is infinite.*

The main ingredient of the proof of Theorem 1.1 is the result in [7] on the Galois representations attached to Hecke eigenforms. This is used to prove that if $f$ and $g$ are nonzero cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ and $R(f, g)$ is finite, then $f = cg$ for some constant $c$. This is the main part of the proof of Theorem 1.1.

**Remark 1.3.** In the same way, the result can be extended to harmonic weak Maass forms. In this case, if $f$ and $g$ are harmonic weak Maass forms whose shadows are cusp forms, then we need to look at the set

$$R\left(f^+, g^+\right) := \left\{x \in \mathbb{P}^1(\mathbb{C}) \mid x = [a_{f^+}(p) : a_{g^+}(p)] \text{ for some prime } p\right\},$$

where $f^+$ (resp. $g^+$) denotes the holomorphic part of $f$ (resp. $g$) and $a_{f^+}(n)$ (resp. $a_{g^+}(n)$) denotes the $n$th Fourier coefficient of $f^+$ (resp. $g^+$).

The remainder of this paper is organized as follows. In Section 2, we review some preliminaries concerning the Fourier coefficients of weakly holomorphic modular forms and the Galois representations attached to Hecke eigenforms. In Section 3, we prove the main theorem for the case of cusp forms. In Section 4, we prove the main theorem: Theorem 1.1.

## 2. Preliminaries

In this section, we review some basic material concerning the Fourier coefficients of weakly holomorphic modular forms and the Galois representations attached to Hecke eigenforms.

### *2.1. Fourier coefficients of weakly holomorphic modular forms*

In this section, we review some results related to the asymptotic of the Fourier coefficients of weakly holomorphic modular forms based on [5] and [6].

Let $f \in M_k^!$. We write $k = 12o_k + k'$ with $o_k \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. Then, by the valence formula, we have

$$\mathrm{ord}_\infty(f) \le o_k \tag{2.1}$$

if $f$ is nonzero. Moreover, for $m \ge -o_k$, there is a unique $f_{k,m} \in M_k^!$ such that

$$f_{k,m}(z) = q^{-m} + O(q^{o_k+1}). \tag{2.2}$$

Then, $\{f_{k,m} \mid m \ge -o_k\}$ forms a basis of $M_k^!$. In [2], Duke and Jenkins studied various properties of this basis.

For positive integers $m, n$, and $c$, let

$$A_{m,c}(n) := \sum_{\substack{-d=0 \\ (c,d)=1}}^{c-1} e^{\frac{2\pi i}{c}(nd-ma)},$$

where $a$ is an integer such that $ad \equiv 1 \pmod{c}$. We introduce the Bessel function of the first kind (for example, see [9])

$$I_n(z) := \sum_{t=0}^{\infty} \frac{(z/2)^{n+2t}}{t!\Gamma(n+t+1)}.$$

Note that this Bessel function satisfies an asymptotic expansion

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}}$$

as $z \to \infty$. Then, we have the following theorem.

**Theorem 2.1.** [5, Theorem 1–3], [6, pp. 149–151] *For $m \ge -o_k$, let $a_{k,m}(n)$ be the nth Fourier coefficient of $f_{k,m}$.*

(1) *If $m > 0$, then*

$$a_{k,m}(n) \sim C_k A_{m,1}(n) \left(\frac{n}{m}\right)^{(k-1)/2} \frac{e^{4\pi\sqrt{mn}}}{(mn)^{1/4}}$$

*as $n \to \infty$, where $C_k$ is a constant dependent on $k$.*

(2) *If $m = 0$, then*

$$a_{k,m}(n) \sim D_k n^{k-1}$$

*as $n \to \infty$, where $D_k$ is a constant dependent on $k$.*

### *2.2. Galois representations attached to Hecke eigenforms*

In this section, we introduce the result in [7] concerning the Galois representations attached to Hecke eigenforms. For a positive integer $n$, let $T_n$ be the $n$th Hecke operator. For a positive even integer $k$, let $S_k$ denote the space of cusp forms of weight $k$ on $\mathrm{SL}_2(\mathbb{Z})$. Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k$ be a normalized Hecke eigenform, i.e. $f|T_p = a_f(p)f$ for every prime $p$ and $a_f(1) = 1$. Let $E_f$ be the field generated by all the Hecke eigenvalues $a_f(p)$ over $\mathbb{Q}$, and $H_f$ be the $\mathbb{Z}$-algebra generated by all the Hecke eigenvalues $a_f(p)$. Let $G_{\mathbb{Q}}$ denote $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. For a prime $\ell$, let

$$\rho_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2\left(E_f \otimes \mathbb{Q}_\ell\right)$$

be the representation of $G_{\mathbb{Q}}$ attached to $f$. Note that if $p$ is a prime not equal to $\ell$ and $\mathrm{Frob}_p \in G_{\mathbb{Q}}$ is a Frobenius element at $p$, then the trace of $\rho_{f,\ell}(\mathrm{Frob}_p)$ is $a_f(p)$ in $E_f \otimes \mathbb{Q}_\ell$ and the determinant of $\rho_{f,\ell}(\mathrm{Frob}_p)$ is $p^{k-1}$ (for example, see page 261 in [7]).

Let $\mathcal{G}_{f,\ell}$ be the image of $\rho_{f,\ell}$ in $\mathrm{GL}_2\left(E_f \otimes \mathbb{Q}_\ell\right)$. If

$$\mathcal{A}_{f,\ell} := \left\{u \in \mathrm{GL}_2\left(E_f \otimes \mathbb{Q}_\ell\right) \mid \det(u) \in (\mathbb{Z}_\ell^\times)^{k-1}\right\},$$

then $\mathcal{A}_{f,\ell}$ contains $\mathcal{G}_{f,\ell}$. Moreover, it was proved in [7] and [8] that for all but finitely many primes $\ell$, we have

$$\mathcal{G}_{f,\ell} = \mathcal{A}_{f,\ell}.$$

Let $f' \in S_{k'}$ be a normalized Hecke eigenform. Suppose that if $k = k'$, then $f$ and $f'$ are not conjugate under the action of $G_{\mathbb{Q}}$. Let $T_{f,f'}$ be the $\mathbb{Z}$-subalgebra of $H_f \times H_{f'}$ generated by the pairs $(a_f(p), a_{f'}(p))$. It should be noted that according to the assumption, $[H_f \times H_{f'} : T_{f,f'}]$ is finite (for example, see lines 8–10 on p. 268 in [7]). Let $\mathcal{A}_{\rho_f \times \rho_{f'}}$ be the image of $\rho_{f,\ell} \times \rho_{f',\ell}$ in $\mathrm{GL}_2\left(E_f \otimes \mathbb{Q}_\ell\right) \times \mathrm{GL}_2\left(E_{f'} \otimes \mathbb{Q}_\ell\right)$. Ribet [7] proved the following theorem.

**Theorem 2.2** *(Theorem 6.1 in [7]). If $\ell$ is a prime such that*

- $\ell \geq k + k'$,
- $\mathcal{G}_{f,\ell} = \mathcal{A}_{f,\ell}$,
- $\mathcal{G}_{f',\ell} = \mathcal{A}_{f',\ell}$,
- $\ell \nmid [H_f \times H_{f'} : T_{f,f'}]$,

*then*

$$\mathcal{A}_{\rho_f \times \rho_{f'}} = \left\{(u, u') \in \mathcal{A}_{f,\ell} \times \mathcal{A}_{f',\ell} \;\middle|\; \det(u) = v^{k-1},\ \det(u') = v^{k'-1} \text{ for some } v \in \mathbb{Z}_\ell^\times\right\}.$$

This theorem implies the following lemma.

**Lemma 2.3.** *For integers $j$, $1 \le j \le m$, suppose that $f_j$ are normalized Hecke eigenforms in $S_{k_j}$ that are not conjugate to each other under the action of $G_{\mathbb{Q}}$. If $\ell$ is a sufficiently large prime, then $H_{f_1} \times \cdots \times H_{f_m}$ is dense in an open subset of $(H_{f_1} \otimes \mathbb{Q}_\ell) \times \cdots \times (H_{f_m} \otimes \mathbb{Q}_\ell)$.*

To prove this lemma, we need the following result.

**Lemma 2.4** *(Lemma 3.4 in [7]). Let $\mathcal{U}_1, \ldots, \mathcal{U}_t$ $(t > 1)$ be profinite groups. Assume that for each $i$ the following condition is satisfied: for each open subgroup $\mathcal{W}$ of $\mathcal{U}_i$, the closure of the commutator subgroup of $\mathcal{W}$ is open in $\mathcal{U}_i$. Let $\mathcal{H}$ be a closed subgroup of*

$$\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_t,$$

*which maps to an open subgroup of each group $\mathcal{U}_i \times \mathcal{U}_j$ $(i \neq j)$. Then, $\mathcal{H}$ is open in $\mathcal{U}$.*

Now, we prove Lemma 2.3.

**Proof.** Let $\mathcal{G}$ be the image of $\rho_{f_1,\ell} \times \cdots \times \rho_{f_m,\ell}$ in $\mathrm{GL}_2(H_{f_1} \otimes \mathbb{Z}_\ell) \times \cdots \times \mathrm{GL}_2(H_{f_m} \otimes \mathbb{Z}_\ell)$. We claim that $\mathcal{G}$ contains $\mathrm{SL}_2(H_{f_1} \otimes \mathbb{Z}_\ell) \times \cdots \times \mathrm{SL}_2(H_{f_m} \otimes \mathbb{Z}_\ell)$, which then provides the proof. Now, we prove the claim. Note that if $\ell$ is a sufficiently large prime, then, for all pairs $(j_1, j_2)$, the prime $\ell$ satisfies the following conditions:

- $\mathcal{G}_{f_{j_1},\ell} = \mathcal{A}_{f_{j_1},\ell}$,
- $\mathcal{G}_{f_{j_2},\ell} = \mathcal{A}_{f_{j_2},\ell}$,
- $\ell \nmid \left[H_{f_{j_1}} \times H_{f_{j_2}} : T_{f_{j_1},f_{j_2}}\right]$.

Therefore, we assume that $\ell$ satisfies these conditions.

Let

$$\mathcal{U} := \mathrm{SL}_2(H_{f_1} \otimes \mathbb{Z}_\ell) \times \cdots \times \mathrm{SL}_2(H_{f_m} \otimes \mathbb{Z}_\ell)$$

and

$$\mathcal{H} := \mathcal{U} \cap \mathcal{G}.$$

Note that $\mathcal{H}$ is closed in $\mathrm{GL}_2(H_{f_1} \otimes \mathbb{Q}_\ell) \times \cdots \times \mathrm{GL}_2(H_{f_m} \otimes \mathbb{Q}_\ell)$ since $G_{\mathbb{Q}}$ is compact and $\rho_{f_1,\ell} \times \cdots \times \rho_{f_m,\ell}$ is continuous. Thus, we see that $\mathcal{H}$ is also closed in $\mathcal{U}$. For all pairs $(j_1, j_2)$, the projection of $\mathcal{H}$ to $\mathrm{SL}_2\left(H_{f_{j_1}} \otimes \mathbb{Z}_\ell\right) \times \mathrm{SL}_2\left(H_{f_{j_2}} \otimes \mathbb{Z}_\ell\right)$ is surjective by Theorem 2.2. For each $j$, we have

$$\mathrm{SL}_2(H_{f_j} \otimes \mathbb{Z}_\ell) \cong \prod_{v|\ell} \mathrm{SL}_2(\mathcal{O}_v),$$

where $v$ denote the prime ideals of $H_{f_j}$ above $\ell$, and $\mathcal{O}_v$ denotes the completion of $H_{f_j}$ at $v$. Note that $\mathrm{SL}_2(\mathcal{O}_v)$ is a $\ell$-adic lie group, and the lie algebra of $\mathrm{SL}_2(\mathcal{O}_v)$ is the same as its own derived algebra. This implies that for each open subgroup $\mathcal{W}$ of $\mathrm{SL}_2(H_{f_j} \otimes \mathbb{Z}_\ell)$, the closure of the commutator subgroup of $\mathcal{W}$ is open in $\mathrm{SL}_2(H_{f_j} \otimes \mathbb{Z}_\ell)$ (see Remark 3 on p. 253 in [7]). Therefore, by Lemma 2.4, we complete the proof of the claim. $\square$

**Remark 2.5.** For the convenience of readers, let us recall the lie algebra of $\mathrm{SL}_2(\mathcal{O}_v)$ and its derived subalgebra. The lie algebra of $\mathrm{SL}_2(\mathcal{O}_v)$ is isomorphic to

$$\mathrm{sl}_2(\mathcal{O}_v) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathcal{O}_v \text{ and } a + c = 0 \right\}.$$

The derived subalgebra $\mathrm{Der}(\mathrm{sl}_2(\mathcal{O}_v))$ of $\mathrm{sl}_2(\mathcal{O}_v)$ is generated by all $[A, B]$ ($A, B \in \mathrm{sl}_2(\mathcal{O}_v)$), where $[A, B] = AB - BA$. Note that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left[\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right],$$

$$\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) = \left[\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right],$$

and

$$\left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right) = \left[\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right].$$

Assume that $\ell \neq 2$. The matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)$ consist a basis of $\mathrm{sl}_2(\mathcal{O}_v)$. Therefore, we have

$$\mathrm{Der}(\mathrm{sl}_2(\mathcal{O}_v)) = \mathrm{sl}_2(\mathcal{O}_v).$$

### *2.3. Lemma for hyperplanes*

For later use, we prove the following lemma.

**Lemma 2.6.** *Let $U$ be a subset of $V = \mathbb{Q}^n$. Suppose that $U$ is dense in an open subset of $V \otimes \mathbb{Q}_\ell$.*

(1) *If $T_1, \ldots, T_m$ are hyperplanes in $V$, then*

$$U \not\subset \cup T_i.$$

(2) *If $L_1, \ldots, L_m$ are hyperplanes in $V \otimes \mathbb{C}$, then we have*

$$U \not\subset \cup L_i.$$

**Proof.** (1) Suppose that

$$U \subset \cup T_i.$$

This implies

$$\overline{U} \subset \cup \overline{T_i}$$

in $V \otimes \mathbb{Q}_\ell$. Then, $\cup \overline{T_i}$ contains an open set in $V \otimes \mathbb{Q}_\ell$. Note that $\overline{T_i}$ is a hyperplane in $V \otimes \mathbb{Q}_\ell$ for each $i$. This gives a contradiction.

(2) Due to (1), it is enough to prove that $L_i \cap V$ is contained in a hyperplane in $V$ for each $i$. Note that $L_i$ can be expressed as

$$\{(y_1, \dots, y_n) \in V \otimes \mathbb{C} |\ a_{i,1}y_1 + \cdots + a_{i,n}y_n = 0\}$$

for $a_{i,j} \in \mathbb{C}$. Since $L_i$ is a hyperplane, we see that $(a_{i,1}, \dots, a_{i,n}) \neq (0, \dots, 0)$. Without loss of generality, we may assume that $a_{i,n} \neq 0$. Then, $(y_1, \dots, y_n) \in L_i$ is equivalent to

$$y_1 = t_1,\ \dots,\ y_{n-1} = t_{n-1},\ y_n = -\frac{a_{i,1}}{a_{i,n}}t_1 - \cdots - \frac{a_{i,n-1}}{a_{i,n}}t_{n-1}$$

for some $t_1, \dots, t_{n-1} \in \mathbb{C}$. Therefore, $(x_1, \dots, x_n) \in L_i \cap V$ is equivalent to

$$\begin{aligned} &x_1 = t_1,\ \dots,\ x_{n-1} = t_{n-1} \text{ for some } t_1, \dots, t_{n-1} \in \mathbb{Q}, \\ &x_n = -\frac{a_{i,1}}{a_{i,n}}t_1 - \cdots - \frac{a_{i,n-1}}{a_{i,n}}t_{n-1} \in \mathbb{Q}. \end{aligned}$$

We can take $\mathbb{Q}$-linearly independent complex numbers $\alpha_1 = 1, \dots, \alpha_{n-1}$ such that

$$\frac{a_{i,1}}{a_{i,n}}, \dots, \frac{a_{i,n-1}}{a_{i,n}}$$

can be expressed as $\mathbb{Q}$-linear combinations of $\alpha_1, \dots, \alpha_{n-1}$. This implies that

$$x_n = F_1(t_1, \dots, t_{n-1}) + \sum_{i=2}^{n-1} \alpha_i F_i(t_1, \dots, t_{n-1})$$

for some degree 1 homogeneous polynomials $F_i$ with coefficients in $\mathbb{Q}$. Therefore, $x_n \in \mathbb{Q}$ is equivalent to $x_n = F_1(t_1, \dots, t_{n-1})$. From this, we see that $L_i \cap V$ is in the hyperplane

$$\{(x_1, \dots, x_n) \in \mathbb{Q}^n \mid x_n = F_1(x_1, \dots, x_{n-1})\}. \quad \square$$

## 3. Coefficients of cusp forms

In this section, we prove that if $f$ and $g$ are cusp forms and $R(f,g)$ is finite, then $f = cg$ for some constant $c$. To prove this, we need the following lemmas.

**Lemma 3.1.** *Suppose that $f_1, \ldots, f_m$ are normalized Hecke eigenforms on $\mathrm{SL}_2(\mathbb{Z})$ such that any two of them are not conjugate under the action of $G_{\mathbb{Q}}$. Then, there are no finite subsets $B$ of $(E_{f_1} \otimes \mathbb{C}) \times \cdots \times (E_{f_m} \otimes \mathbb{C})$ such that $(0, \ldots, 0) \notin B$ and for any primes $p$, we have*

$$A_1 a_{f_1}(p) + \cdots + A_m a_{f_m}(p) = 0$$

*for some $(A_1, \ldots, A_m) \in B$.*

**Proof.** Suppose that $B$ is a finite subset of $(E_{f_1} \otimes \mathbb{C}) \times \cdots \times (E_{f_m} \otimes \mathbb{C})$ such that $(0, \ldots, 0) \notin B$ and for any primes $p$, we have

$$A_1 a_{f_1}(p) + \cdots + A_m a_{f_m}(p) = 0$$

for some $(A_1, \ldots, A_m) \in B$. Then, the set

$$C = \bigcup_{(A_1, \ldots, A_m) \in B} \{(x_1, \ldots, x_m) \in (E_{f_1} \otimes \mathbb{C}) \times \cdots \times (E_{f_m} \otimes \mathbb{C}) \mid A_1 x_1 + \cdots + A_m x_m = 0\}$$

is a finite union of hyperplanes $(E_{f_1} \otimes \mathbb{C}) \times \cdots \times (E_{f_m} \otimes \mathbb{C})$.

By Lemma 2.3, we see that there is a prime $\ell$ such that $H_{f_1} \times \cdots \times H_{f_m}$ is dense in an open subset of $(H_{f_1} \otimes \mathbb{Q}_\ell) \times \cdots \times (H_{f_m} \otimes \mathbb{Q}_\ell)$. Since the trace of $\rho_{f_i,\ell}(\mathrm{Frob}_p)$ is $a_{f_i}(p)$ for primes $p \neq \ell$ and $\{\mathrm{Frob}_p \mid p \text{ is a prime}\}$ is dense in $G_{\mathbb{Q}}$ by Chebotarev's density theorem, we see that the set

$$T = \{(a_{f_1}(p), \ldots, a_{f_n}(p)) \mid p \text{ is a prime with } p \neq \ell\}$$

is a dense subset of $H_{f_1} \times \cdots \times H_{f_m}$. Since we have

$$(H_{f_1} \otimes \mathbb{Q}_\ell) \times \cdots \times (H_{f_m} \otimes \mathbb{Q}_\ell) \cong (E_{f_1} \times \cdots \times E_{f_m}) \otimes \mathbb{Q}_\ell$$

and

$$(E_{f_1} \otimes \mathbb{C}) \times \cdots \times (E_{f_n} \otimes \mathbb{C}) \cong (E_{f_1} \times \cdots \times E_{f_m}) \otimes \mathbb{C},$$

by Lemma 2.6, the set $T$ is not contained in any finite union of hyperplanes in $(E_{f_1} \otimes \mathbb{C}) \times \cdots \times (E_{f_m} \otimes \mathbb{C})$. This is a contradiction since $T$ is contained in $C$ by the assumption. $\square$

Let $f$ be a normalized Hecke eigenform on $\mathrm{SL}_2(\mathbb{Z})$. Let $m = [E_f : \mathbb{Q}]$ and $\sigma_1, \ldots, \sigma_m$ be the embeddings from $E_f$ to $\bar{\mathbb{Q}}$. Note that $E_f \otimes \mathbb{Q}_\ell \cong (\mathbb{Q}_\ell)^m$. Let $\{\tau_1, \ldots, \tau_m\}$ be a basis of $E_f$ over $\mathbb{Q}$. Then, for each $n > 0$, the coefficient $a_f(n)$ can be written as a linear combination of $\tau_1, \ldots, \tau_m$, i.e.,

$$a_f(n) = a_1(n)\tau_1 + \cdots + a_m(n)\tau_m$$

for $a_i(n) \in \mathbb{Q}$. This means that

$$a_f(n)^\sigma = a_1(n)\tau_1^\sigma + \cdots + a_m(n)\tau_m^\sigma \tag{3.1}$$

for $\sigma \in G_\mathbb{Q}$. From this, we prove the following lemma.

**Lemma 3.2.** *There are no finite subsets $B$ of $\mathbb{C}^m$ such that $(0, \ldots, 0) \notin B$ and for any primes $p$, we have*

$$A_1 a_f(p)^{\sigma_1} + \cdots + A_m a_f(p)^{\sigma_m} = 0 \tag{3.2}$$

*for some $(A_1, \ldots, A_m) \in B$.*

**Proof.** Suppose that $B$ is a finite subset of $\mathbb{C}^m$ such that $(0, \ldots, 0) \notin B$ and for any primes $p$, we have

$$A_1 a_f(p)^{\sigma_1} + \cdots + A_m a_f(p)^{\sigma_m} = 0$$

for some $(A_1, \ldots, A_m) \in B$. By (3.1), we see that the equation (3.2) is equivalent to

$$(A_1, \ldots, A_m) \begin{pmatrix} \tau_1^{\sigma_1} & \cdots & \tau_m^{\sigma_1} \\ \vdots & \ddots & \vdots \\ \tau_1^{\sigma_m} & \cdots & \tau_m^{\sigma_m} \end{pmatrix} \begin{pmatrix} a_1(p) \\ \vdots \\ a_m(p) \end{pmatrix} = 0.$$

Since $\{\tau_1, \ldots, \tau_m\}$ is a basis of $E_f$ over $\mathbb{Q}$, the matrix

$$\begin{pmatrix} \tau_1^{\sigma_1} & \cdots & \tau_m^{\sigma_1} \\ \vdots & \ddots & \vdots \\ \tau_1^{\sigma_m} & \cdots & \tau_m^{\sigma_m} \end{pmatrix}$$

is invertible. Therefore, we have

$$(A_1', \ldots, A_m') = (A_1, \ldots, A_m) \begin{pmatrix} \tau_1^{\sigma_1} & \cdots & \tau_m^{\sigma_1} \\ \vdots & \ddots & \vdots \\ \tau_1^{\sigma_m} & \cdots & \tau_m^{\sigma_m} \end{pmatrix} \neq (0, \ldots, 0)$$

since $(A_1, \ldots, A_m) \neq (0, \ldots, 0)$. This means that the set

$$\{(a_1(p), \ldots, a_m(p)) \mid p \text{ is a prime}\}$$

is a subset of the set

$$C = \bigcup_{(A_1,\ldots,A_m)\in B} \{(x_1,\ldots,x_m) \in \mathbb{C}^m \mid A_1'x_1 + \cdots + A_m'x_m = 0\}, \tag{3.3}$$

which is a finite union of hyperplanes in $E_f \otimes \mathbb{C}$.

By Lemma 2.3, we see that there is a prime $\ell$ such that $H_f$ is dense in an open subset of $H_f \otimes \mathbb{Q}_\ell$. Since the trace of $\rho_{f,\ell}(\mathrm{Frob}_p)$ is $a_f(p)$ for primes $p \neq \ell$ and $\{\mathrm{Frob}_p \mid p \text{ is a prime}\}$ is dense in $G_\mathbb{Q}$ by Chebotarev's density theorem, we see that the set

$$T = \{a_f(p) \mid p \text{ is a prime with } p \neq \ell\}$$

is a dense subset of $H_f$. Then, by Lemma 2.6, the set $T$ is not contained in any finite union of hyperplanes in $E_f \otimes \mathbb{C}$. We consider the isomorphism from $E_f$ to $\mathbb{Q}^m$ defined by $a \mapsto (a_1,\ldots,a_m)$, where $a_1,\ldots,a_m$ is determined by the decomposition

$$a = a_1\tau_1 + \cdots + a_m\tau_m.$$

By using this isomorphism, we see that the set

$$T' = \{(a_1(p),\ldots,a_m(p)) \mid p \text{ is a prime with } p \neq \ell\}$$

is not contained in any finite union of hyperplanes in $\mathbb{C}^m$. This is a contradiction since $T'$ is contained in $C$ by the assumption. $\square$

Suppose that $f_1,\ldots,f_m$ are normalized Hecke eigenforms such that any two of them are not conjugate under the action of $G_\mathbb{Q}$. Let $a_i(n)$ be the $n$th Fourier coefficient of $f_i$, i.e.,

$$f_i(z) = \sum_{n>0} a_i(n)q^n.$$

For each $i$, let $t_i = [E_{f_i} : \mathbb{Q}]$ and $\{\sigma_{i,1},\ldots,\sigma_{i,t_i}\}$ be the embeddings of $E_{f_i}$ to $\bar{\mathbb{Q}}$. Let $\{\tau_{i,1},\ldots,\tau_{i,t_i}\}$ be a basis of $E_{f_i}$ over $\mathbb{Q}$. Therefore, $a_i(n)$ can be written as a linear combination of $\tau_{i,1},\ldots,\tau_{i,t_i}$, i.e.,

$$a_i(n) = a_{i,1}(n)\tau_{i,1} + \cdots + a_{i,t_i}(n)\tau_{i,t_i}$$

for $a_{i,j}(n) \in \mathbb{Q}$. From this, we see that

$$a_i(n)^\sigma = a_{i,1}(n)\tau_{i,1}^\sigma + \cdots + a_{i,t_i}(n)\tau_{i,t_i}^\sigma \tag{3.4}$$

for $\sigma \in G_\mathbb{Q}$. Then, we have the following lemma.

**Lemma 3.3.** *Let $t = t_1 + \cdots + t_m$. There are no finite subsets $B$ of $\mathbb{C}^t$ such that $(0, \ldots, 0) \notin B$ and for any primes $p$, we have*

$$\sum_{i=1}^{m} \sum_{j=1}^{t_i} A_{i,j} a_i(p)^{\sigma_{i,j}} = 0 \tag{3.5}$$

*for some $(A_{1,1}, \ldots, A_{1,t_1}, \ldots, A_{m,1}, \ldots, A_{m,t_m}) \in B$.*

**Proof.** Suppose that $B$ is a finite subset of $\mathbb{C}^t$ such that $(0, \ldots, 0) \notin B$ and for any primes $p$, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{t_i} A_{i,j} a_i(p)^{\sigma_{i,j}} = 0$$

for some $(A_{1,1}, \ldots, A_{1,t_1}, \ldots, A_{m,1}, \ldots, A_{m,t_m}) \in B$.

By (3.4), we see that the equation (3.5) is equivalent to

$$(A_{1,1}, \ldots, A_{1,t_1}, \ldots, A_{m,1}, \ldots, A_{m,t_m}) \begin{pmatrix} \tau_{1,1}^{\sigma_{1,1}} & \cdots & \tau_{1,t_1}^{\sigma_{1,1}} & & & & & \\ \vdots & \ddots & \vdots & & 0 & & 0 & \\ \tau_{1,1}^{\sigma_{1,t_1}} & \cdots & \tau_{1,t_1}^{\sigma_{1,t_1}} & & & & & \\ & & & \ddots & & & & \\ & 0 & & & \ddots & & 0 & \\ & & & & & \ddots & & \\ & & & & & \tau_{m,1}^{\sigma_{m,1}} & \cdots & \tau_{m,t_m}^{\sigma_{m,1}} \\ & 0 & & 0 & & \vdots & \ddots & \vdots \\ & & & & & \tau_{m,1}^{\sigma_{m,t_m}} & \cdots & \tau_{m,t_m}^{\sigma_{m,t_m}} \end{pmatrix}$$

$$\times \begin{pmatrix} a_{1,1}(p) \\ \vdots \\ a_{1,t_1}(p) \\ \vdots \\ a_{m,1}(p) \\ \vdots \\ a_{m,t_m}(p) \end{pmatrix} = 0.$$

Note that the matrix

$$\begin{pmatrix} \tau_{1,1}^{\sigma_{1,1}} & \cdots & \tau_{1,t_1}^{\sigma_{1,1}} & & & & & \\ \vdots & \ddots & \vdots & 0 & & 0 & & \\ \tau_{1,1}^{\sigma_{1,t_1}} & \cdots & \tau_{1,t_1}^{\sigma_{1,t_1}} & & & & & \\ & & & \ddots & & & & \\ & 0 & & & \ddots & 0 & & \\ & & & & & \ddots & & \\ & & & & & \tau_{m,1}^{\sigma_{m,1}} & \cdots & \tau_{m,t_m}^{\sigma_{m,1}} \\ & 0 & & 0 & & \vdots & \ddots & \vdots \\ & & & & & \tau_{m,1}^{\sigma_{m,t_m}} & \cdots & \tau_{m,t_m}^{\sigma_{m,t_m}} \end{pmatrix}$$

is invertible since the matrix

$$\begin{pmatrix} \tau_{i,1}^{\sigma_{i,1}} & \cdots & \tau_{i,t_i}^{\sigma_{i,1}} \\ \vdots & \ddots & \vdots \\ \tau_{i,1}^{\sigma_{i,t_i}} & \cdots & \tau_{i,t_i}^{\sigma_{i,t_i}} \end{pmatrix}$$

is invertible for each $i$. Therefore, if we let

$$\begin{aligned} &(A'_{1,1}, \ldots, A'_{1,t_1}, \ldots, A'_{m,1}, \ldots, A'_{m,t_m}) \\ &= (A_{1,1}, \ldots, A_{1,t_1}, \ldots, A_{m,1}, \ldots, A_{m,t_m}) \begin{pmatrix} \tau_{1,1}^{\sigma_{1,1}} & \cdots & \tau_{1,t_1}^{\sigma_{1,1}} & & & & & \\ \vdots & \ddots & \vdots & 0 & & 0 & & \\ \tau_{1,1}^{\sigma_{1,t_1}} & \cdots & \tau_{1,t_1}^{\sigma_{1,t_1}} & & & & & \\ & & & \ddots & & & & \\ & 0 & & & \ddots & 0 & & \\ & & & & & \ddots & & \\ & & & & & \tau_{m,1}^{\sigma_{m,1}} & \cdots & \tau_{m,t_m}^{\sigma_{m,1}} \\ & 0 & & 0 & & \vdots & \ddots & \vdots \\ & & & & & \tau_{m,1}^{\sigma_{m,t_m}} & \cdots & \tau_{m,t_m}^{\sigma_{m,t_m}} \end{pmatrix}, \end{aligned}$$

then $(A'_{1,1}, \ldots, A'_{1,t_1}, \ldots, A'_{m,1}, \ldots, A'_{m,t_m}) \neq (0, \ldots, 0)$ since $(A_{1,1}, \ldots, A_{1,t_1}, \ldots, A_{m,1}, \ldots, A_{m,t_m}) \neq (0, \ldots, 0)$. This means that the set

$$\{(a_{1,1}(p), \ldots, a_{1,t_1}(p), \ldots, a_{m,1}(p), \ldots, a_{m,t_m}(p)) \mid p \text{ is a prime}\}$$

is a subset of the set

$$C = \bigcup_{(A_{1,1}, \ldots, A_{1,t_1}, \ldots, A_{m,1}, \ldots, A_{m,t_m}) \in B}$$

$$\times \left\{ (x_{1,1}, \ldots, x_{1,t_1}, \ldots, x_{m,1}, \ldots, x_{m,t_m}) \in \mathbb{C}^t \;\middle|\; \sum_{i=1}^{m} \sum_{j=1}^{t_i} A'_{i,j} x_{i,j} = 0 \right\}, \quad (3.6)$$

which is a finite union of hyperplanes in $\mathbb{C}^t$.

As in the proof of Lemma 3.1, there is a prime $\ell$ such that the set

$$T = \{(a_{f_1}(p), \ldots, a_{f_n}(p)) \mid p \text{ is a prime with } p \neq \ell\}$$

is not contained in any finite union of hyperplanes in $(E_{f_1} \otimes \mathbb{C}) \times \cdots \times (E_{f_m} \otimes \mathbb{C})$. As in the proof of Lemma 3.2, we consider the isomorphism from $E_{f_1} \times \cdots \times E_{f_m}$ to $\mathbb{Q}^t$ defined by

$$(a_1, \ldots, a_m) \mapsto (a_{1,1}, \ldots, a_{1,t_1}, \ldots, a_{m,1}, \ldots, a_{m,t_m}),$$

where $a_{1,1}, \ldots, a_{1,t_1}, \ldots, a_{m,1}, \ldots, a_{m,t_m}$ are determined by the decomposition

$$a_i = a_{i,1}\tau_{i,1} + \cdots + a_{i,t_i}\tau_{i,t_i}$$

for each $i$. By this isomorphism, we see that

$$T' = \{(a_{1,1}(p), \ldots, a_{1,t_1}(p), \ldots, a_{m,1}(p), \ldots, a_{m,t_m}(p)) \mid p \text{ is a prime with } p \neq \ell\}$$

is not contained in any finite union of hyperplanes in $\mathbb{C}^t$. This is a contradiction since $T'$ is contained in $C$ by the assumption. $\square$

From Lemma 3.3, we can prove the following theorem.

**Theorem 3.4.** *Suppose that $f$ and $g$ are nonzero cusp forms on* $\mathrm{SL}_2(\mathbb{Z})$. *If $f$ is not a constant multiple of $g$, then $R(f, g)$ defined in (1.1) is not finite.*

**Proof.** Since $f$ and $g$ are cusp forms, each function can be written as a linear combination of finitely many normalized Hecke eigenforms $\{f_1, \ldots, f_m\}$, i.e.,

$$f = \sum_{i=1}^{m} a_i f_i, \;\; g = \sum_{i=1}^{m} b_i f_i$$

for $a_i, b_i \in \mathbb{C}$. For each $i$, we consider embeddings $\sigma_{i,1}, \ldots, \sigma_{i,t_i}$ from $E_{f_i}$ to $\bar{\mathbb{Q}}$, where $t_i = [E_{f_i} : \mathbb{Q}]$. Consider the set

$$A = \{f_i^{\sigma_{i,j}} \mid 1 \leq i \leq m, \; 1 \leq j \leq t_i\}.$$

We write

$$t = |A|, \;\; A = \{h_1, \ldots, h_t\}.$$

Since $A$ contains $\{f_1, \ldots, f_m\}$, both $f$ and $g$ can be written as linear combinations of elements of $A$, i.e.,

$$f = \sum_{i=1}^{t} \alpha_i h_i, \ g = \sum_{i=1}^{t} \beta_i h_i$$

for $\alpha_i, \beta_i \in \mathbb{C}$.

Suppose that $R(f,g)$ is finite and that $f$ is not a constant multiple of $g$. Let $a_i(n)$ be the $n$th Fourier coefficient of $h_i$. Then, there is a finite subset $B$ of $\mathbb{C}^t$ such that $(0, \ldots, 0) \notin B$ and for each prime $p$, we have

$$A_1 a_1(p) + \cdots + A_t a_t(p) = 0$$

for some $(A_1, \ldots, A_t) \in B$. This is a contradiction due to Lemma 3.3. $\square$

**Remark 3.5.** Suppose that $f$ and $g$ are cusp forms. They may be zero. Then, Theorem 3.4 implies that if $R(f,g)$ is finite, then there are constants $\alpha$ and $\beta$ such that $\alpha f = \beta g$.

## 4. Proof of the main result

In this section, we prove Theorem 1.1. Suppose that $f$ is not a cusp form and $g$ is a cusp form. For a weakly holomorphic modular form $h \in M^!$, let $m_h = |\mathrm{ord}_\infty(h)|$ and $k_h$ be the weight of $h$. Since $g$ is a cusp form, the Fourier coefficients of $g$ should satisfy the Hecke bound

$$a_g(n) = O(n^{k_g/2})$$

as $n \to \infty$. Then, by Theorem 2.1, we have

$$\lim_{n\to\infty} \left| \frac{a_f(n)}{a_g(n)} \right| = \infty,$$

which means that the set $R(f,g)$ cannot be finite. This is a contradiction. Therefore, both $f$ and $g$ are cusp forms or none of them are cusp forms. If both $f$ and $g$ are cusp forms, then by Theorem 3.4, $f = cg$ for some constant $c$.

Suppose that neither $f$ nor $g$ is a cusp form. If $m_f \neq m_g$, then we may assume that $m_f > m_g$. By Theorem 2.1, we have

$$\lim_{n\to\infty} \left| \frac{a_f(n)}{a_g(n)} \right| = \infty,$$

which means that $R(f,g)$ is an infinite set. This is a contradiction. Therefore, $m_f = m_g$. In the same way, we see that $k_f = k_g$. By multiplying a nonzero constant to $g$, we may assume that

$$a_f(-m_f) = a_g(-m_g). \tag{4.1}$$

Therefore, it is enough to show that $f = g$.

Suppose that $[\alpha : \beta] \in R(f, g)$ satisfying

$$\beta a_f(p) = \alpha a_g(p) \tag{4.2}$$

for infinitely many primes $p$. Such $[\alpha : \beta]$ exists since $R(f, g)$ is finite. By the asymptotic expansion given in Theorem 2.1, we see that both $f$ and $g$ have only finitely many zero coefficients. Therefore, both $\alpha$ and $\beta$ are nonzero.

Then, we have a strictly increasing sequence $\{p_i\}$ of primes satisfying (4.2). By Theorem 2.1 and (4.2), we have

$$\frac{\alpha}{\beta} = \lim_{i\to\infty} \frac{a_f(p_i)}{a_g(p_i)} = \lim_{i\to\infty} \frac{a_f(-m_f)e^{4\pi\sqrt{p_i m_f}}}{a_g(-m_g)e^{4\pi\sqrt{p_i m_g}}} = 1 \tag{4.3}$$

since $m_f = m_g$ and $k_f = k_g$. Therefore, we see that $[\alpha : \beta] = [1 : 1] \in R(f, g)$.

This implies that if $[\alpha : \beta] \in R(f, g)$ and $[\alpha : \beta] \neq [1 : 1]$, then the number of primes satisfying (4.2) is finite. Since $R(f, g)$ is finite, we see that

$$a_f(p) = a_g(p)$$

for all but finitely many primes $p$.

Now, we prove that $f - g$ is a cusp form. Suppose that $f - g$ is not a cusp form. This means that the principal parts of $f$ and $g$ are not the same. Let

$$m_0 = \max\{n \geq 0 \mid a_f(-n) \neq a_g(-n)\}.$$

Then, by (2.1), we see that $-m_0 \leq o_k$. Let

$$\hat{f} = \sum_{m>m_0} a_f(-m) f_{k_f,m}, \;\; \tilde{f} = f - \hat{f},$$

$$\hat{g} = \sum_{m>m_0} a_g(-m) f_{k_f,m}, \;\; \tilde{g} = g - \hat{g},$$

where $f_{k_f,m}$ is a weakly holomorphic modular form as in (2.2). We denote by $a_{\hat{f}}(n)$ (resp. $a_{\tilde{f}}(n), a_{\hat{g}}(n)$, and $a_{\tilde{g}}(n)$) the $n$th Fourier coefficient of $\hat{f}$ (resp. $\tilde{f}, \hat{g}$, and $\tilde{g}$). By the definition of $m_0$, we see that $\hat{f} = \hat{g}$. Then, for all but finitely many primes $p$, we have

$$a_{\tilde{f}}(p) = a_{\tilde{g}}(p). \tag{4.4}$$

This implies that $R(\tilde{f}, \tilde{g})$ is finite.

If $m_0 > 0$, then at least one of $\tilde{f}$ and $\tilde{g}$ is not a cusp form. Since $R(\tilde{f}, \tilde{g})$ is finite, in the same way as above, we see that $m_{\tilde{f}} = m_{\tilde{g}} = m_0$. Note that we have a strictly increasing sequence $\{p_i\}$ of primes satisfying (4.4). Then, by the same argument as in (4.3), we have

$$1 = \lim_{i\to\infty} \frac{a_{\tilde{f}}(p_i)}{a_{\tilde{g}}(p_i)} = \lim_{i\to\infty} \frac{a_{\tilde{f}}(-m_0)e^{4\pi\sqrt{p_i m_0}}}{a_{\tilde{g}}(-m_0)e^{4\pi\sqrt{p_i m_0}}} = \frac{a_{\tilde{f}}(-m_0)}{a_{\tilde{g}}(-m_0)}.$$

Therefore, we obtain $a_{\tilde{f}}(-m_0) = a_{\tilde{g}}(-m_0)$ which implies that $a_f(-m_0) = a_g(-m_0)$. This is a contradiction due to the definition of $m_0$.

If $m_0 = 0$, then both $\tilde{f}$ and $\tilde{g}$ are holomorphic modular forms and $a_{\tilde{f}}(0) \neq a_{\tilde{g}}(0)$. Then, we see that

$$\tilde{f} = a_{\tilde{f}}(0) f_{k_f,0} + f_c,$$
$$\tilde{g} = a_{\tilde{g}}(0) f_{k_f,0} + g_c$$

for some cusp forms $f_c, g_c \in S_{k_f}$. By the Hecke bound, cusp forms $f_c(z) = \sum_{n>0} a_{f_c}(n)q^n$ and $g_c(z) = \sum_{n>0} a_{g_c}(n)q^n$ satisfy

$$a_{f_c}(n), a_{g_c}(n) = O(n^{k_f/2}). \tag{4.5}$$

By (4.4), we have a strictly increasing sequence $\{p_i\}$ of primes satisfying

$$(a_{\tilde{f}}(0) - a_{\tilde{g}}(0))a_{k_f,0}(p_i) = a_{g_c}(p_i) - a_{f_c}(p_i),$$

where $a_{k_f,0}(n)$ is the $n$th Fourier coefficient of $f_{k_f,0}$. This is a contradiction since

$$\lim_{i\to\infty} \left| \frac{(a_{\tilde{f}}(0) - a_{\tilde{g}}(0))a_{k_f,0}(p_i)}{a_{g_c}(p_i) - a_{f_c}(p_i)} \right| = \infty$$

by Theorem 2.1 and (4.5). In conclusion, $f - g$ should be a cusp form.

If $S_{k_f} = \{0\}$, the proof is completed. Otherwise, we consider

$$F = f - \sum_{m\geq 0} a_f(-m) f_{k_f,m},$$
$$G = g - \sum_{m\geq 0} a_g(-m) f_{k_f,m}.$$

Both $F$ and $G$ are cusp forms and

$$a_F(p) = a_G(p) \tag{4.6}$$

for all but finitely many primes $p$, where $a_F(n)$ (resp. $a_G(n)$) denotes the $n$th Fourier coefficient of $F$ (resp. $G$). Therefore, $R(F, G)$ is finite. By Theorem 3.4, there are constants $\alpha', \beta'$ such that $\beta' F = \alpha' G$. By (4.6), we see that $F = G$, and this implies that $f = g$ since $a_f(-m) = a_g(-m)$ for all $m \geq 0$. This completes the proof.

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