Journal of Number Theory 202 (2019) 278–297

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

General Section

On a sum involving the Euler function

Olivier Bordellès [a], Lixia Dai [b], Randell Heyman [c,*], Hao Pan [d], Igor E. Shparlinski [e]

[a] *2 Allée de la combe, 43000 Aiguilhe, France*
[b] *School of Mathematical Sciences, Nanjing Normal University, Nanjing 210046, People's Republic of China*
[c] *Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia*
[d] *School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210046, People's Republic of China*
[e] *Department of Pure Mathematics, University of New South Wales, 2052 NSW, Australia*

A R T I C L E   I N F O

Article history:
Received 9 August 2018
Received in revised form 8 January 2019
Accepted 8 January 2019
Available online 19 February 2019
Communicated by S.J. Miller

Dedicated to the bright memory of Jean Bourgain

MSC:
11A25
11L07
11N37

Keywords:
Euler function
Integer part

A B S T R A C T

We obtain reasonably tight upper and lower bounds on the sum $\sum_{n \leqslant x} \varphi\left(\lfloor x/n \rfloor\right)$, involving the Euler functions φ and the integer parts $\lfloor x/n \rfloor$ of the reciprocals of integers. For slower growing arithmetic functions f we obtain asymptotic formulas for similar sums of $f\left(\lfloor x/n \rfloor\right)$. These are analogues of a series of previous results for sequences involving the integer part functions such Beatty and Piatetski–Shapiro sequences.

* Corresponding author.
E-mail addresses: borde43@wanadoo.fr (O. Bordellès), lilidainjnu@163.com (L. Dai), randell@unsw.edu.au (R. Heyman), haopan79@zoho.com (H. Pan), igor.shparlinski@unsw.edu.au (I.E. Shparlinski).

https://doi.org/10.1016/j.jnt.2019.01.006
0022-314X/

Reciprocals
Exponent pair

1. Background and motivation

Let, as usual, for an integer $n \geqslant 1$, $\varphi(n)$ denote the Euler function, that is, the number of units in the residue ring $\mathbb{Z}/n\mathbb{Z}$.

By a classical result of Walfisz [22], we have the following asymptotic formula for the summary function of the Euler function

$$\sum_{n\leqslant x} \varphi(n) = \frac{x^2}{2\zeta(2)} + O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right),$$

see also [8, Theorem 6.44].

Furthermore, for any real number x we denote by $\lfloor x \rfloor$ its integer part, that is, the greatest integer that does not exceed x. The most straightforward sum of the floor function is related to the divisor summatory function since

$$\sum_{n\leqslant x} \lfloor x/n \rfloor = \sum_{n\leqslant x}\sum_{k\leqslant x/n} 1 = \sum_{n\leqslant x} \tau(n),$$

where $\tau(n)$ is the number of divisors of n. From [11, Theorem 2] we infer

$$\sum_{n\leqslant x} \lfloor x/n \rfloor = x\log x + x(2\gamma - 1) + O\left(x^{517/1648+o(1)}\right),$$

where γ is the *Euler–Mascheroni* constant, in particular $\gamma \approx 0.57\,722$.

Here we combine both functions and consider an apparently new type of sums, namely,

$$S(x) = \sum_{n\leqslant x} \varphi\left(\lfloor x/n \rfloor\right).$$

The sum $S(x)$ is also a mean value of a certain divisor function, as it may be seen by interchanging the summations. More precisely, if τ_x is the divisor function defined by

$$\tau_x(n) = \sum_{\substack{d\mid n \\ \gcd(d,\lfloor dx/n\rfloor)=1}} 1$$

then

$$S(x) = \sum_{n\leqslant x} \tau_x(n).$$

Note that, for each fixed real number $x \geqslant 1$, the arithmetic function τ_x is not multiplicative, which explains why an asymptotic formula for $S(x)$ is quite difficult to obtain.

However, the aim of this work is to obtain reasonably tight upper and lower bounds for this sum.

We also consider more general sums of arithmetic functions with $\lfloor x/n \rfloor$, and in the case of functions growing slower than the Euler function we obtain asymptotic formulas for such sums.

We remark our work is partially motivated by the extensive body of research on arithmetic functions with integer parts of real-valued functions, most commonly, with Beatty $\lfloor \alpha n + \beta \rfloor$ sequences, see, for example, [1,3,6,12,13], and Piatetski–Shapiro $\lfloor n^\gamma \rfloor$ sequences, see, for example, [2,4,5,7,14,15], with real α, β and γ. In particular, we obtain an analogue of the result of Morgenbesser [15] on the sum of digits of $\lfloor n^c \rfloor$ for the sequence $\lfloor x/n \rfloor$, see Example 3.4 below.

2. Main results

2.1. The Euler function

We start with upper and lower bounds on $S(x)$.

Theorem 2.1. *Uniformly, for all $x \geqslant 3$,*

$$\left(\frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + \frac{1380}{4009} + o(1)\right) x \log x$$
$$\geqslant S(x) \geqslant \left(\frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + o(1)\right) x \log x,$$

as $x \to \infty$.

The proofs of both lower and upper bounds of Theorem 2.1 rely on the theory of exponent pairs, see [8, Chapter 6]. In particular, to obtain the numerically strongest result, we use the recently discovered exponent pair of Bourgain [10] combined with so called A- and B-processes, see [8, Sections 6.4.2 and 6.6.2]. We remark that in the lower bound of Theorem 2.1 the quantity in $o(1)$ is negative.

We note it is natural to ask the following:

Question 2.2. *Is it true that*

$$S(x) = \left(\frac{1}{\zeta(2)} + o(1)\right) x \log x$$

as $x \to \infty$?

In Section 7 we present some numerical data which makes us rather cautiously believe that the answer to Question 2.2 is positive.

2.2. Slowly growing arithmetic functions

One of the difficulties in investigating the sum $S(x)$ is a large size of $\varphi(n)$. In particular, some individual terms of the sum $S(x)$ are only logarithmically smaller than the entire sum. However, for slowly growing arithmetic functions $f(n)$ in similar sums,

$$S_f(x) = \sum_{n \leqslant x} f\left(\lfloor x/n \rfloor\right),$$

we are able to get an asymptotic formula.

Let $\tau_k(n)$ denote the generalised divisor function, which is defined as the number of ordered representations $n = d_1 \dots d_k$ with integer numbers $d_1, \dots, d_k \geqslant 1$. In particular $\tau_1(n) = 1$.

We also define $\varepsilon_1(x) = 0$ and

$$\varepsilon_k(x) = \sqrt{\frac{k \log\log\log x}{\log\log x}} \left(k - 1 + \frac{30}{\log\log\log x}\right), \tag{2.1}$$

for $k \geqslant 2$. Now we have the obvious estimate $\varepsilon_k(x) = o(1)$ as $x \to \infty$.

We write O_k to indicate that in the relations $U = O_k(V)$ the implied constant may depend on k. We also write $U \ll V$ and $U \ll_k V$ is equivalents of $U = O(V)$ and $U = O_k(V)$, respectively.

Finally, we use $\mathbb{Z}_{\geqslant k}$ to denote the set

$$\mathbb{Z}_{\geqslant k} = \mathbb{Z} \cap [k, \infty).$$

Theorem 2.3. *Let f be a complex-valued arithmetic function such that there exist $A > 0$ and $k \in \mathbb{Z}_{\geqslant 1}$ such that $|f| \leqslant A\tau_k$. Then*

$$\sum_{n \leqslant x} f\left(\lfloor x/n \rfloor\right) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O_k\left(Ax^{1/2} (\log x)^{\delta_k + \varepsilon_{k+1}(x)/2}\right),$$

where $\delta_1 = 0$ if $k = 1$, $\delta_k = k - 1/2$ if $k \geqslant 2$, and where $\varepsilon_{k+1}(x)$ is defined in (2.1).

In particular, applying Theorem 2.3 to $f(n) = \varphi(n)/n$ (and using $k = 1$) we obtain:

Corollary 2.4. *We have*

$$\sum_{n \leqslant x} \frac{\varphi\left(\lfloor x/n \rfloor\right)}{\lfloor x/n \rfloor} = \kappa x + O\left(x^{1/2}\right),$$

where

$$\kappa = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2(n+1)} \approx 0.78\,838.$$

Finally, the method of proof of Theorem 2.3 can be extended to more general and faster growing arithmetic functions at the cost of a weaker error term.

We use

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61\,803$$

to denote the Golden ratio.

Theorem 2.5. *Let f be a complex-valued arithmetic function and assume that there exists $A > 0$ such that*

$$|f(n)| \ll n^{\phi-1} (\log en)^{-A}.$$

Then

$$\sum_{n \leqslant x} f\left(\lfloor x/n \rfloor\right) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O\left(x (\log x)^{-A(\phi-1)}\right).$$

We also have a result which depends on the average behaviour of arithmetic functions, which is very useful for functions with irregular behaviour. We give several examples of such functions in Section 3.

Theorem 2.6. *Let f be a complex-valued arithmetic function and assume that there exists $0 < \alpha < 2$ such that*

$$\sum_{n \leqslant x} |f(n)|^2 \ll x^{\alpha}. \tag{2.2}$$

Then

$$\sum_{n \leqslant x} f\left(\lfloor x/n \rfloor\right) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O\left(x^{(\alpha+1)/3} (\log x)^{(1+\alpha)(2+\varepsilon_2(x))/6}\right)$$

where $\varepsilon_2(x)$ is given in (2.1).

In particular, if in Theorem 2.1 one replaces the sum $S(x)$ with $\varphi(n)$ with a similar sum with $\varphi(n)^{\beta}$ for some $\beta < 1/2$, then Theorem 2.6 immediately applies and implies an asymptotic formula.

3. Some applications

Here we give some examples of interesting arithmetic functions to which we can apply our results.

Example 3.1. Let $f(n) = \sqrt{3}^{\,\Omega(n)}$. From [18, Chapter I.3, Exercise 58(f)], we have

$$\sum_{n \leqslant x} f(n)^2 \ll x^{\log 3/\log 2}$$

(indeed one verifies that the functions $f(\vartheta)$ and $g(\vartheta)$ involved in the asymptotic formulas of [18, Chapter I.3, Exercise 58(f)] are both monotonically decreasing). Hence, Theorem 2.6 gives

$$\sum_{n \leqslant x} \sqrt{3}^{\,\Omega(\lfloor x/n \rfloor)} = x \sum_{n=1}^{\infty} \frac{\sqrt{3}^{\,\Omega(n)}}{n(n+1)} + O\left(x^{\log 6/\log 8} (\log x)^{\log 6/\log 8 + o(1)}\right).$$

Note that

$$\sum_{n=1}^{\infty} \frac{\sqrt{3}^{\,\Omega(n)}}{n(n+1)} \approx 1.77\,694.$$

Clearly in Example 3.1 one can take $f(n) = \lambda^{\Omega(n)}$ with any $\lambda \leqslant \sqrt{3}$ and still have an asymptotic formula. We now show that one can also take a slightly larger values of λ.

Example 3.2. Let $f(n) = \lambda^{\Omega(n)}$ with $\sqrt{3} \leqslant \lambda < 2$. Combining the trivial bound

$$\Omega(n) \leqslant \frac{\log n}{\log 2}$$

with [18, Chapter I.3, Exercise 58(f)], we derive

$$\begin{aligned} \sum_{n \leqslant x} f(n)^2 &= \sum_{n \leqslant x} 3^{\Omega(n)} (\lambda^2/3)^{\Omega(n)} \\ &\leqslant x^{\log(\lambda^2/3)/\log 2} \sum_{n \leqslant x} 3^{\Omega(n)} \ll x^{2\log \lambda/\log 2}. \end{aligned}$$

Hence by Theorem 2.6, for any positive $\lambda < 2$ there exists some $\kappa > 0$ such that

$$\sum_{n \leqslant x} \lambda^{\Omega(\lfloor x/n \rfloor)} = x \sum_{n=1}^{\infty} \frac{\lambda^{\Omega(n)}}{n(n+1)} + O\left(x^{1-\kappa}\right).$$

We now give an application of Theorem 2.6 to a very different function.

Example 3.3. Let $k \in \mathbb{Z}_{\geqslant 2}$ and define $M_k(n)$ to be the maximal k-full divisor of n (see [17]). Since

$$L\left(s, M_k^2\right) = \zeta(s) \prod_p \left(1 + \frac{p^{2k}-1}{p^{ks}}\right) \qquad \left(\sigma > 2 + \tfrac{1}{k}\right)$$

we infer that

$$\sum_{n \leqslant x} M_k(n)^2 \leqslant x^{2+1/k+o_k(1)},$$

where $o_k(1)$ denotes a quantity which for a fixed k tends to zero as $x \to \infty$. Now let $f_k(n) = n^{-1/k} M_k(n)$. By partial summation, we obtain from the above estimate

$$\sum_{n \leqslant x} f_k(n)^2 \leqslant x^{2-1/k+o_k(1)}.$$

Applying Theorem 2.6 we derive

$$\sum_{n \leqslant x} \lfloor x/n \rfloor^{-1/k} M_k\left(\lfloor x/n \rfloor\right) = x \sum_{n=1}^{\infty} \frac{M_k(n)}{n^{1+1/k}(n+1)} + O\left(x^{1-1/(3k)+o_k(1)}\right).$$

Furthermore, if for an integer $q \geq 2$ we use $\sigma_q(n)$ to denote the sum of q-ary digits of n, then we see that Theorem 2.6 immediately implies:

Example 3.4. For any integer $q \geq 2$, we have

$$\sum_{n \leqslant x} \sigma_q\left(\lfloor x/n \rfloor\right) = x \sum_{n=1}^{\infty} \frac{\sigma_q(n)}{n(n+1)} + O\left(x^{2/3+o(1)}\right).$$

4. Proof of Theorem 2.1

4.1. Preliminaries

4.1.1. Vaaler polynomials

For a real $z \in \mathbb{R}$ we denote

$$\psi(z) = z - \lfloor z \rfloor - \frac{1}{2} \qquad \text{and} \qquad \mathbf{e}(z) = \exp(2\pi i z). \tag{4.1}$$

We need a result of Vaaler [21], approximating $\psi(z)$ via trigonometric polynomials which we present in the form given by [8, Theorem 6.1]. For this, for any $0 < |t| < 1$ we put $\Phi(t) = \pi t(1 - |t|)\cot(\pi t) + |t|$. Note that $0 < \Phi(t) < 1$ for $0 < |t| < 1$.

Lemma 4.1. *For any real number $x \geqslant 1$ and any positive integer H,*

$$\psi(z) = -\sum_{0<|h|\leqslant H} \Phi\left(\frac{h}{H+1}\right)\frac{\mathbf{e}(hz)}{2\pi ih} + \mathcal{R}_H(z),$$

where the error term $\mathcal{R}_H(z)$ satisfies

$$|\mathcal{R}_H(z)| \leqslant \frac{1}{2H+2}\sum_{|h|\leqslant H}\left(1-\frac{|h|}{H+1}\right)\mathbf{e}(hz).$$

4.1.2. Initial transformation

Let x be sufficiently large and J be any real number satisfying $x^{1/2} < J \leqslant x$. Clearly

$$\begin{aligned} S(x) &= \sum_{n\leqslant x/J}\varphi\left(\lfloor x/n\rfloor\right) + \sum_{x/J<n\leqslant x}\varphi\left(\lfloor x/n\rfloor\right) \\ &\leqslant \sum_{n\leqslant x/J}\varphi\left(\lfloor x/n\rfloor\right) + \sum_{n\leqslant J}\varphi(n)\left(\left\lfloor\frac{x}{n}\right\rfloor - \left\lfloor\frac{x}{n+1}\right\rfloor\right) \\ &= \sum_{n\leqslant x/J}\varphi\left(\lfloor x/n\rfloor\right) + x\sum_{n\leqslant J}\frac{\varphi(n)}{n(n+1)} \\ &\qquad + \sum_{n\leqslant J}\varphi(n)\left(\psi\left(\frac{x}{n+1}\right) - \psi\left(\frac{x}{n}\right)\right), \end{aligned}$$

where $\psi(z)$ is given by (4.1) Now, splitting the last sum into two ranges $n \leqslant x^{1/2}$ and $x^{1/2} < n \leqslant J$ we obtain

$$S(x) = S_0(x) + S_1(x) + S_2(x) + S_3(x), \tag{4.2}$$

where

$$\begin{aligned} S_0(x) &= \sum_{n\leqslant x/J}\varphi\left(\lfloor x/n\rfloor\right), \\ S_1(x) &= x\sum_{n\leqslant J}\frac{\varphi(n)}{n(n+1)}, \\ S_2(x) &= \sum_{n\leqslant x^{1/2}}\varphi(n)\left(\psi\left(\frac{x}{n+1}\right) - \psi\left(\frac{x}{n}\right)\right), \\ S_3(x) &= \sum_{x^{1/2}<n\leqslant J}\varphi(n)\left(\psi\left(\frac{x}{n+1}\right) - \psi\left(\frac{x}{n}\right)\right). \end{aligned}$$

4.2. The lower bound

4.2.1. Exponential sums twisted by the Euler totient

We refer to [8, Sections 6.6.3] for the definition and basic properties of exponent pairs.

Lemma 4.2. *Let $K, N \in \mathbb{Z}_{\geqslant 1}$ and $x > 0$ such that $N < K \leqslant 2N$ and $N \leqslant x$. If (k, ℓ) is an exponent pair, then*

$$\sum_{N<n\leqslant K} \varphi(n)\mathbf{e}\left(\frac{x}{n}\right) \ll x^k N^{1+\ell-2k} \log N + N^3 x^{-1} + N.$$

Furthermore, if $(k, \ell) \neq \left(\frac{1}{2}, \frac{1}{2}\right)$, then the factor $\log N$ may be omitted.

Proof. For any arithmetic functions f and g, $f \star g$ is the usual Dirichlet convolution product of f and g, defined as

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

Using $\varphi = \mu \star \mathrm{Id}$, see [8, Equation (4.7)], we obtain

$$\begin{aligned}\sum_{N<n\leqslant K} \varphi(n)\mathbf{e}\left(\frac{x}{n}\right) &= \sum_{N<n\leqslant K} (\mu \star \mathrm{Id})(n)\mathbf{e}\left(\frac{x}{n}\right) \\ &= \sum_{n\leqslant K} \mu(n) \sum_{N/n<m\leqslant K/n} m\mathbf{e}\left(\frac{x}{mn}\right) \\ &= \sum_{n\leqslant N} \mu(n) \sum_{N/n<m\leqslant K/n} m\mathbf{e}\left(\frac{x}{mn}\right) + O(N).\end{aligned} \tag{4.3}$$

For all $L, M \in \mathbb{Z}_{\geqslant 1}$ such that $M < L \leqslant 2M$, if (k, ℓ) is an exponent pair, then by Abel summation

$$\begin{aligned}\sum_{M<m\leqslant L} m\mathbf{e}\left(\frac{x}{mn}\right) &\ll M \max_{M\leqslant n\leqslant L} \left| \sum_{M\leqslant m\leqslant n} \mathbf{e}\left(\frac{x}{mn}\right)\right| \\ &\ll M\left\{\left(\frac{x}{Mn}\right)^k M^{\ell-k} + \frac{M^2 n}{x}\right\} \\ &\ll \left(\frac{x}{n}\right)^k M^{1+\ell-2k} + \frac{M^3 n}{x}.\end{aligned} \tag{4.4}$$

Inserting (4.4) with

$$M = \frac{N}{n} \qquad \text{and} \qquad L = \frac{K}{n}$$

in (4.3), we obtain

$$\sum_{N<n\leqslant K} \varphi(n)\mathbf{e}\left(\frac{x}{n}\right) \ll x^k N^{1+\ell-2k} \sum_{n\leqslant N} \frac{1}{n^{1+\ell-k}} + \frac{N^3}{x} \sum_{n\leqslant N} \frac{1}{n^2} + N$$
$$\ll x^k N^{1+\ell-2k} (\log N)^{\alpha} + \frac{N^3}{x} + N,$$

where $\alpha = 1$ if $(k,\ell) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\alpha = 0$ otherwise, giving the asserted result. □

4.2.2. Exponent pairs and a lower bound on $S(x)$

The desired lower bound on $S(x)$ is a particular case of the following more general result, which may have its own interest.

Lemma 4.3. *Let $x \geqslant e$ be sufficiently large and let J be any real number satisfying $x^{1/2} < J \leqslant x$ and (k,ℓ) be an exponent pair. Then*

$$S(x) \geqslant \frac{x \log J}{\zeta(2)} + O\left(\mathfrak{J}(\log J)^2 + x\right),$$

where

$$\mathfrak{J} = \left(J^{\ell+1} x^{k+1}\right)^{1/(k+2)} + \left(J^{2(\ell+1)} x^k\right)^{1/(k+2)}$$
$$+ \left(J^{3k-\ell+5} x^{-k-1}\right)^{1/(k+2)} + J^3/x.$$

Proof. Recalling (4.2) and using that $S_0(x) \geq 0$ we write

$$S(x) \geq S_1(x) + S_2(x) + S_3(x).$$

Now

$$S_1(x) = x \sum_{n\leqslant J} \frac{\varphi(n)}{n^2} + O(x) = \frac{x \log J}{\zeta(2)} + O(x)$$

and obviously

$$S_2(x) \ll x.$$

It remains to estimate S_3. Covering the interval $[x^{1/2}, J]$ by $L \ll \log J$ dyadic intervals of the form $[N, 2N]$, we have

$$S_3(x) \leqslant \left| \sum_{x^{1/2} < n \leqslant J} \varphi(n) \psi\left(\frac{x}{n+1}\right) \right| + \left| \sum_{x^{1/2} < n \leqslant J} \varphi(n) \psi\left(\frac{x}{n}\right) \right|$$
$$\ll L \max_{\vartheta \in \{0,1\}} \max_{x^{1/2} < N \leqslant J} \left| \sum_{N < n \leqslant 2N} \varphi(n) \psi\left(\frac{x}{n+\vartheta}\right) \right|.$$

Now, by Lemma 4.1, for any integer $H \geqslant 1$,

$$S_3(x) \ll L \max_{\vartheta \in \{0,1\}} \max_{x^{1/2} < N \leqslant J} \left(\frac{N^2}{H} + \sum_{h \leqslant H} \frac{1}{h} \left| \sum_{N < n \leqslant 2N} \varphi(n) \mathbf{e}\left(\frac{hx}{n+\vartheta}\right) \right| \right).$$

Note that the function

$$n \longmapsto \frac{hx}{n(n+1)}$$

is non-increasing and bounded by HxN^{-2} so that, by partial summation,

$$\sum_{N < n \leqslant 2N} \varphi(n) \mathbf{e}\left(\frac{hx}{n+1}\right) = \sum_{N < n \leqslant 2N} \varphi(n) \mathbf{e}\left(\frac{hx}{n}\right) \mathbf{e}\left(-\frac{hx}{n(n+1)}\right)$$
$$\ll \left(1 + \frac{Hx}{N^2}\right) \max_{N < K \leqslant 2N} \left| \sum_{N < n \leqslant K} \varphi(n) \mathbf{e}\left(\frac{hx}{n}\right) \right|$$

and therefore

$$S_3(x) \ll L \max_{x^{1/2} < N \leqslant J} \left(\frac{N^2}{H} + \left(1 + \frac{Hx}{N^2}\right) W \right), \tag{4.5}$$

where

$$W = \sum_{h \leqslant H} \frac{1}{h} \max_{N < K \leqslant 2N} \left| \sum_{N < n \leqslant K} \varphi(n) \mathbf{e}\left(\frac{hx}{n}\right) \right|.$$

The estimate of Lemma 4.2 yields

$$W \ll (Hx)^k N^{1+\ell-2k} \log N + \frac{N^3}{x} + N \log H$$
$$\ll (Hx)^k N^{1+\ell-2k} \log N + \frac{N^3}{x} \log H,$$

where we have used the fact that $N > x^{1/2}$ implies that the term N is absorbed by $N^3 x^{-1}$.

Inserting this estimate in (4.5), we derive

$$\begin{aligned} S_3(x) \ll L \max_{x^{1/2}<N\leqslant J} \Bigg(\frac{N^2}{H} \\ + \left(1+\frac{Hx}{N^2}\right)\left((Hx)^k N^{1+\ell-2k}\log N + \frac{N^3}{x}\log H\right)\Bigg) \\ \ll L \max_{x^{1/2}<N\leqslant J} \Bigg(\frac{N^2}{H} + (Hx)^k N^{1+\ell-2k}\log N + \frac{N^3}{x}\log H \\ + (Hx)^{k+1}N^{-1+\ell-2k}\log N + NH\log H \Bigg). \end{aligned}$$

Now equalizing the first and the fourth term, we choose

$$H = \left\lfloor \left(N^{2k-\ell+3}x^{-k-1}\right)^{1/(k+2)} \right\rfloor.$$

Note that the condition $N > x^{1/2}$ ensures that $H \geqslant 1$. We then obtain

$$\begin{aligned} S_3(x) \ll L \max_{x^{1/2}<N\leqslant J} \Bigg(\left(N^{\ell+1}x^{k+1}\right)^{\frac{1}{k+2}}\log N + \left(N^{2(\ell+1)}x^k\right)^{\frac{1}{k+2}}\log N \\ + \frac{N^3}{x}\log H + \left(N^{3k-\ell+5}x^{-(k+1)}\right)^{\frac{1}{k+2}}\log H \Bigg) \\ \ll \mathfrak{J}(\log J)^2, \end{aligned}$$

where we have used that $\log H \ll \log N \ll \log J$, concluding the proof. □

4.2.3. Concluding the proof of the lower bound

The lower bound of Theorem 2.1 follows from Lemma 4.3 at once by using the exponent pair of Bourgain [10, Theorem 6], coupled with several applications of van der Corput's A- and B-processes, see [8, Sections 6.4.2 and 6.6.2]:

$$(k,\ell) = BA^3\left(BA^2\right)^2\left(\frac{13}{84}+\varepsilon, \frac{55}{84}+\varepsilon\right) = \left(\frac{3071}{7887}+\varepsilon, \frac{1380}{2629}+\varepsilon\right)$$

and choosing $J = x^{2629/4009-\varepsilon}$.

4.3. The upper bound

4.3.1. Some explicit estimates

The following estimate is a well-known result. For a proof, see [16, Lemma 2.1].

Lemma 4.4. *For all $x \geqslant 1$*

$$\left| \sum_{n \leqslant x} \frac{1}{n} - \log x - \gamma \right| \leqslant \frac{6}{11x}.$$

We also need some bounds on sums involving the Euler function, which follows from [20, Lemma 2.2] and partial summation.

Lemma 4.5. *For all $x \geqslant 1$ we have*

$$\sum_{n \leqslant x} \frac{\varphi(n)}{n^2} \leqslant \frac{\log x}{\zeta(2)} + 2 + \frac{1}{\zeta(2)}.$$

4.3.2. Concluding the proof

We recall the representation of $S(x)$ as (4.2).

First, using Lemma 4.4

$$S_0(x) \leqslant x \sum_{n \leqslant x/J} \frac{1}{n} \leqslant x \log(x/J) + O(x). \tag{4.6}$$

Next, using Lemma 4.5 we derive

$$S_1(x) \leqslant x \sum_{n \leqslant J} \frac{\varphi(n)}{n(n+1)} \leqslant x \sum_{n \leqslant J} \frac{\varphi(n)}{n^2} = x \frac{\log J}{\zeta(2)} + O(x). \tag{4.7}$$

Furthermore, we obviously have

$$S_2(x) \leqslant \sum_{n \leq x^{1/2}} \varphi(n) \leqslant \sum_{n \leq x^{1/2}} n \ll x. \tag{4.8}$$

Now, choosing $J = x^{2629/4009-\varepsilon}$ for a sufficiently small $\varepsilon > 0$ as in Section 4.2.3, we see that

$$S_3(x) = o(x). \tag{4.9}$$

Now substituting the bound (4.6), (4.7), (4.8) and (4.9) in (4.2) implies the asserted upper bound Theorem 2.1.

5. Proof of Theorem 2.3

5.1. Initial transformation

Let $T \in (2\sqrt{x}, x]$ be a parameter at our disposal. Then

$$\begin{aligned} S_f(x) &= \sum_{n \leqslant x} f(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \\ &= \sum_{n \leqslant T} f(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \\ &\qquad + \sum_{T < n \leqslant x} f(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \\ &= \sum_{n \leqslant T} f(n) \left(\frac{x}{n(n+1)} + O(1) \right) \\ &\qquad + O\left(\sum_{T < n \leqslant x} |f(n)| \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \right). \end{aligned}$$

Hence

$$S_f(x) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O\left(R_1(x) + R_2(x) + R_3(x)\right), \tag{5.1}$$

where

$$\begin{aligned} R_1(x) &= x \sum_{n > T}^{\infty} \frac{|f(n)|}{n(n+1)}, \\ R_2(x) &= \sum_{n \leqslant T} |f(n)|, \\ R_3(x) &= \sum_{T < n \leqslant x} |f(n)| \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right). \end{aligned}$$

5.2. Bounding error terms

Using that

$$\sum_{n \leqslant T} |f(n)| \leqslant A \sum_{n \leqslant T} \tau_k(n) \ll_k AT(\log T)^{k-1},$$

see [8, Section 4.8, Exercise 13], and by partial summation we obtain

$$R_1(x) \leqslant Ax \sum_{n>T}^{\infty} \frac{\tau_k(n)}{n^2} \ll_k AT^{-1}x(\log T)^{k-1}, \tag{5.2}$$

where the implied constant in $U \ll_k V$ (which is equivalent to $U = O_k(V)$) may depend on k.

We also have

$$R_2(x) \leqslant \sum_{n \leqslant T} |f(n)| \leqslant A \sum_{n \leqslant T} \tau_k(n) \ll_k AT(\log T)^{k-1}. \tag{5.3}$$

If $k = 1$, then

$$\begin{aligned} R_3(x) &\leqslant A \sum_{T<n\leqslant x} \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \\ &= A \left(\left\lfloor \frac{x}{\lceil T \rceil} \right\rfloor - \left\lfloor \frac{x}{\lfloor x \rfloor + 1} \right\rfloor \right) \ll AxT^{-1}. \end{aligned} \tag{5.4}$$

Choosing $T = 3\sqrt{x}$ and substituting (5.2), (5.3) and (5.4) in (5.1) we obtain the desired result for $k = 1$.

Now we assume that $k \geqslant 2$ and estimate $R_3(x)$ in this case. We have

$$R_3(x) \ll A \max_{T<N\leqslant x} \Sigma_N(x) \log(x/T), \tag{5.5}$$

where

$$\Sigma_N(x) = \sum_{N<d\leqslant 2N} \tau_k(d) \left(\left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{x}{d+1} \right\rfloor \right).$$

Interverting the summations we obtain

$$\begin{aligned} \Sigma_N(x) &= \sum_{N<d\leqslant 2N} \tau_k(d) \left(\left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{x}{d} - \frac{x}{d(d+1)} \right\rfloor \right) \\ &\leqslant \sum_{N<d\leqslant 2N} \tau_k(d) \left(\left\lfloor \frac{x}{d} \right\rfloor - \left\lfloor \frac{x}{d} - \frac{x - xN^{-1}}{d} \right\rfloor \right) \\ &= \sum_{N<d\leqslant 2N} \tau_k(d) \sum_{x-xN^{-1}<md\leqslant x} 1 \\ &= \sum_{x-xN^{-1}<n\leqslant x} \sum_{\substack{d|n \\ N<d\leqslant 2N}} \tau_k(d). \end{aligned} \tag{5.6}$$

Now [9, Lemma 5.2] yields

$$\Sigma_N(x) \ll_k (\log N)^{k-1} \sum_{x-xN^{-1}<n\leqslant x} \Delta_{k+1}(n),$$

where Δ_r is the rth Hooley divisor function (see [9] and the references therein) defined for each $r \in \mathbb{Z}_{\geqslant 2}$ as

$$\Delta_r(n) = \max_{u_1,\dots,u_{r-1} \in \mathbb{R}} \sum_{\substack{d_1 d_2 \cdots d_{r-1} \mid n \\ e^{u_1} < d_1 \leqslant e^{u_1+1}, \dots, e^{u_{r-1}} < d_{r-1} \leqslant e^{u_{r-1}+1}}} 1. \tag{5.7}$$

Now using [9, Lemma 5.1], for an arbitrary fixed $\varepsilon > 0$, we obtain

$$\Sigma_N(x) \ll_{k,\varepsilon} \left(xN^{-1} (\log x)^{\varepsilon_{k+1}(x)} + x^{\varepsilon} \right) (\log N)^{k-1},$$

where $\varepsilon_{k+1}(x)$ is defined in (2.1) (and the implied constant is now allowed to depend on ε as well).

5.3. Concluding the proof

Collecting the previous estimates finally gives

$$\sum_{n \leqslant x} f(\lfloor x/n \rfloor) = x \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} + O_{k,\varepsilon} \left\{ AT(\log x)^{k-1} + \frac{Ax}{T} (\log x)^{k+\varepsilon_k(x)} + Ax^{\varepsilon} \right\},$$

and choosing $T = 3\sqrt{x(\log x)^{1+\varepsilon_{k+1}(x)}}$, and $\varepsilon = 1/2$, we see that the last term never dominates, which completes the proof.

6. Proofs of Theorems 2.5 and 2.6

6.1. Proof of Theorem 2.5

The proof follows closely that of Theorem 2.3 in the case $k = 1$ above. It is only sufficient to note that, since $|f(n)| \ll n^{\phi-1}(\log en)^{-A}$, the bounds (5.2), (5.3) and (5.4) become here

$$R_1(x) \ll xT^{\phi-2}, \qquad R_2(x) \ll T^{\phi}, \qquad R_3(x) \ll x^{\phi} T^{-1} (\log T)^{-A},$$

respectively. Choosing $T = x^{\phi/(\phi+1)}(\log x)^{-A/(\phi+1)}$ and replacing $\log T$ and with $\log x$ in the bound on $R_3(x)$ yields the asserted result.

6.2. Proof of Theorem 2.6

Again, the proof follows closely that of Theorem 2.3 in the case $k \geqslant 2$. Firstly, note that, using the assumption (2.2) and the Cauchy–Schwarz inequality, we obtain

$$\sum_{n \leqslant x} |f(n)| \leqslant x^{1/2} \left(\sum_{n \leqslant x} |f(n)|^2 \right)^{1/2} \ll x^{(\alpha+1)/2}.$$

Hence, for any $T \in \left(2x^{1/2}, x\right]$, the bounds for $R_1(x)$ and $R_2(x)$ become

$$R_1(x) \ll x T^{(\alpha-3)/2} \qquad \text{and} \qquad R_2(x) \ll T^{(\alpha+1)/2},$$

respectively.

Now similarly to our treatment in (5.5) and (5.6), we obtain

$$\begin{aligned} R_3(x) &= \sum_{T < n \leqslant x} |f(n)| \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n+1} \right\rfloor \right) \\ &\ll \max_{T < N \leqslant x} \left(\sum_{N < n \leqslant 2N} |f(n)| \left(\left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n} - \frac{xN^{-1}}{n} \right\rfloor \right) \right) \log x \\ &\ll \max_{T < N \leqslant x} \left(\sum_{x - x/N < n \leqslant x} \sum_{\substack{d \mid n \\ N < d \leqslant 2N}} |f(d)| \right) \log x. \end{aligned}$$

Now using the Cauchy–Schwarz inequality and (2.2), we derive

$$\sum_{\substack{d \mid n \\ N < d \leqslant 2N}} |f(d)| \leqslant \left(\sum_{d \leqslant 2N} |f(d)|^2 \right)^{1/2} \Delta(n)^{1/2} \ll N^{\alpha/2} \Delta(n)^{1/2},$$

where we set $\Delta(n) = \Delta_2(n)$, see (5.7). Using the Cauchy–Schwarz inequality again we then get

$$\begin{aligned} \sum_{x - x/N < n \leqslant x} \sum_{\substack{d \mid n \\ N < d \leqslant 2N}} |f(d)| &\ll N^{\alpha/2} \sum_{x - x/N < n \leqslant x} \Delta(n)^{1/2} \\ &\ll x^{1/2} N^{(\alpha-1)/2} \left(\sum_{x - x/N < n \leqslant x} \Delta(n) \right)^{1/2} \end{aligned}$$

and [9, Lemma 5.1] yields, if $0 < \alpha < 2$

Table 1
Approximate values of $\rho(x)$.

x	$\rho(x)$
10^6	0.5844
10^7	0.5849
10^8	0.5896
10^9	0.5909
10^{10}	0.5940

$$R_3(x) \ll \max_{T<N\leqslant x} \left(N^{-1+\alpha/2} x(\log x)^{1+\varepsilon_2(x)/2} + x^{1/2+o(1)} N^{(\alpha-1)/2}\right)$$
$$\ll xT^{-1+\alpha/2}(\log x)^{1+\varepsilon_2(x)/2} + \begin{cases} x^{1/2+o(1)}T^{(\alpha-1)/2}, & \text{if } 0<\alpha<1, \\ x^{\alpha/2+o(1)}, & \text{if } 1<\alpha<2, \end{cases}$$

and choosing $T = x^{2/3}(\log x)^{(2+\varepsilon_2(x))/3}$ gives

$$R_1(x) + R_2(x) + R_3(x) \ll x^{(\alpha+1)/3}(\log x)^{(1+\alpha)(2+\varepsilon_2(x))/6} + x^{\alpha/2+o(1)}$$

implying the asserted result since $0 < \alpha < 2$.

7. Numerical results

As we have mentioned, is not clear that a limit for

$$\rho(x) = \frac{S(x)}{x \log x}$$

exists and if it exists whether it coincides with

$$\frac{1}{\zeta(2)} \approx 0.60\,793.$$

Using Maple we can calculate approximate values of $\rho(x)$ for various values of x as shown in Table 1.

Unlike the normal totient summation, increasing x by 1 changes all the previous summands. So the ratio $\rho(x)$ can meaningfully change for a small change in x as shown in Table 2.

The meaningful change in the ratio $\rho(x)$ is also evident for small changes in x above 10^{10} (Table 3).

8. Comments

In the proof of the lower bound of Theorem 2.1, one can take the slightly weaker exponent pair

Table 2
Changes in the values of $\rho(x)$ near $x = 10^6$.

x	$\rho(x)$
10^6	0.5844
$10^6 + 1$	0.6274
$10^6 + 2$	0.5965
$10^6 + 3$	0.6447
$10^6 + 4$	0.6108

Table 3
Changes in the values of $\rho(x)$ near $x = 10^{10}$.

x	$\rho(x)$
10^{10}	0.5940
$10^{10} + 1$	0.6200
$10^{10} + 2$	0.6001
$10^{10} + 3$	0.6270
$10^{10} + 4$	0.6144

$$(k, \ell) = BA^3 \left(BA^2\right)^2 B\,(0,1) = \left(\frac{97}{251}, \frac{132}{251}\right)$$

and choose $J = x^{251/383}(\log x)^{-1198/383}$ to obtain

$$\sum_{n \leqslant x} \varphi\left(\lfloor x/n \rfloor\right) \geqslant \left(\frac{251}{383} - \frac{1198 \log\log x}{383 \log x}\right) \frac{x \log x}{\zeta(2)} + O(x)$$

and similarly for the upper bound. Note that

$$\frac{251}{383\zeta(2)} \approx 0.39\,841 \qquad \text{and} \qquad \frac{2629}{4009\zeta(2)} \approx 0.39\,866$$

while in the upper bound we have

$$\frac{251}{383} \cdot \frac{1}{\zeta(2)} + \frac{132}{383} \approx 0.74\,305 \quad \text{and} \quad \frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + \frac{1380}{4009} \approx 0.74\,289.$$

We also note that one can obtain an asymptotically weaker but fully explicit form of the upper bound on $S(x)$, which does not rely on exponent pairs. Namely, taking $J = x^{1/2}$ so that the sum $S_3(x)$ in (4.2) becomes trivial, and using the explicit bound

$$\sum_{n \leqslant x} \varphi(n) \leqslant \frac{x^2}{2\zeta(2)} + x \log x + 2x,$$

which can be derived by combining [19, Lemma 3.1] with [20, Lemma 2.2], one obtains

$$S(x) \leqslant \frac{1}{2}\left(1 + \frac{1}{\zeta(2)}\right) x \log x + 4x + \frac{\sqrt{x}\log x}{4} + \sqrt{x},$$

for any $x \geq 3$. For comparison,

$$\frac{1}{2}\left(1+\frac{1}{\zeta(2)}\right) \approx 0.80\,396.$$

Acknowledgments

The authors are grateful to the referee for many very useful comments and suggestions.

During the preparation of this work, L. Dai was supported by National Natural Science Foundation of China (Grant 11571174) and Qing Lan Project of Nanjing Normal University, H. Pan by National Natural Science Foundation of China (Grant 11671197) and I. E. Shparlinski by the Australian Research Council (Grants DP170100786).

References

[1] A.G. Abercrombie, W.D. Banks, I.E. Shparlinski, Arithmetic functions on Beatty sequences, Acta Arith. 136 (2009) 81–89.
[2] Y. Akbal, Friable values of Piatetski-Shapiro sequences, Proc. Amer. Math. Soc. 145 (2017) 4255–4268.
[3] R.C. Baker, W.D. Banks, Character sums with Piatetski–Shapiro sequences, Quart. J. Math. 66 (2015) 393–416.
[4] R.C. Baker, W.D. Banks, J. Brüdern, I.E. Shparlinski, A. Weingartner, Piatetski–Shapiro sequences, Acta Arith. 157 (2013) 37–68.
[5] R.C. Baker, W.D. Banks, V.Z. Guo, A.M. Yeager, Piatetski–Shapiro primes from almost primes, Monatsh. Math. 174 (2014) 357–370.
[6] R.C. Baker, L. Zhao, Gaps between primes in Beatty sequences, Acta Arith. 172 (2016) 207–242.
[7] W.D. Banks, V.Z. Guo, I.E. Shparlinski, Almost primes of the form $\lfloor p^c \rfloor$, Indag. Math. (N.S.) 27 (2016) 423–436.
[8] O. Bordellès, Arithmetic Tales, Springer, 2012.
[9] O. Bordellès, Short interval results for certain arithmetic functions, Int. J. Number Theory 14 (2018) 535–548.
[10] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, J. Amer. Math. Soc. 30 (2017) 205–224.
[11] J. Bourgain, N. Watt, Mean square of zeta function, circle problem and divisor problem revisited, Preprint, see http://arxiv.org/abs/1709.04340, 2017.
[12] A.M. Güloğlu, C.W. Nevans, Sums with multiplicative functions over a Beatty sequences, Bull. Aust. Math. Soc. 78 (2008) 327–334.
[13] G. Harman, Primes in Beatty sequences in short intervals, Mathematika 62 (2016) 572–586.
[14] K. Liu, I.E. Shparlinski, T. Zhang, Squares in Piatetski–Shapiro sequences, Acta Arith. 181 (2017) 239–252.
[15] J.F. Morgenbesser, The sum of digits of $\lfloor n^c \rfloor$, Acta Arith. 148 (2011) 367–393.
[16] O. Ramaré, P. Akhilesh, Explicit averages of non-negative multiplicative functions: going beyond the main term, Colloq. Math. 147 (2017) 1–39.
[17] D. Suryanarayana, P. Subrahmanyam, The maximal k-full divisor of an integer, Indian J. Pure Appl. Math. 12 (1981) 175–190.
[18] G. Tenenbaum, Introduction to Analytic and Probabilistic Number Theory, Grad. Studies Math., vol. 163, Amer. Math. Soc., 2015.
[19] E. Treviño, The least k-th power non-residue, J. Number Theory 149 (2015) 201–224.
[20] E. Treviño, The Burgess inequality and the least k-th power non-residue, Int. J. Number Theory 11 (2015) 1653–1678.
[21] J.D. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc. 12 (1985) 183–215.
[22] A. Walfisz, Weylsche Exponentialsummen in der neueren Zahlentheorie, Berlin, 1963.