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Large and moderate deviation principles for alternating Engel expansions

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A B S T R A C T

In this paper, we investigate the large and moderate deviation principles for alternating Engel expansions, a classical representation of real numbers in number theory.

1. Introduction

For any real number $x \in (0, 1]$ and $n \geq 1$, we define

$$x = x_1, \qquad d_n := d_n(x) = \left\lfloor \frac{1}{x_n} \right\rfloor \quad \text{and} \quad x_{n+1} = 1 - x_n d_n, \tag{1.1}$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Then for every $x \in (0, 1]$, the algorithm (1.1) uniquely generates a finite or infinite series. That is,

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$$x = \frac{1}{d_1} - \frac{1}{d_1 d_2} + \cdots + (-1)^{n-1} \frac{1}{d_1 d_2 \cdots d_n} + \cdots, \tag{1.2}$$

where $d_n \geq 1$ are positive integers with $d_{n+1} \geq d_n + 1$ for all $n \geq 1$. The representation (1.2) is said to be the *alternating Engel expansion (or, Pierce expansion)* of x and $d_n(x)$, $n \geq 1$ are called the *digits* of the alternating Engel expansion of x. We sometimes write the form (1.2) as $x = ((d_1, d_2, \cdots, d_n, \cdots))$. This expansion was first considered by Pierce [10] in 1929. Furthermore, some arithmetic and statistical properties of the alternating Engel expansion, such as the representation of rational numbers, law of large numbers, central limit theorem and law of the iterated logarithm were studied by Remez [11], Shallit [12] and Valēēv and Zlēbov [15]. Later, Shallit [13] applied this expansion to proposing a very nice method for determining leap years which generalizes those existent in 1994. For more details about the alternating Engel expansion, we refer the reader to [3,4,6,17] and the references therein.

Now we turn to introducing the large and moderate deviation principles. Let $\{X_n : n \geq 1\}$ be a sequence of real-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A function $I : \mathbb{R} \to [0, \infty]$ is called a *good rate function* if it is lower semi-continuous and has compact level sets. Let $\{\lambda_n : n \geq 1\}$ be a sequence of positive real numbers with $\lim_{n\to\infty} \lambda_n = \infty$. We say that the sequence $\{X_n : n \geq 1\}$ satisfies a *large deviation principle* (LDP for short) with speed λ_n and good rate function I under $\mathbf{P}$, if for any Borel set Γ,

$$-\inf_{x\in\Gamma^\circ} I(x) \leq \liminf_{n\to\infty} \frac{1}{\lambda_n} \log \mathbf{P}(X_n \in \Gamma) \leq \limsup_{n\to\infty} \frac{1}{\lambda_n} \log \mathbf{P}(X_n \in \Gamma) \leq -\sup_{x\in\bar{\Gamma}} I(x),$$

where Γ° and $\bar{\Gamma}$ denotes the interior and the closure of Γ respectively. Formally, there is no distinction between the large deviation principle and the moderate deviation principle (MDP for short). Usually LDP characterizes the convergence speed of the law of large numbers, while MDP describes the speed of convergence between the law of large numbers and the central limit theorem. For an introduction to the theory of large and moderate deviation principles, we refer the reader to Dembo and Zeitouni [1], Touchette [14] and Varadhan [16].

We here denote by $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space, where $\Omega = (0, 1]$, $\mathcal{F}$ is the Borel σ-algebra on $(0, 1]$ and $\mathbf{P}$ denotes the Lebesgue measure on $(0, 1]$. In [12], Shallit established a relation between Stirling numbers of the first kind (see Jordan [7]) and the distribution of the digit d_n occurring in the alternating Engel expansion to estimate the expectation and variance of quantities connected with d_n. Using these estimates, Shallit showed a strong law of large numbers for the digit sequence $\{d_n : n \geq 1\}$, i.e., for $\mathbf{P}$-almost all $x \in (0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \log d_n(x) = 1.$$

And he also proved the central limit theorem holds for the digit sequence $\{d_n : n \geq 1\}$. That is, for every $y \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\log d_n - n}{\sqrt{n}} \leq y\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt.$$

A natural question is to consider the probability of the event that $\frac{\log d_n}{n}$ deviates away from its ergodic mean 1. These probabilities are exponentially small and follow a large deviation principle in general. This leads to the study of large deviations for the alternating Engel expansion.

Theorem 1.1. *Let $\{d_n : n \geq 1\}$ be the digit sequence of the alternating Engel expansion. Then $\left\{\frac{\log d_n - n}{n} : n \geq 1\right\}$ satisfies a LDP with speed n and good rate function*

$$I(x) = \begin{cases} x - \log(x+1), & \text{if } x > -1; \\ +\infty, & \text{otherwise}, \end{cases} \tag{1.3}$$

under $\mathbf{P}$.

Remark 1. As we have seen in Theorem 1.1, the rate function $I(x)$ coincides completely with the rate function (see [1, Exercise 2.2.23]) for the empirical mean of independent and identically distributed (i.i.d.) exponential random variables with mean 1. However, Theorem 1.2 of Zhu [18] has showed that the rate function of LDP for the Engel expansion (i.e., the alternating Engel expansion with all positive terms) does not consist with the rate function for the empirical mean of these i.i.d. exponential random variables with mean 1. This is a difference between Engel expansion and the alternating Engel expansion in the context of large deviations.

As a complement of Theorem 1.1, we give the following MDP result for the alternating Engel expansion.

Theorem 1.2. *Let $\{d_n : n \geq 1\}$ be the digit sequence of the alternating Engel expansion and $\{a_n : n \geq 1\}$ be a positive sequence satisfying*

$$a_n \to \infty, \qquad \frac{a_n}{\sqrt{n \log n}} \to \infty \qquad \text{and} \qquad \frac{a_n}{n} \to 0. \tag{1.4}$$

Then $\left\{\frac{\log d_n - n}{a_n} : n \geq 1\right\}$ satisfies an MDP with speed $n^{-1}a_n^2$ and good rate function $I(x) = x^2/2$ under $\mathbf{P}$.

Remark 2. We may obtain the result if the second condition of a_n in (1.4) is replaced by $a_n/\sqrt{n} \to \infty$ as $n \to \infty$. However, we need the condition $\frac{a_n}{\sqrt{n \log n}} \to \infty$ to avoid some technical difficulties. The main reason is that we cannot obtain the finer estimates when we make use of the Gärtner–Ellis theorem (see [1, Theorem 2.3.6]).

As an application of Theorem 1.2, we immediately get the following corollary.

Corollary 1. *Let $\{d_n : n \geq 1\}$ be the digit sequence of the alternating Engel expansion and $a_n = n^p$ with $p \in (1/2, 1)$. Then $\left\{\frac{\log d_n - n}{n^p} : n \geq 1\right\}$ satisfies an MDP with speed n^{2p-1} and good rate function $I(x) = x^2/2$ under $\mathbf{P}$.*

Recently, some authors began to consider the large and moderate deviations related to number theory. For example, Mehrdad and Zhu [9] established the large and moderate deviations for Erdős–Kac theorem, which is a celebrated result about the number of distinct prime factors of a uniformly chosen random integer in number theory. Later, Zhu [18] and Hu [5] studied, respectively, the large deviations and moderate deviations for Engel, Sylvester and Cantor product expansions considered by Erdős et al. [2], which are the classical representations of real numbers in number theory. Although Engel expansion and alternating Engel expansion have some similar properties (see [2,3,12,17]), our Remark 1 still indicates that there is a difference between these two expansions in the context of large deviations. Moreover, we emphasize that the proofs of Zhu [18] and Hu [5] follow from an explicit computation of the Mellin transform of the digit occurring in the Engel expansion and its asymptotic analysis. However, there are no such properties of d_n in the case of the alternating Engel expansion. Fortunately, we overcome these difficulties by observing that the digit sequence $\{d_n : n \geq 1\}$ of the alternating Engel expansion is strictly increasing. Therefore, we complete the proofs of our theorems and make them more clear by using the key Lemmas 3.1 and 3.3 in Section 3.

2. Preliminary

In this section, we recall some definitions and several arithmetic and metric properties of the alternating Engel expansion, see [3,4,6,12,13,15,17] for details. We use the notation $\mathbf{E}(\xi)$ to denote the expectation of a random variable ξ with respect to the probability measure $\mathbf{P}$.

Recall that the alternating Engel expansion of $x \in (0, 1]$,

$$x = \frac{1}{d_1} - \frac{1}{d_1 d_2} + \cdots + (-1)^{n-1} \frac{1}{d_1 d_2 \cdots d_n} + \cdots, \tag{2.5}$$

where $d_n \geq 1$ are defined as (1.1) and $d_{n+1} \geq d_n + 1$ for all $n \geq 1$. We sometimes write the representation (2.5) as $x = ((d_1, d_2, \cdots, d_n, \cdots))$. We first give an elementary arithmetic property of the alternating Engel expansion in representing of real numbers, which was obtained by Remez [11].

Proposition 2.1. *(See [11].) Any real number $x \in (0,1]$ can be represented in the form (2.5). Moreover, if x is irrational then the alternating Engel expansion of x is unique and infinite; If x is a rational number then it can be expanded into a finite alternating Engel expansion (i.e., $x_n = 0$ for some $n \geq 1$) in the following different ways:*

$$x = ((d_1, d_2, \cdots, d_{n-1}, d_n, d_n + 1)) = ((d_1, d_2, \cdots, d_{n-1}, d_n + 1)).$$

For any irrational $x \in (0,1]$ and $n \geq 1$, the truncated alternating Engel expansion

$$\frac{P_n(x)}{Q_n(x)} = \frac{1}{d_1} - \frac{1}{d_1 d_2} + \cdots + \frac{(-1)^{n-1}}{d_1 d_2 \cdots d_n}$$

is called the *n-th convergent* of the alternating Engel expansion of x. With the conventions $P_0 = 0$ and $Q_0 = 1$, the quantities P_n and Q_n satisfy the following recursive formula:

$$P_n = d_n P_{n-1} + (-1)^{n-1} \qquad \text{and} \qquad Q_n = d_n Q_{n-1} = d_1 d_2 \cdots d_n.$$

Clearly these convergents are rational numbers and $P_n(x)/Q_n(x) \to x$ as $n \to \infty$ for all irrational $x \in (0,1]$. More precisely,

$$\left| x - \frac{P_n(x)}{Q_n(x)} \right| \leq \frac{1}{Q_{n+1}(x)} \leq \frac{1}{(n+1)!}$$

since $d_{n+1} \geq d_n + 1$ with $d_1 \geq 1$ for all $n \geq 1$. That is to say, the approximation of irrational number x by alternating Engel expansion is much faster than the usually used approximations by continued fraction expansion (see [8, Theorems 9, 12]).

It is worth pointing out that not all positive integer sequences can occur in the alternating Engel expansion. Next we give the following definition, which describes the positive real sequences occurring in the alternating Engel expansion.

Definition 2.1. An n-block $(d_1, d_2, \cdots, d_n)$ is said to be admissible for alternating Engel expansions if there exists $x \in (0,1]$ such that $d_j(x) = d_j$ for all $1 \leq j \leq n$. An infinite sequence $(d_1, d_2, \cdots, d_n, \cdots)$ is called an admissible sequence if $(d_1, d_2, \cdots, d_n)$ is admissible for all $n \geq 1$.

The following proposition gives a characterization of all admissible sequences for the alternating Engel expansion.

Proposition 2.2. *A sequence of positive integers $(d_1, d_2, \cdots, d_n, \cdots)$ is admissible for alternating Engel expansions if and only if for all $n \geq 1$,*

$$d_{n+1} \geq d_n + 1.$$

Proof. The necessity is obvious by the definition of x_n and the algorithm (1.1). In fact, by the algorithm (1.1), we get

$$x_{n+1} < 1 - \frac{1}{d_n + 1} d_n = \frac{1}{d_n + 1}$$

and hence that for all $n \geq 1$,

$$d_{n+1} \geq d_n + 1.$$

To prove the sufficiency, for all $n \geq 1$, we take

$$x = x_1 = \frac{1}{d_1} - \frac{1}{d_1 d_2} + \cdots + (-1)^{n-1} \frac{1}{d_1 d_2 \cdots d_n}.$$

Since $d_{n+1} \geq d_n + 1$ for all $n \geq 1$, we have

$$x_1 > \frac{1}{d_1} - \frac{1}{d_1 d_2} \geq \frac{1}{d_1} - \frac{1}{d_1(d_1+1)} = \frac{1}{d_1+1}.$$

Therefore,

$$\frac{1}{d_1+1} < x_1 < \frac{1}{d_1}.$$

And hence by the algorithm (1.1), $d_1(x) = d_1$ and

$$x_2 = \frac{1}{d_2} - \frac{1}{d_2 d_3} + \cdots + (-1)^{n-1} \frac{1}{d_1 d_2 \cdots d_n}.$$

Repeating the above procedure, we can get $d_i(x) = d_i$ for all $1 \leq i \leq n$. Thus, we get the desired result. □

Definition 2.2. Let $(d_1, d_2, \cdots, d_n)$ be an admissible sequence. We call

$$B(d_1, d_2, \cdots, d_n) = \{x \in (0,1) : d_1(x) = d_1, d_2(x) = d_2, \cdots, d_n(x) = d_n\},$$

the n-th order cylinder. In other words, it is the set of points beginning with $(d_1, \cdots, d_n)$ in their alternating Engel expansions.

The following proposition is about the structure and the length of the cylinder.

Proposition 2.3. *Let $(d_1, d_2, \cdots, d_n)$ be an admissible sequence. Then $B(d_1, d_2, \cdots, d_n)$ is an interval with two endpoints*

$$((d_1, d_2, \cdots, d_{n-1}, d_n)) \quad \text{and} \quad ((d_1, d_2, \cdots, d_{n-1}, d_n + 1)).$$

Consequently, for all $n \geq 1$,

$$\mathbf{P}(B(d_1, d_2, \cdots, d_n)) = \frac{1}{d_1 d_2 \cdots d_{n-1} d_n (d_n + 1)}. \tag{2.6}$$

Proof. Let

$$A_n = ((d_1, d_2, \cdots, d_{n-1}, d_n)) \quad \text{and} \quad B_n = ((d_1, d_2, \cdots, d_{n-1}, d_n + 1)).$$

Clearly, $B(d_1, d_2, \cdots, d_n)$ is an interval with two endpoints A_n and B_n. More precisely, $B(d_1, d_2, \cdots, d_n) = [A_n, B_n)$ when n is even and $B(d_1, d_2, \cdots, d_n) = (B_n, A_n]$ as n is odd. Therefore, (2.6) holds for all $n \geq 1$. □

A further result was obtained by Shallit [12]; it states that the digit sequence $\{d_n : n \geq 1\}$ occurring in the alternating Engel expansion forms a homogeneous Markov chain.

Proposition 2.4. *(See [12, Theorem 2].) The sequence $\{d_n : n \geq 1\}$ forms a homogeneous Markov chain with initial distribution*

$$\mathbf{P}(d_1 = j) = \frac{1}{j(j+1)} \quad \text{for all } j \geq 1 \tag{2.7}$$

and transition probabilities

$$\mathbf{P}(d_{n+1} = k \mid d_n = j) = \frac{j+1}{k(k+1)} \quad \text{for all } k \geq j+1 \text{ and } j \geq 1. \tag{2.8}$$

At the end of this section, we introduce a useful theorem called Gärtner–Ellis theorem (see [1, Theorem 2.3.6]) in the theory of large deviations. Let $\{X_n : n \geq 1\}$ be a sequence of the real-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For any $\lambda \in \mathbb{R}$ and $n \geq 1$, we define the *logarithmic moment generating function* of X_n,

$$\Lambda_n(\lambda) := \log \mathbf{E}(e^{\lambda X_n}).$$

Proposition 2.5 *(Gärtner–Ellis theorem). Assume that $\Lambda(\lambda) := \lim_{n\to\infty} (1/n)\Lambda_n(n\lambda)$ exists over the domain $\mathcal{D}_\Lambda := \{\lambda \in \mathbb{R} : \Lambda(\lambda) < \infty\}$. If the following hold:*

1. *The interior of $\mathcal{D}_\Lambda$ contains the origin.*
2. *$\Lambda(\cdot)$ is a lower semicontinuous function.*
3. *$\Lambda(\cdot)$ is differentiable throughout the interior of $\mathcal{D}_\Lambda$.*
4. *$\lim_{n\to\infty} |\Lambda'(\lambda_n)| = \infty$ whenever $\{\lambda_n : n \geq 1\}$ is a sequence in the interior of $\mathcal{D}_\Lambda$ converging to a boundary point of interior of $\mathcal{D}_\Lambda$.*

Then the sequence $\{X_n : n \geq 1\}$ *satisfies a LDP with speed* n *and good rate function*

$$I(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda(\lambda)\}, \quad \forall x \in \mathbb{R}.$$

Remark 3. When $1/n$ is replaced by a positive real number sequence a_n with $a_n \to 0$ as $n \to \infty$ and the function $\Lambda(\cdot)$ is properly modified, the results of Gärtner–Ellis theorem are also valid.

3. The proofs of the theorems

In this section, we will give the proofs of Theorems 1.1 and 1.2. The following Lemma 3.1 is the key lemma in the proofs of our theorems.

Lemma 3.1. *Let* $\theta < 1$*. Then for all* $j \geq 1$*, we have*

$$\left(1 + \frac{1}{j}\right)^{\theta - 1} \cdot \frac{1}{1-\theta} \leq \sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta} \leq \left(1 + \frac{1}{j}\right) \cdot \frac{1}{1-\theta}.$$

Proof. Let $\theta < 1$. Notice that for any $j \geq 1$,

$$\int_j^{\infty} \frac{1}{x^{2-\theta}} dx = \frac{1}{1-\theta} \cdot j^{\theta - 1} \tag{3.9}$$

and

$$\sum_{k=j+1}^{\infty} \frac{k^{\theta}}{k(k+1)} \leq \sum_{k=j+1}^{\infty} \frac{1}{k^{2-\theta}} \leq \int_j^{\infty} \frac{1}{x^{2-\theta}} dx, \tag{3.10}$$

by (3.9) and (3.10), we have that

$$\sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta} = \frac{j+1}{j^{\theta}} \sum_{k=j+1}^{\infty} \frac{k^{\theta}}{k(k+1)} \leq \left(1 + \frac{1}{j}\right) \cdot \frac{1}{1-\theta}.$$

On the other hand, for any $j \geq 1$, we obtain that

$$\sum_{k=j+1}^{\infty} \frac{1}{k^{2-\theta}} \geq \int_{j+1}^{\infty} \frac{1}{x^{2-\theta}} dx = \frac{1}{1-\theta} \cdot (j+1)^{\theta - 1} \tag{3.11}$$

and

$$\sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta} = \frac{j+1}{j^{\theta}} \sum_{k=j+1}^{\infty} \frac{k^{\theta}}{k(k+1)} = \frac{j+1}{j^{\theta}} \sum_{k=j+1}^{\infty} \frac{1}{k^{2-\theta}} \cdot \frac{k}{k+1}. \tag{3.12}$$

Since $k/(k+1) \geq j/(j+1)$ for all $k \geq j$, in view of (3.11) and (3.12), we deduce that

$$\sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta} \geq \frac{j+1}{j^{\theta}} \sum_{k=j+1}^{\infty} \frac{1}{k^{2-\theta}} \cdot \frac{j}{j+1} \geq \frac{1}{1-\theta} \cdot \left(1+\frac{1}{j}\right)^{\theta-1}. \qquad \square$$

3.1. Proof of Theorem 1.1

To prove Theorem 1.1, we also need the following Lemma 3.2.

Lemma 3.2. *Let $\{d_n : n \geq 1\}$ be the digit sequence of the alternating Engel expansion. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}(d_n^{\theta}) = \begin{cases} \log(\frac{1}{1-\theta}), & \text{if } \theta < 1; \\ +\infty, & \text{if } \theta \geq 1. \end{cases}$$

Proof. Let $\theta \geq 1$. Notice that $d_{n+1} > d_n$ with $d_1 \geq 1$ for all $n \geq 1$, the initial distribution (2.7) yields that for any $n \geq 1$,

$$\mathbf{E}(d_n^{\theta}) \geq \mathbf{E}(d_1^{\theta}) = \sum_{k=1}^{\infty} \mathrm{P}(d_1 = k) \cdot k^{\theta} = \sum_{k=1}^{\infty} \frac{k^{\theta}}{k(k+1)} = +\infty.$$

Therefore, $\lim\limits_{n\to\infty} \frac{1}{n} \log E(d_n^{\theta}) = +\infty$.

Let $\theta < 1$. Since $d_{n+1} \geq d_n + 1$ with $d_1 \geq 1$, we have that $d_n \geq n$ for all $n \geq 1$. By the definition of expectation, we deduce that

$$\begin{aligned} \mathbf{E}(d_n^{\theta}) &= \sum_{k=n}^{\infty} \mathbf{P}(d_n = k) \cdot k^{\theta} \\ &= \sum_{k=n}^{\infty} \sum_{j=n-1}^{k-1} \mathbf{P}(d_n = k \mid d_{n-1} = j) \cdot \mathbf{P}(d_{n-1} = j) \cdot k^{\theta} \\ &= \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1} = j) \cdot j^{\theta} \cdot \sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta}, \end{aligned} \tag{3.13}$$

where the second equality is from the conditional probability and the last equality follows from the transition probabilities (2.8). By Lemma 3.1, we have

$$\left(1+\frac{1}{j}\right)^{\theta-1} \cdot \frac{1}{1-\theta} \leq \sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta} \leq \left(1+\frac{1}{j}\right) \cdot \frac{1}{1-\theta}. \tag{3.14}$$

Therefore, in view of (3.13) and the first inequality of (3.14), we obtain that

$$\begin{aligned}
\mathbf{E}(d_n^\theta) &= \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1}=j)\cdot j^\theta \cdot \sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)}\left(\frac{k}{j}\right)^\theta \\
&\geq \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1}=j)\cdot j^\theta \cdot \frac{1}{1-\theta}\cdot\left(\frac{j}{j+1}\right)^{1-\theta} \\
&\geq \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1}=j)\cdot j^\theta \cdot \frac{1}{1-\theta}\cdot\left(\frac{n-1}{n}\right)^{1-\theta} \\
&\geq \cdots\cdots \\
&\geq \sum_{j=1}^{\infty} \mathbf{P}(d_1=j)\cdot j^\theta \cdot\left(\frac{1}{1-\theta}\right)^{n-1}\cdot\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\cdots\cdot\frac{n-2}{n-1}\cdot\frac{n-1}{n}\right)^{1-\theta},
\end{aligned} \tag{3.15}$$

where the second inequality follows from the fact $j/(j+1) \geq i/(i+1)$ for all $j \geq i$. In view of (3.15), we deduce that

$$\mathbf{E}(d_n^\theta) \geq M n^{\theta-1}\left(\frac{1}{1-\theta}\right)^{n-1}, \tag{3.16}$$

where $M = \sum_{j=1}^{\infty} \frac{j^\theta}{j(j+1)}$ is a positive constant. Thus, we complete the lower bounded estimate of $\mathbf{E}(d_n^\theta)$.

Next, we will give the upper bounded estimate of $\mathbf{E}(d_n^\theta)$. By (3.13) and the second inequality of (3.14), we obtain that

$$\begin{aligned}
\mathbf{E}(d_n^\theta) &= \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1}=j)\cdot j^\theta \cdot \sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)}\left(\frac{k}{j}\right)^\theta \\
&\leq \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1}=j)\cdot j^\theta \cdot \frac{1}{1-\theta}\cdot\left(1+\frac{1}{j}\right) \\
&\leq \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1}=j)\cdot j^\theta \cdot \frac{1}{1-\theta}\cdot\left(\frac{n}{n-1}\right) \\
&\leq \cdots\cdots \\
&\leq \sum_{j=1}^{\infty} \mathbf{P}(d_1=j)\cdot j^\theta \cdot\left(\frac{1}{1-\theta}\right)^{n-1}\cdot\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{n-1}{n-2}\cdot\frac{n}{n-1}\right),
\end{aligned} \tag{3.17}$$

where the second inequality follows from the fact $1+1/j \leq 1+1/i$ for all $j \geq i$. Thus, by (3.17), we have that

$$\mathbf{E}(d_n^\theta) \leq nM\left(\frac{1}{1-\theta}\right)^{n-1}, \tag{3.18}$$

where $M = \sum_{j=1}^{\infty} \frac{j^\theta}{j(j+1)}$ is a positive constant.

By (3.16) and (3.18), we have that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{E}(d_n^\theta) \geq \liminf_{n\to\infty} \left\{ \frac{\log M}{n} + (\theta - 1)\frac{\log n}{n} + \frac{n-1}{n} \log\left(\frac{1}{1-\theta}\right) \right\}$$
$$= \log\left(\frac{1}{1-\theta}\right)$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{E}(d_n^\theta) \leq \limsup_{n\to\infty} \left\{ \frac{\log M}{n} + \frac{\log n}{n} + \frac{n-1}{n} \log\left(\frac{1}{1-\theta}\right) \right\}$$
$$= \log\left(\frac{1}{1-\theta}\right).$$

That is, for $\theta < 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}(d_n^\theta) = \log\left(\frac{1}{1-\theta}\right). \qquad \square$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.2, we know that

$$\Lambda(\theta) = -\theta + \lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}(d_n^\theta) = \begin{cases} -\theta - \log(1-\theta), & \text{if } \theta < 1; \\ +\infty, & \text{if } \theta \geq 1. \end{cases}$$

It is not difficult to check that $\Lambda(\cdot)$ satisfies all the conditions of Proposition 2.5. By Gärtner–Ellis theorem, we obtain that the sequence $\left\{\frac{\log d_n - n}{n} : n \geq 1\right\}$ satisfies a LDP with speed n and good rate function

$$I(x) = \sup_{\theta\in\mathbb{R}} \{\theta x - \Lambda(\theta)\} = \sup_{\theta<1} \{\theta x + \theta + \log(1-\theta)\}$$
$$= \begin{cases} x - \log(1+x), & \text{if } x > -1; \\ +\infty, & \text{if } x \leq -1. \end{cases} \qquad \square$$

3.2. Proof of Theorem 1.2

As an application of Lemma 3.1, we obtain the following lemma.

Lemma 3.3. *Let $\theta \in (-1, 1)$. Then for any $j \geq 1$, we have*

$$\frac{1}{1-\theta} \cdot \left(\frac{j}{j+1}\right)^2 \leq \sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^\theta \leq \frac{1}{1-\theta} \cdot \left(\frac{j+1}{j}\right). \tag{3.19}$$

Now we are going to give the proof of Theorem 1.2.

Proof of Theorem 1.2. For any $\lambda \in \mathbb{R}$, we consider the logarithmic moment generating function of $\frac{\log d_n - n}{a_n}$,

$$\Lambda_n(\lambda) = \log \mathbf{E}\left(\exp\left(\lambda \cdot \frac{\log d_n - n}{a_n}\right)\right).$$

From the Gärtner–Ellis theorem, in order to obtaining the desired result, it suffices to show that for any $\lambda \in \mathbb{R}$,

$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{n}{a_n^2} \Lambda_n\left(\frac{a_n^2}{n}\lambda\right) = \frac{\lambda^2}{2}.$$

That is,

$$\lim_{n\to\infty} \frac{n}{a_n^2} \log \mathbf{E}\left(\exp\left\{\frac{a_n}{n}\left(\log d_n - n\right)\lambda\right\}\right) = \frac{\lambda^2}{2}. \tag{3.20}$$

For any $\lambda \in \mathbb{R}$ and $n \geq 1$, let

$$\theta_n := \theta_n(\lambda) = \frac{a_n}{n}\lambda \quad \text{and} \quad \Upsilon_n(\lambda) = \mathbf{E}(\exp\{\theta_n(\log d_n - n)\}).$$

In view of (1.4), it is clear that $\theta_n \to 0$ as $n \to \infty$ and $\Upsilon_n(\lambda)$ can be rewritten as

$$\Upsilon_n(\lambda) = e^{-n\theta_n}\mathbf{E}(d_n^{\theta_n}). \tag{3.21}$$

To get (3.20), we only need to estimate the expectation $\mathbf{E}(d_n^{\theta_n})$. Since $\theta_n \to 0$ as $n \to \infty$, there exists $N > 0$ such that $n \geq N$, we have $\theta_n \in (-\frac{1}{2}, \frac{1}{2})$. Now we give the lower and upper bounded estimates of $\mathbf{E}(d_n^{\theta_n})$. By (3.13) and the first inequality of (3.19), we deduce that for all $n \geq N$,

$$\begin{aligned}
\mathbf{E}(d_n^{\theta_n}) &= \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1} = j) \cdot j^{\theta_n} \cdot \sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)}\left(\frac{k}{j}\right)^{\theta_n} \\
&\geq \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1} = j) \cdot j^{\theta_n} \cdot \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{1-\theta_n} \\
&\geq \cdots\cdots \\
&\geq \sum_{j=1}^{\infty} \mathbf{P}(d_1 = j) \cdot j^{\theta_n} \cdot \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \cdots \cdot \frac{n-2}{n-1} \cdot \frac{n-1}{n}\right)^2 \left(\frac{1}{1-\theta_n}\right)^{n-1}.
\end{aligned}$$

Let $M = \frac{1}{2}\sum_{j=1}^{\infty} \frac{1}{j^{3/2}(j+1)}$, then M is a positive constant. Since $\theta_n \in (-\frac{1}{2}, \frac{1}{2})$, we deduce that

$$\mathbf{E}(d_n^{\theta_n}) \geq \frac{M}{n^2}\left(\frac{1}{1-\theta_n}\right)^n. \tag{3.22}$$

On the other hand, in view of (3.13) and the second inequality of (3.19), we have that for all $n \geq N$,

$$\begin{aligned}
\mathbf{E}(d_n^{\theta_n}) &= \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1} = j) \cdot j^{\theta_n} \cdot \sum_{k=j+1}^{\infty} \frac{j+1}{k(k+1)} \left(\frac{k}{j}\right)^{\theta_n} \\
&\leq \sum_{j=n-1}^{\infty} \mathbf{P}(d_{n-1} = j) \cdot j^{\theta_n} \cdot \left(\frac{n}{n-1}\right) \cdot \frac{1}{1-\theta_n} \\
&\leq \cdots\cdots \\
&\leq \sum_{j=1}^{\infty} \mathbf{P}(d_1 = j) \cdot j^{\theta_n} \cdot \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \cdots \cdot \frac{n-1}{n-2} \cdot \frac{n}{n-1}\right) \left(\frac{1}{1-\theta_n}\right)^{n-1}.
\end{aligned}$$

Let $M' = \frac{3}{2}\sum_{j=1}^{\infty} \frac{1}{j^{1/2}(j+1)}$, then M' is a positive constant. Notice that $\theta_n \in (-\frac{1}{2}, \frac{1}{2})$, we obtain that

$$\mathbf{E}(d_n^{\theta_n}) \leq nM'\left(\frac{1}{1-\theta_n}\right)^n. \tag{3.23}$$

Combining (1.4), (3.21), (3.22) and (3.23), we deduce that by Taylor expansion

$$\begin{aligned}
\liminf_{n\to\infty} \frac{n}{a_n^2} \log \Upsilon_n(\lambda) &= \liminf_{n\to\infty} \left\{ \frac{n^2}{a_n^2}(-\theta_n) + \frac{n}{a_n^2} \log \mathbf{E}(d_n^{\theta_n}) \right\} \\
&\geq \liminf_{n\to\infty} \left\{ \frac{n}{a_n^2}(-\theta_n) + \frac{n}{a_n^2} \log M - \frac{2n\log n}{a_n^2} - \frac{n^2}{a_n^2} \log(1-\theta_n) \right\} \\
&= \frac{\lambda^2}{2}
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
\limsup_{n\to\infty} \frac{n}{a_n^2} \log \Upsilon_n(\lambda) &= \limsup_{n\to\infty} \left\{ \frac{n^2}{a_n^2}(-\theta_n) + \frac{n}{a_n^2} \log \mathbf{E}(d_n^{\theta_n}) \right\} \\
&\leq \limsup_{n\to\infty} \left\{ \frac{n^2}{a_n^2}(-\theta_n) + \frac{n}{a_n^2} \log M' + \frac{n\log n}{a_n^2} - \frac{n}{a_n^2} \log(1-\theta_n) \right\} \\
&= \frac{\lambda^2}{2}.
\end{aligned} \tag{3.25}$$

By (3.24) and (3.25), the equation (3.20) is established. From the Gärtner–Ellis theorem, we know that the sequence $\{\frac{\log d_n - n}{a_n} : n \geq 1\}$ satisfies an MDP with speed $n^{-1}a_n^2$ and good rate function

$$I(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda(\lambda)\} = \frac{x^2}{2}, \quad \forall\, x \in \mathbb{R}. \qquad \square$$

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