Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Counting rational points on smooth cubic curves

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ARTICLE INFO

Article history:
Received 26 October 2016
Received in revised form 6 December 2017
Accepted 7 December 2017
Available online xxxx
Communicated by A. Pal

Keywords:
Cubic curves
Rational points
Counting function
Elliptic curves
Global determinant method
Descent

ABSTRACT

We use a global version of Heath-Brown's p-adic determinant method developed by Salberger to give upper bounds for the number of rational points of height at most B on non-singular cubic curves defined over $\mathbb{Q}$. The bounds are uniform in the sense that they only depend on the rank of the corresponding Jacobian.

1. Introduction

Let $F(X_0, X_1, X_2) \in \mathbb{Z}[X_0, X_1, X_2]$ be a non-singular cubic form, so that $F = 0$ defines a smooth plane cubic curve C in $\mathbb{P}^2$. We want to study the asymptotic behaviour of the counting function

$$N(B) = \sharp\{P \in C(\mathbb{Q}) : H(P) \leq B\},$$

with respect to the naive height function $H(P) := \max\{|x_0|, |x_1|, |x_2|\}$ for $P = [x_0, x_1, x_2]$ with co-prime integer values of x_0, x_1, x_2.

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https://doi.org/10.1016/j.jnt.2017.12.001
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It is known that if the rank r of the Jacobian $\mathrm{Jac}(C)$ is positive, then we have

$$N(B) \sim c_F(\log B)^{r/2} \tag{1}$$

as $B \to \infty$. This result was shown by Néron. Moreover, if $r = 0$ then $N(B) \leq 16$ by Mazur's theorem (see Mazur [5], Theorem 8) on torsion groups of elliptic curves. But (1) is not a uniform upper bound as the constant c_F depends on C. The aim of this paper is to give uniform upper bounds for $N(B)$ which only depend on the rank of $\mathrm{Jac}(C)$.

In this direction, Heath-Brown and Testa (see [4], Corollary 1.3) established the uniform bound

$$N(B) \ll (\log B)^{3+r/2} \tag{2}$$

by using the p-adic determinant method developed by the first author (see [3]). In [4], they also used a result of David [1] about the successive minima of the quadratic form given by the canonical height pairing on $\mathrm{Jac}(C)$ to prove the sharper uniform bounds $N(B) \ll (\log B)^{1+r/2}$ for all r and $N(B) \ll (\log B)^{r/2}$ if r is sufficiently large.

We shall in this paper give a direct proof of the bound

$$N(B) \ll (\log B)^{2+r/2}, \tag{3}$$

based on the determinant method, which does not depend on any deep result about the canonical height pairing.

To do this, we follow the approach in [4] with descent. But we replace the p-adic determinant method by a global determinant method developed by Salberger [6]. The main result of this paper is the following

Theorem 1. *Let $F(X_0, X_1, X_2) \in \mathbb{Z}[X_0, X_1, X_2]$ be a non-singular cubic form, so that $F = 0$ defines a smooth plane cubic curve C. Let r be the rank of Jac(C). Then for any $B \geq 3$ and any positive integer m we have*

$$N(B) \ll m^r \left(B^{\frac{2}{3m^2}} + m^2\right) \log B$$

uniformly in C, with an implied constant independent of m.

This bound improves upon the estimate

$$N(B) \ll m^{r+2} \left(B^{\frac{2}{3m^2}} \log B + \log^2 B\right)$$

in [4] (see Theorem 1.2). Taking $m = 1 + [\sqrt{\log B}]$ we immediately obtain the following result.

Please cite this article in press as: M.H. Tran, Counting rational points on smooth cubic curves, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.12.001

Corollary 2. *Under the conditions above we have*

$$N(B) \ll (\log B)^{2+r/2}$$

uniformly in C.

In the appendix we include for comparison a short account of the bounds for $N(B)$ that can be deduced from David's result.

2. The descent argument

We shall in this section recall the argument in [4], where the study of $N(B)$ is reduced to a counting problem for a biprojective curve.

Let $\psi : C \times C \to \operatorname{Jac}(C)$ be the morphism to the Jacobian of C defined by $\psi(P, Q) = [P] - [Q]$. Let m be a positive integer and define an equivalence relation on $C(\mathbb{Q})$ as follows: $P \sim_m Q$ if $\psi(P, Q) \in m(\operatorname{Jac}(C)(\mathbb{Q}))$. The number of equivalence classes is at most $16m^r$ by the theorems of Mazur and Mordell–Weil. There is therefore a class K such that

$$N(B) \ll m^r \sharp\{P \in K : H(P) \leq B\}.$$

If we fix a point R in K then for any other point P in K, there will be a further point Q in $C(\mathbb{Q})$ such that $[P] = m[Q] - (m-1)[R]$ in the divisor class group of C. We define the curve $X = X_R$ by

$$X_R := \{(P, Q) \in C \times C : [P] = m[Q] - (m-1)[R]\}$$

in $\mathbb{P}^2 \times \mathbb{P}^2$. Then $N(B) \ll m^r \sharp\mathcal{K}$, where

$$\mathcal{K} := \{(P, Q) \in X(\mathbb{Q}) : H(P) \leq B\}.$$

We have thus reduced the counting problem for C to a counting problem for a biprojective curve X in $\mathbb{P}^2 \times \mathbb{P}^2$. We shall also need the following lemma from [4] (see Lemma 2.1).

Lemma 3. *Let* C *be a smooth plane cubic curve defined by a primitive form* F *with* $\|F\| \ll B^{30}$, *and* R *be a point in* $C(\mathbb{Q})$. *There exists an absolute constant* A *with the following property. Suppose that* (P, Q) *is a point in* $X_R(\mathbb{Q})$ *and that* $B \geq 3$. *Then if* $H(P), H(R) \leq B$ *we have* $H(Q) \leq B^A$.

3. The global determinant method

We shall in this section apply Salberger's global determinant method in [6] to X and consider congruences between integral points on X modulo all primes of good reduction for C and X. It is a refinement of the p-adic determinant method used in [3] and [4].

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We will label the points in $\mathcal{K}$ as (P_j, Q_j) for $1 \le j \le N$, say, and fix integers $a, b \ge 1$. Let I_1 be the vector space of all bihomogeneous forms in $(x_0, x_1, x_2; y_0, y_1, y_2)$ of bidegree (a, b) with coefficients in $\mathbb{Q}$ and I_2 be the subspace of such forms which vanish on X. Since the monomials

$$x_0^{e_0} x_1^{e_1} x_2^{e_2} y_0^{f_0} y_1^{f_1} y_2^{f_2}$$

with

$$e_0 + e_1 + e_2 = a \text{ and } f_0 + f_1 + f_2 = b$$

form a basis for I_1, there is a subset of monomials $\{F_1, ..., F_s\}$ whose corresponding cosets form a basis for I_1/I_2. As in [4] (see Lemma 3.1), if $\frac{1}{a} + \frac{m^2}{b} < 3$, then $s = 3(m^2a + b)$. Thus we shall always assume that $a \ge 1$ and $b \ge m^2$ to make sure that $s = 3(m^2a + b)$. Consider the $N \times s$ matrix

$$M = \begin{pmatrix} F_1(P_1, Q_1) & F_2(P_1, Q_1) & \dots & F_s(P_1, Q_1) \\ F_1(P_2, Q_2) & F_2(P_2, Q_2) & \dots & F_s(P_2, Q_2) \\ \vdots & \vdots & \dots & \vdots \\ F_1(P_N, Q_N) & F_2(P_N, Q_N) & \dots & F_s(P_N, Q_N) \end{pmatrix}.$$

If we can choose a and b such that $\operatorname{rank}(M) < s$, then there is a non-zero column vector $\underline{c}$ such that $M\underline{c} = \underline{0}$. This will produce a bihomogeneous form G, say, of bidegree (a, b) such that $G(P_j, Q_j) = 0$ for all $1 \le j \le N$. Hence all points in $\mathcal{K}$ will lie on the variety $Y \subset \mathbb{P}^2 \times \mathbb{P}^2$ given by $G = 0$, while the irreducible curve X does not lie on Y. Thus

$$N \le \sharp(X \cap Y) \le 3(m^2a + b) \tag{4}$$

by the Bézout-type argument in [4] (see Lemma 5.1).

In order to show that $\operatorname{rank}(M) < s$, we may clearly suppose that $N \ge s$. We will show that each $s \times s$ minor $\det(\Delta)$ of M vanishes. Without loss of generality, let Δ be the $s \times s$ matrix formed by the first s rows of M.

$$\Delta = \begin{pmatrix} F_1(P_1, Q_1) & F_2(P_1, Q_1) & \dots & F_s(P_1, Q_1) \\ F_1(P_2, Q_2) & F_2(P_2, Q_2) & \dots & F_s(P_2, Q_2) \\ \vdots & \vdots & \dots & \vdots \\ F_1(P_s, Q_s) & F_2(P_s, Q_s) & \dots & F_s(P_s, Q_s) \end{pmatrix}.$$

The idea is now to give an upper bound for $\det(\Delta)$ which is smaller than a certain integral factor of $\det(\Delta)$. To do this, we first recall a result from [3] (see Theorem 4).

Lemma 4. *For a plane cubic curve C defined by a primitive integral form F, either $N(B) \le 9$ or $\|F\| \ll B^{30}$.*

Thus from now on, we may and shall always suppose that $\|F\| \ll B^{30}$. It is not difficult to see that every entry in Δ has modulus at most $B^a B^{Ab}$, where A is the absolute constant in Lemma 3. Since Δ is an $s \times s$ matrix, we get that

$$\log|\det(\Delta)| \leq s\log\, s + s\log\, B^{a+Ab}. \tag{5}$$

Now we find a factor of $\det(\Delta)$ of the form p^{N_p}, where p is a prime of good reduction for C. In order to do that, we divide Δ into blocks such that elements in each block have the same reduction modulo p.

Let p be a prime number and Q^* be a point on $C(\mathbb{F}_p)$. Then we define the set

$$S(Q^*, p, \Delta) = \{(P_j, Q_j) : 1 \leq j \leq s,\ \overline{Q_j} = Q^*\},$$

where $\overline{Q_j}$ denotes the reduction from $C(\mathbb{Q})$ to $C(\mathbb{F}_p)$. Suppose $\sharp S(Q^*, p, \Delta) = E$. We consider any $E \times E$ sub-matrix Δ^* of Δ corresponding to $S(Q^*, p, \Delta)$ and recall a result from [4] (see Lemma 4.2). Note that our set $S(Q^*, p, \Delta)$ has fewer elements than the set $S(Q'; p, B)$ defined at the beginning of Section 3 in [4] but the proof still works.

Lemma 5. *If p is a prime of good reduction for C, then $p^{E(E-1)/2}$ divides* $\det(\Delta^*)$.

From this lemma we obtain a factor of $\det(\Delta)$ of the form p^{N_p} by means of Laplace expansion. Moreover, we can do the same argument for all primes of good reduction for C and then obtain a very large factor of $\det(\Delta)$. That is the idea of the global determinant method in [6].

Lemma 6. *Let p be a prime of good reduction for C. There exists a non-negative integer $N_p \geq \frac{s^2}{2n_p} + O(s)$ such that $p^{N_p}|\det(\Delta)$, where n_p is the number of $\mathbb{F}_p$-points on $C(\mathbb{F}_p)$.*

Proof. Let P be a point on $C(\mathbb{F}_p)$ and s_P be the number of elements in $S(P, p, \Delta)$. Then by Lemma 5, there exists an integer $N_P = s_P(s_P - 1)/2$ such that $p^{N_P}|\det(\Delta^*)$ for each $s_P \times s_P$ sub-matrix Δ^* of Δ corresponding to $S(P, p, \Delta)$.

If we apply this to all points on $C(\mathbb{F}_p)$ and use Laplace expansion, then we get that $p^{N_p}|\det(\Delta)$ for

$$N_p = \sum_P N_P = \frac{1}{2}\sum_P {s_P}^2 - \frac{s}{2} \geq \frac{s^2}{2n_p} + O(s)$$

in case C has good reduction at p. This completes the proof of Lemma 6.

We now give a bound for the product of primes of bad reduction for C. Since $\|F\| \ll B^{30}$, the discriminant D_F of F will satisfy $\log|D_F| \ll \log\, B$. Thus $\log\, \Pi_C \ll \log\, B$, where Π_C is the product of all primes of bad reduction for C. We have therefore the following bound.

Lemma 7. *Suppose that* $\|F\| \ll B^{30}$. *The product* Π_C *of all primes of bad reduction for* C *satisfies* $\log \Pi_C = O(\log B)$.

We need one more lemma from [6] (see Lemma 1.10).

Lemma 8. *Let* $\Pi > 1$ *be an integer and* p *run over all prime factors of* Π. *Then*

$$\sum_{p|\Pi} \frac{\log p}{p} \leq \log \log \Pi + 2.$$

Proof. We may and shall assume that Π is a square-free. Let l be a positive integer such that $l \leq \Pi$ and $v_p(n)$ be the highest integer such that $p^{v_p(n)}|n$. We then have (see Tenenbaum [7], pp. 13–14)

$$\begin{aligned} l \sum_{p|\Pi} \frac{\log p}{p} - \sum_{p|\Pi} \log p &\leq \sum_{p|\Pi} v_p(l!) \log p \\ &\leq \sum_{p \leq \Pi} v_p(l!) \log p = \log l! \leq l \log l, \\ \Rightarrow \sum_{p|\Pi} \frac{\log p}{p} \leq \log l + \frac{1}{l} \sum_{p|\Pi} \log p &\leq \log l + (1/l) \log \Pi. \end{aligned}$$

To obtain the assertion, let $l = [\log \Pi]$ for $\Pi > 2$.

4. Proof of Theorem 1

We now use the lemmas in Section 3 to prove that $\det(\Delta)$ vanishes if s is large enough. Let Π_C be the product of all primes p of bad reduction for C. Then

$$\sum_{p|\Pi_C} \frac{\log p}{p} \leq \log \log B + O(1) \tag{6}$$

by Lemma 7 and Lemma 8. We apply Lemma 6 to the primes $p \leq s$ of good reduction for C and write $\sum_{p \leq s}^{*}$ for a sum over these primes. We then obtain a positive factor T of $\det(\Delta)$ which is relatively prime to Π_C such that

$$\log T \geq \frac{s^2}{2} \sum_{p \leq s}^{*} \frac{\log p}{n_p} + O(s) \sum_{p \leq s}^{*} \log p.$$

The last term is $O(s^2)$ since $\sum_{p \leq s} \log p = O(s)$ (see [7], p. 31). Also,

$$\frac{\log p}{n_p} \geq \frac{\log p}{p} - \frac{(n_p - p) \log p}{p^2}.$$

Moreover, it is well-known that if p is a prime of good reduction for C, then $n_p = p + O(\sqrt{p})$. Thus we conclude that

$$\frac{\log\, p}{n_p} \geq \frac{\log\, p}{p} + O\left(\frac{\log\, p}{p^{3/2}}\right)$$

for all primes p of good reduction for C. Therefore,

$$\sum_{p\leq s}^{*} \frac{\log\, p}{n_p} \geq \sum_{p\leq s}^{*} \frac{\log\, p}{p} + O(1)$$

and then

$$\log\, T \geq \frac{s^2}{2}\sum_{p\leq s}^{*} \frac{\log\, p}{p} + O(s^2).$$

But by (6),

$$\sum_{p\leq s} \frac{\log\, p}{p} - \sum_{p\leq s}^{*} \frac{\log\, p}{p} \leq \log\,\log\, B + O(1)$$

and $\sum_{p\leq s} \frac{\log\, p}{p} = \log\, s + O(1)$ (see [7], p. 14). Hence,

$$\log\, T \geq \frac{s^2}{2}\log\left(\frac{s}{\log\, B}\right) + O(s^2). \tag{7}$$

Thus from (5) and (7) we obtain

$$\begin{aligned}\log\left(\frac{|\det(\Delta)|}{T}\right) &\leq s\log\, s + s\log\, B^{a+Ab} - \frac{s^2}{2}\log\left(\frac{s}{\log\, B}\right) + O(s^2)\\ &= \frac{s^2}{2}\left(\log B^{\frac{2(a+Ab)}{s}} - \log\left(\frac{s}{\log\, B}\right)\right) + O(s^2).\end{aligned}$$

There is therefore an absolute constant $u \geq 1$ such that

$$\log\left(\frac{|\det(\Delta)|}{T}\right) \leq \frac{s^2}{2}\left(\log B^{\frac{2(a+Ab)}{s}} - \log\left(\frac{s}{u\log\, B}\right)\right).$$

If

$$s > uB^{\frac{2(a+Ab)}{s}}\log\, B \tag{8}$$

we have in particular that $\log\left(\frac{|\det(\Delta)|}{T}\right) < 0$ and hence $\det(\Delta) = 0$ as $\frac{|\det(\Delta)|}{T} \in \mathbb{Z}_{\geq 0}$.

Remember that $s = 3(m^2a + b)$ if $a \geq 1$ and $b \geq m^2$. We now choose $b = m^2$ and

$$a = 1 + \left[\frac{uB^{\frac{2}{3m^2}} \log B}{m^2} + A \log B \right].$$

Then

$$uB^{\frac{2(a+Ab)}{s}} \log\, B = uB^{\frac{2(a+Am^2)}{3m^2(a+1)}} \log\, B$$

$$< uB^{\frac{2}{3m^2}} B^{\frac{2A}{3a}} \log B < s.$$

Thus (8) holds and hence $\det(\Delta) = 0$. Then $\operatorname{rank}(M) < s$ such that there is a bihomogeneous form in $\mathbb{Q}[x_0, x_1, x_2, y_0, y_1, y_2]$ which vanishes at all $(P_j, Q_j) \in X(Q),\ 1 \leq j \leq N$, with $H(P_j) \leq B$ but not everywhere on X. Hence (see (4))

$$N \leq 3(m^2a + b) \ll \left(B^{\frac{2}{3m^2}} + m^2\right) \log B$$

$$\Rightarrow N(B) \ll m^r \left(B^{\frac{2}{3m^2}} + m^2\right) \log B.$$

This completes the proof of Theorem 1.

Acknowledgment

I wish to thank my supervisor Per Salberger for introducing me to the problem and giving me many useful suggestions.

Appendix A

In this appendix we record the following more precise version of a result in [4].

Theorem 9. *Let C be any smooth plane cubic curve and r be the rank of Jac(C). Let $m_l = \frac{l^2-4l-4}{8l^2+8l}$ for $l \geq 1$. Then*

$$N(B) \ll \begin{cases} (\log B)^{-(m_1+\ldots+m_r)+r/2}, & \text{if } 1 \leq r < 16; \\ (\log B)^{r/2}, & \text{if } r \geq 16, \end{cases}$$

with an absolute implied constant. In particular, $N(B) \ll (\log B)^{1+r/2}$ for all r.

Proof. The proof is just a careful re-examination of the argument of Heath-Brown and Testa [4]. This argument is based on a result of David [1] about successive minima for the quadratic form Q corresponding to the canonical height on Jac(C). As in [4] (see (11)),

$$N(B) \ll \prod_{j \leq r} \max\left\{1, 4\frac{\sqrt{c\log\, B}}{M_j}\right\}, \tag{9}$$

where c is an absolute constant and M_j, $j = 1, ..., r$ are successive minima of $\sqrt{Q}$.

We now recall Corollary 1.6 from [1], which shows that if D is the discriminant of $\text{Jac}(C)$ then for all $l \leq r$, $M_l \gg (\log|D|)^{m_l}$, where $m_l = \frac{l^2-4l-4}{8l^2+8l}$. Note that David's result refers to the successive minima for Q, while we have given the corresponding results for $\sqrt{Q}$.

In Lemma 4 we saw that $\|F\| \ll B^{30}$ if $N(B) > 9$. There is, therefore, in that case an absolute constant k such that

$$\max\left\{1, 4\frac{\sqrt{c\log B}}{M_j}\right\} \leq k(\log B)^{1/2}(\log|D|)^{-m_j}$$

for $j = 1, ..., r$ since $|m_j| < 1/2$ and $\log|D| \ll \log B$. Hence, if $N(B) > 9$, then from (9) we obtain

$$N(B) \ll k^r(\log B)^{r/2}(\log|D|)^{-(m_1+...+m_r)}. \tag{10}$$

If $1 \leq r < 16$, then $-(m_1 + ... + m_r) > 0$ and the assertion holds. If $r \geq 16$, let $D_0 = \exp(k^{1/m_{16}})$. Then $k(\log|D|)^{-m_j} \leq 1$ for $j > 16$ and $|D| \geq D_0$. Hence

$$N(B) \ll (\log B)^{r/2}(\log|D|)^{-(m_1+...+m_{16})} \ll (\log B)^{r/2}$$

as $-(m_1 + ... + m_{16}) < 0$. When $|D| \leq D_0$ the rank r is bounded and we get the same assertion by (10).

So in any case, $N(B) \ll (\log\, B)^{r/2}$, if $r \geq 16$. It should thereby be noted that Elkies (see [2]) has shown that there exist elliptic curves of rank $r \geq 28$.

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