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Computational Section

On perfect powers that are sums of cubes of a five term arithmetic progression

Alejandro Argáez-García [a,b,*,1]

[a] *Facultad de Ingeniería Química, Universidad Autónoma de Yucatán, Periférico Norte Kilómetro 33.5, Tablaje Catastral 13615 Chuburna de Hidalgo Inn, Mérida, Yucatán, C.P. 97200, Mexico*

[b] *Centro de Invetigación en Matemáticas, A.C. - Unidad Mérida, Parque Científico y Tecnológico de Yucatán Km 5.5, Carretera Sierra Papacal - Chuburná Puerto Sierra Papacal, Mérida, Yucatán, C.P. 97302 Mexico*

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A B S T R A C T

We prove that the equation $(x-2r)^3+(x-r)^3+x^3+(x+r)^3+(x+2r)^3=y^p$ only has solutions which satisfy $xy=0$ for $1 \le r \le 10^6$ and $p \ge 5$ prime.

* Correspondence to: Facultad de Ingeniería Química, Universidad Autónoma de Yucatán, Periférico Norte Kilómetro 33.5, Tablaje Catastral 13615 Chuburna de Hidalgo Inn, Mérida, Yucatán, C.P. 97200, Mexico.

E-mail addresses: alejandroargaezg@gmail.com, alejandro.argaez@correo.uady.mx, alejandro.argaez@cimat.mx.

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1. Introduction

Finding perfect powers that are sums of terms in an arithmetic progression has received much interest; recent contributions can be found in [1], [2], [3], [4], [5], [9], [10], [14], [15], [16], [17], [19], [21], [22] and [23].

We are interested in solutions (x, y, r) where x, y and r are coprime.

In this paper, we prove the following:

Theorem 1.1. *Let $p \geq 5$ be a prime. The equation*

$$(x-2r)^3 + (x-r)^3 + x^3 + (x+r)^3 + (x+2r)^3 = y^p \quad x, r, y, p \in \mathbb{Z},\ \gcd(x, r) = 1, \tag{1}$$

with $0 < r \leq 10^6$ only has solutions which satisfy $xy = 0$.

The restriction $\gcd(x, r) = 1$ is natural one, for otherwise it is easy to construct artificial solutions by scaling. We use a combination of techniques in the resolution of (1), the main ones being; a result of Mignotte based on linear form in logarithms ([8, Chapter 12, p. 423]); the method of Chabauty ([18,20,11]), the theorem due to Bilu, Hanrot and Voutier on primitive divisors ([6]), as well as some various elementary techniques.

2. Background

Here, we record some essential Theorems and Lemmas which are necessary for the computations in Section 4.

Theorem 2.1. *(Mignotte) Assume that the exponential Diophantine inequality*

$$| ax^n - by^n | \leq c, \qquad \text{with } a, b, c \in \mathbb{Z}_{\geq 0} \text{ and } a \neq b$$

has a solution in strictly positive integers x and y with $\max\{x, y\} > 1$. Let $A = \max\{a, b, 3\}$. Then

$$n \leq \max\left\{3\log(1.5 \mid c/b \mid), \frac{7400 \log A}{\log\left(1 + \log A/ \log(\mid a/b \mid)\right)}\right\}.$$

2.1. Criteria for eliminating equations of signature $(p, 2p, 2)$

We first apply a descent to equation (1) in Section 3. We are left with equations of the form:

$$aw_2^p - bw_1^{2p} = cr^2 \tag{2}$$

where p is an odd prime and a, b, c are positive integers satisfying $\gcd(a, b, c) = 1$.

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The criteria to follow in order to determine whether (2) has solutions was previously presented in [13,4,1]. We start with the following lemma that gives us a criterion for the nonexistence of solutions.

Lemma 2.2. *Let $p \geq 3$ be a prime. Let a, b and c be positive integers such that* $\gcd(a, b, c) = 1$. *Let $q = 2kp + 1$ be a prime that does not divide a. Define*

$$\mu(p, q) = \{\eta^{2p} \ : \ \eta \in \mathbb{F}_q\} = \{0\} \cup \{\zeta \in \mathbb{F}_q^* \ : \ \zeta^k = 1\} \tag{3}$$

and

$$B(p, q) = \left\{\zeta \in \mu(p, q) \ : \ ((b\zeta + c)/a)^{2k} \in \{0, 1\}\right\} .$$

If $B(p, q) = \emptyset$, then equation (2) does not have integral solutions.

2.2. Local solubility

After reducing the number of equations using Lemma 2.2, we give the next step and use classical local solubility methods to conclude nonexistence of solutions for many tuples (a, b, c, p) in equation (2).

These methods work as follows: let $g = \mathrm{Rad}(\gcd(a, c))$ and suppose that $g > 1$. Recall the condition $\gcd(a, b, c) = 1$. Then $g \mid w_1$, and we can write $w_1 = gw_1'$. Thus

$$aw_2^p - bg^{2p}{w_1'}^{2p} = c.$$

Removing a factor of g from the coefficients, we obtain

$$a'{w_2}^p - b'{w_1'}^{2p} = c',$$

where $a' = a/c$ and $c' = c/g < c$. Similarly, if $h = \gcd(b, c) > 1$, we obtain

$$a'{w_2'}^p - b'{w_1}^{2p} = c',$$

where $c' = c/h < c$. Applying these operations repeatedly, we arrive at an equation of the form

$$A\rho^p - B\sigma^{2p} = C \tag{4}$$

where A, B, C are now pairwise coprime. A necessary condition for the existence of solutions is that for any odd prime $q \mid A$, the residue $-BC$ modulo q is a square. Besides this basic test, we also check for local solubility at the primes dividing A, B, C, and all primes $q \leq 19$.

2.3. A descent

If local techniques previously presented failed to rule out solutions to equation (2) for particular coefficients and exponent (a, b, c, p) then we may perform a further descent to rule out solutions. With A, B, C as in (4) we let

$$B' = \prod_{\mathrm{ord}_q(B) \text{ is odd}} q.$$

Thus $BB' = v^2$. Write $AB' = u$ and $CB' = mn^2$ with m squarefree. Rewrite (4) as

$$(v\sigma^p + n\sqrt{-m})(v\sigma^p - n\sqrt{-m}) = u\rho^p.$$

Let $K = \mathbb{Q}(\sqrt{-m})$ and $\mathcal{O}$ be its ring of integers. Let $\mathfrak{S}$ contain the prime ideals of $\mathcal{O}$ that divide u or $2n\sqrt{-m}$. Clearly $(v\sigma^p + n\sqrt{-m})K^{*p}$ belongs to the "p-Selmer group"

$$K(\mathfrak{S}, p) = \{\epsilon \in K^*/K^{*p} \ : \ \mathrm{ord}_{\mathcal{P}}(\epsilon) \equiv 0 \mod p \text{ for all } \mathcal{P} \notin \mathfrak{S}\}.$$

This is an $\mathbb{F}_p$-vector space of finite dimension can be computed by `Magma` using the command `pSelmerGroup`. Let

$$\mathcal{E} = \{\epsilon \in K(\mathfrak{S}, p) \ : \ Norm(\epsilon)/u \in \mathbb{Q}^{*p}\}.$$

It follows that

$$v\sigma^p + n\sqrt{-m} = \epsilon\eta^p, \tag{5}$$

where $\eta \in K^*$ and $\epsilon \in \mathcal{E}$.

We end up with the last criteria.

Lemma 2.3. *Let $\mathfrak{q}$ be a prime ideal of K. Suppose one of the following holds:*

(i) $\mathrm{ord}_{\mathfrak{q}}(v)$, $\mathrm{ord}_{\mathfrak{q}}(n\sqrt{-m})$, $\mathrm{ord}_{\mathfrak{q}}(\epsilon)$ *are pairwise distinct modulo p;*
(ii) $\mathrm{ord}_{\mathfrak{q}}(2v)$, $\mathrm{ord}_{\mathfrak{q}}(\epsilon)$, $\mathrm{ord}_{\mathfrak{q}}(\overline{\epsilon})$ *are pairwise distinct modulo p;*
(iii) $\mathrm{ord}_{\mathfrak{q}}(2n\sqrt{-m})$, $\mathrm{ord}_{\mathfrak{q}}(\epsilon)$, $\mathrm{ord}_{\mathfrak{q}}(\overline{\epsilon})$ *are pairwise distinct modulo p.*

Then there is no $\sigma \in \mathbb{Z}$ and $\eta \in K$ satisfying (5).

Lemma 2.4. *Let $q = 2kp + 1$ be a prime. Suppose $q\mathcal{O} = \mathfrak{q}_1\mathfrak{q}_2$ where $\mathfrak{q}_1$, $\mathfrak{q}_2$ are distinct, and such that $\mathrm{ord}_{\mathfrak{q}_j}(\epsilon) = 0$ for $j = 1, 2$. Let*

$$\chi(p, q) = \{\eta^p \ : \ \eta \in \mathbb{F}_q\}.$$

Let

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$$C(p,q) = \{\zeta \in \chi(p,q) \ : \ ((v\zeta + n\sqrt{-m})/\epsilon)^{2k} \equiv 0 \ or \ 1 \mod \mathfrak{q}_j \ for \ j = 1, \ 2\}.$$

Suppose $C(p,q) = \emptyset$. *Then there is no* $\sigma \in \mathbb{Z}$ *and* $\eta \in K$ *satisfying* (5).

2.4. Thue equations

Finally for the remaining equations that couldn't be ruled out, they can be considered Thue equations by letting $\sigma = w_2$ and $\tau = w_1^2$

$$a\sigma^p - b\tau^p = c$$

where the exponent is a prime p. We use `Magma's` Thue solver [7] and `PARI/GP's` *thueinit*, *thue* commands [12] as the final test to determine whether the equations have solutions.

2.5. Prime divisors of Lehmer sequences

A *Lehmer pair* is a pair α, β of algebraic integers such that $(\alpha+\beta)^2$ and $\alpha\beta$ are nonzero coprime rational integers and α/β is not a root of unity. The *Lehmer sequence* associated to the Lehmer pair (α, β) is

$$\tilde{u}_n = \tilde{u}_n(\alpha,\beta) = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & n \text{ is odd} \\ \dfrac{\alpha^n - \beta^n}{\alpha^2 - \beta^2} & n \text{ is even} \end{cases} \tag{6}$$

A prime p is called a *primitive divisor* of $\tilde{u}_n$ if it divides $\tilde{u}_n$ but does not divide $(\alpha^2 - \beta^2) \cdot \tilde{u}_1 \cdots \tilde{u}_{n-1}$. We now state the following celebrated theorem due to Bilu, Hanrot and Voutier [6].

Theorem 2.5. *Let* α, β *be a Lehmer pair. Then* $\tilde{u}_n(\alpha,\beta)$ *has a primitive root for all* $n > 30$ *and for all prime* $n > 13$.

It will be necessary to use Lehmer pairs to prove for certain values of r that (1) has no solutions.

3. Descent to eight cases

Equation (1) can be rewritten as: $5x(x^2 + 6r^2) = y^p$. Letting $y = 5w$, we can rewrite as:

$$x(x^2 + 6r^2) = 5^{p-1}w^p. \tag{7}$$

Note that $\gcd(x, x^2 + 6r^2) \in \{1, 2, 3, 6\}$ depending on whether $2, 3$ divides x or not. Thus, we are now able to divide into eight cases.

Case	Conditions on x	Descent equations	Equation of signature $(p, p, 2)$
1	$5 \mid x$ and $6 \nmid x$	$x = 5^{p-1}w_1^p$ $x^2 + 6r^2 = w_2^p$	$w_2^p - 5^{2p-2}w_1^{2p} = 6r^2$
2	$5 \mid x$ and $2 \mid x$ and $3 \nmid x$	$x = 2^{p-1}5^{p-1}w_1^p$ $x^2 + 6r^2 = 2w_2^p$	$w_2^p - 2^{2p-3}5^{2p-2}w_1^{2p} = 3r^2$
3	$5 \mid x$ and $3 \mid x$ and $2 \nmid x$	$x = 3^{p-1}5^{p-1}w_1^p$ $x^2 + 6r^2 = 3w_2^p$	$w_2^p - 3^{2p-3}5^{2p-2}w_1^{2p} = 2r^2$
4	$5 \mid x$ and $6 \mid x$	$x = 6^{p-1}5^{p-1}w_1^p$ $x^2 + 6r^2 = 6w_2^p$	$w_2^p - 6^{2p-3}5^{2p-2}w_1^{2p} = r^2$
5	$5 \nmid x$ and $6 \nmid x$	$x = w_1^p$ $x^2 + 6r^2 = 5^{p-1}w_2^p$	$5^{p-1}w_2^p - w_1^{2p} = 6r^2$
6	$5 \nmid x$ and $2 \mid x$ and $3 \nmid x$	$x = 2^{p-1}w_1^p$ $x^2 + 6r^2 = 2 \cdot 5^{p-1}w_2^p$	$5^{p-1}w_2^p - 2^{2p-3}w_1^{2p} = 3r^2$
7	$5 \nmid x$ and $3 \mid x$ and $2 \nmid x$	$x = 3^{p-1}w_1^p$ $x^2 + 6r^2 = 3 \cdot 5^{p-1}w_2^p$	$5^{p-1}w_2^p - 3^{2p-3}w_1^{2p} = 2r^2$
8	$5 \nmid x$ and $6 \mid x$	$x = 6^{p-1}w_1^p$ $x^2 + 6r^2 = 6 \cdot 5^{p-1}w_2^p$	$5^{p-1}w_2^p - 6^{2p-3}w_1^{2p} = r^2$

4. Tables of computations

In this section, we apply the Theorems and Lemmas of Section 2 in order to fully resolve equation (1) and prove Theorem 1.1.

4.1. Case 1

We first apply Theorem 2.1 to obtain the bound $p \le 34365$ when $|r| \le 1.7 \times 10^{2486}$. Thus, when we focus on $5 \le p \le 34365$ and $1 \le r \le 10^6$ we have the following table containing the obtained information after computational calculations.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
5	258519	33667	52	0
7	102711	46794	5	0
11	2690	1364	1	0
13	5855	3044	1	0
17	752	415	0	0
19	1644	858	0	0
23	10	6	0	0
29	23	9	0	0
31	61	32	0	0
37	1	1	0	0
41	2	1	0	0
$43 \le p \le 34365$	0	0	0	0

4.2. Case 2

For $p = 5$ we have $w_2^5 - 2^7 \cdot 5^8 w_1^{10} = 3r^2$. Letting $X = w_2/w_1^2$ and $Y = 3r/w_1^5$ we obtain the hyperelliptic curve

$$Y^2 = 3X^5 - 3 \cdot 2^7 \cdot 5^8$$

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whose Jacobian has rank 1. We can show, using Magma, that $C(\mathbb{Q}) = \{\infty\}$ which implies the curve only has trivial solutions.

Using Theorem 2.1 we bound $p \leq 56565$ for $|r| \leq 6.8 \times 10^{4092}$. Thus, when we focus on $7 \leq p \leq 56565$ and $1 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
7	60261	21654	2	0
11	895	347	1	0
13	783	401	1	0
17	126	69	1	1
19	656	296	1	1
23	3	1	1	1
31	4	3	0	0
$37 \leq p \leq 56565$	0	0	0	0

The Thue equations that could not be solved by Magma, were solved by PARI/GP

p	r	Thue equation	Solution
17	3^8	$w_2^{17} - 2^{31} \cdot 5^{32} w_1^{34} = 3^{17}$	$(3, 0)$
19	3^9	$w_2^{19} - 2^{35} \cdot 5^{36} w_1^{38} = 3^{19}$	$(3, 0)$
23	3^{11}	$w_2^{23} - 2^{43} \cdot 5^{44} w_1^{46} = 3^{23}$	$(3, 0)$

using the commands *thueinit* and *thue*. Observe that the solutions found contradict the condition $w_1 \cdot w_2 \neq 0$. Moreover, observe that all the equations that were solved are of signature $(p, 2p, p)$.

4.3. Case 3

For $p = 5$ we have $w_2^5 - 3^7 \cdot 5^8 w_1^{10} = 2r^2$. Choosing the change of variable $X = w_2/w_1^2$ and $Y = 2r/w_1^5$ we obtain the hyperelliptic curve

$$Y^2 = 2X^5 - 2 \cdot 3^7 \cdot 5^8$$

whose Jacobian has rank 1. We can show, using Magma, that $C(\mathbb{Q}) = \{\infty\}$ which implies the curve only has trivial solutions.

Using Theorem 2.1 we bound $p \leq 69551$ for $|r| \leq 3.8 \times 10^{5881}$. Thus, when we focus on $7 \leq p \leq 69551$ and $1 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
7	18911	8045	3	0
11	1639	690	2	0
13	4059	3314	2	0
17	137	76	1	1
19	271	141	1	1
23	11	5	1	1
29	5	2	1	1
31	9	6	1	1
37	1	1	1	1
$41 \leq p \leq 69551$	0	0	0	0

The Thue equations that could not be solved by Magma, were solved by PARI/GP

p	r	Thue equation	Solution
17	2^8	$w_2^{17} - 3^{31} \cdot 5^{32} w_1^{34} = 2^{17}$	$(2, 0)$
19	2^9	$w_2^{19} - 3^{35} \cdot 5^{36} w_1^{38} = 2^{19}$	$(2, 0)$
23	2^{11}	$w_2^{23} - 3^{43} \cdot 5^{44} w_1^{38} = 2^{23}$	$(2, 0)$
29	2^{14}	$w_2^{29} - 3^{55} \cdot 5^{56} w_1^{38} = 2^{29}$	$(2, 0)$
31	2^{15}	$w_2^{31} - 3^{59} \cdot 5^{60} w_1^{38} = 2^{31}$	$(2, 0)$
37	2^{18}	$w_2^{37} - 3^{71} \cdot 5^{72} w_1^{38} = 2^{37}$	$(2, 0)$

using the commands *thueinit* and *thue*. Observe that the solutions found contradict the condition $w_1 \cdot w_2 \neq 0$. Moreover, observe that all the equations that were solved are of signature $(p, 2p, p)$.

4.4. Case 4

For $p = 5$ we have $w_2^5 - 6^7 \cdot 5^8 w_1^{10} = r^2$. Choosing the change of variable $X = w_2/w_1^2$ and $Y = r/w_1^5$ we obtain the hyperelliptic curve

$$Y^2 = X^5 - 6^7 \cdot 5^8$$

whose Jacobian has rank 1. We can show, using Magma, that $C(\mathbb{Q}) = \{\infty\}$ which implies the curve only has trivial solutions.

Using Theorem 2.1 we bound $p \leq 91751$ for $|r| \leq 1.9 \times 10^{6639}$. Thus, when we focus on $7 \leq p \leq 91751$ and $2 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
7	22887	12311	1	0
11	1469	588	0	0
13	237	97	0	0
17	205	118	0	0
19	2480	1273	0	0
23	2	2	0	0
29	2	2	0	0
31	9	4	0	0
37	2	1	0	0
$41 \leq p \leq 91751$	0	0	0	0

We noticed a remarkable behaviour when $r = 1$ and $7 \leq p \leq 91751$.

We observed that application of Lemma 2.2 and the local solubility tests were always failing for every p. Due to this situation, we decided to use *Lehmer pairs* to solve it.

Let $K = \mathbb{Q}(\sqrt{-6})$ and write $\mathcal{O}_K$ for its ring of integers. This has class group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We consider the equation $x^2 + 6 = 6w_2^p$ where $x = 30^{p-1}w_1^p$ and w_1, w_2 as before. Take

$$\begin{aligned} 6w_2^p &= x^2 + 6 \\ &= (x + \sqrt{-6})(x - \sqrt{-6}). \end{aligned}$$

It follows that

$$(x+\sqrt{-6})\mathcal{O}_K = \mathfrak{p}_2\mathfrak{p}_3\zeta^p$$

where $\mathfrak{p}_2 = (2, \sqrt{-6})$, $\mathfrak{p}_3 = (3, \sqrt{-6})$ are the primes above 2, 3 and ζ is an ideal in $\mathcal{O}_K$. We write

$$(x+\sqrt{-6})\mathcal{O}_K = 2^{(1-p)/2}3^{(1-p)/2}(\mathfrak{p}_2\mathfrak{p}_3\zeta)^p$$

and $\mathfrak{q} = \mathfrak{p}_2\mathfrak{p}_3$. It follows that $\mathfrak{q}\zeta = (\gamma) \in \mathcal{O}_K$ is principal and $\gamma = u + v\sqrt{-6} \in \mathcal{O}_K$ with $u, v \in \mathbb{Z}$. After a possible change of sign we obtain

$$x + \sqrt{-6} = \frac{\gamma^p}{6^{(p-1)/2}}$$

Subtracting and conjugating we obtain

$$\frac{\gamma^p}{6^{(p-1)/2}} - \frac{\overline{\gamma}^p}{6^{(p-1)/2}} = 2\sqrt{-6} \tag{8}$$

or equivalently

$$\frac{\gamma^p}{6^{p/2}} - \frac{\overline{\gamma}^p}{6^{p/2}} = 2i.$$

Let $L = \mathbb{Q}(\sqrt{-1}, \sqrt{6})$. Taking $\alpha = \gamma/\sqrt{6}$ and $\beta = \overline{\gamma}/\sqrt{6}$.

Lemma 4.1. *Let α and β as above. Then α and β are algebraic integers. Moreover $(\alpha+\beta)^2$ and $\alpha\beta$ are nonzero coprime rational integers and α/β is not a unit.*

Proof. Let $\mathfrak{q}\mathcal{O}_L = \sqrt{6}\mathcal{O}_L$. By definition $\mathfrak{q}|\gamma, \overline{\gamma}$ which implies that α and β are algebraic integers. Now we compute $(\alpha+\beta)^2 = 4u^2/3$.

Since $\mathfrak{p}_3|\sqrt{-6}$ we conclude that $\mathfrak{p}_3|u$ and so $3|u$. So $(\alpha+\beta)^2$ is a rational integer, i.e., $(\alpha+\beta)^2 \in \mathbb{Z}$. If $(\alpha+\beta)^2 = 0$ then $u = 0$, however this will imply that $30^p w_1^p = 0$ contradicting $w_1 \cdot w_2 \neq 0$. Thus $(\alpha+\beta)^2$ is a nonzero ration integer. Moreover $\alpha\beta = \gamma\overline{\gamma}/6$ is a nonzero rational integer since $3|u$ and $\mathfrak{p}_2|\gamma, \overline{\gamma}$.

We now check that $(\alpha+\beta)^2$ and $\alpha\beta$ are coprime. Suppose they are not coprime. Then there exists a prime $\mathfrak{p}$ of $\mathcal{O}_L$ which divides both. Then $\mathfrak{p}$ divides α and β and from the equation above $\mathfrak{p}$ divides $(w_2)\mathcal{O}_L$ and $2\sqrt{-6}$. We contradicted our assumption of (w_1, w_2) being a nontrivial coprime solution.

Finally, $\alpha/\beta = \gamma/\overline{\gamma} \in \mathcal{O}_K$ is not a root of unity because the only roots of unity in K are ±1 which implies $\gamma = \pm\overline{\gamma}$ implying either $u = 0$ or $v = 0$ which is a contradiction. □

From Lemma 4.1 we have that the pair (α, β) is Lehmer pair and we denote by $\tilde{u}_k$ the associate Lehmer sequence. Substituting we see that

$$\left(\frac{\alpha-\beta}{2i}\right)\left(\frac{\alpha^p-\beta^p}{\alpha-\beta}\right)=1.$$

Hence, we get

$$\left(\frac{\alpha^p-\beta^p}{\alpha-\beta}\right)=1,$$

thus $v = \pm 1$. By Theorem 2.5, we immediately deduce that $p \in \{5, 7, 11, 13\}$. For a given prime p, using equation (8), we see that u is a root of the polynomial:

$$\frac{1}{2\cdot\sqrt{-6}\cdot\sqrt{6}^{(p-1)/2}}((u+v\sqrt{-6})^p-(u-v\sqrt{-6})^p)-1.$$

Computing the roots of these polynomials, we find that there are no solutions.

4.5. Case 5

For $p = 5$ we have $5^4 w_2^5 - w_1^{10} = 6r^2$. Choosing the change of variable $X = w_2/w_1^2$ and $Y = 6r/w_1^5$ we obtain the hyperelliptic curve

$$Y^2 = 6\cdot 5^4 X^5 - 6$$

whose Jacobian has rank 0. We can show, using Magma, that $C(\mathbb{Q}) = \{\infty\}$ which implies the curve only has trivial solutions.

Using Theorem 2.1 we bound $p \le 17183$ for $|r| \le 8.3 \times 10^{1242}$. Thus, when we focus on $7 \le p \le 17183$ and $1 \le r \le 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
7	99944	55754	71	0
11	3345	1417	0	0
13	871	446	0	0
17	1042	552	0	0
19	892	487	0	0
23	20	9	0	0
29	2	2	0	0
31	23	11	0	0
37	1	1	0	0
$41 \le p \le 17183$	0	0	0	0

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4.6. Case 6

Using Theorem 2.1 we bound $p \leq 9101$ for $|r| \leq 9.4 \times 10^{657}$. Thus, when we focus on $5 \leq p \leq 9101$ and $1 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
5	94300	10723	35	0
7	28798	22060	0	0
11	757	388	0	0
13	7923	4238	0	0
17	95	47	0	0
19	686	310	0	0
23	1	1	0	0
29	5	4	0	0
31	14	7	0	0
37	2	2	0	0
$41 \leq p \leq 9101$	0	0	0	0

4.7. Case 7

Using Theorem 2.1 we bound $p \leq 22515$ for $|r| \leq 5.4 \times 10^{1628}$. Thus, when we focus on $5 \leq p \leq 22515$ and $1 \leq r \leq 10^6$ we have the following table.

Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
5	36897	5233	94	0
7	26109	12665	16	0
11	4079	1637	0	0
13	1649	854	0	0
17	945	606	0	0
19	686	459	0	0
23	12	5	0	0
29	5	3	0	0
31	35	21	0	0
37	1	0	0	0
41	1	0	0	0
$43 \leq p \leq 22515$	0	0	0	0

4.8. Case 8

For $p = 5$ we have $5^4 w_2^5 - 6^7 w_1^{10} = r^2$. Choosing the change of variable $X = w_2/w_1^2$ and $Y = r/w_1^5$ we obtain the hyperelliptic curve

$$Y^2 = 5^4 X^5 - 6^7$$

whose Jacobian has rank 0. We can show, using Magma, that $C(\mathbb{Q}) = \{\infty\}$ which implies the curve only has trivial solutions.

Using Theorem 2.1 we bound $p \leq 44855$ for $|r| \leq 2.8 \times 10^{3245}$. Thus, when we focus on $7 \leq p \leq 44855$ and $1 \leq r \leq 10^6$ we have the following table.

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Exponent p	Number of eqns surviving Lemma 2.2	Number of eqns surviving local solubility tests	Number of eqns surviving further descent	Thue eqns not solved by Magma
7	18672	6518	0	0
11	904	381	0	0
13	561	225	0	0
17	122	89	0	0
19	620	375	0	0
23	2	1	0	0
29	1	0	0	0
31	20	11	0	0
41	1	0	0	0
$43 \leq p \leq 44855$	0	0	0	0

This completes the proof of Theorem 1.1.

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References

[1] A. Argáez-García, V. Patel, Perfect powers that are sums of cubes of a three term arithmetic progression, arXiv:1711.06407, 2017.

[2] M.A. Bennett, K. Győry, Á. Pintér, On the Diophantine equation $1^k + 2^k + \cdots + x^k = y^n$, Compos. Math. 140 (6) (2004) 1417–1431.

[3] M.A. Bennett, V. Patel, S. Siksek, Superelliptic equations arising from sums of consecutive powers, Acta Arith. 172 (4) (2016) 377–393.

[4] M.A. Bennett, V. Patel, S. Siksek, Perfect powers that are sums of consecutive cubes, Mathematika 63 (1) (2017) 230–249.

[5] A. Bérczes, I. Pink, G. Savaş, G. Soydan, On the Diophantine equation $(x+1)^k + (x+2)^k + .. + (2x)^k = y^n$, J. Number Theory 183 (2018) 326–351.

[6] Yu. Bilu, G. Hanrot, P.M. Voutier, Existence of primitive divisors of Lucas and Lehmer numbers, J. Reine Angew. Math. 539 (2001) 75–122.

[7] W. Bosma, J. Cannon, C. Playoust, The magma algebra system I: the user language, J. Symbolic Comput. 24 (1997) 235–265, See also, http://magma.maths.usyd.edu.au/magma/.

[8] H. Cohen, Number Theory, vol. II. Analytic and Modern Tools, Graduate Texts in Mathematics, vol. 240, Springer, New York, 2007.

[9] A. Koutsianas, V. Patel, Perfect powers that are sums of squares in a three term arithmetic progression, Int. J. Number Theory 14 (10) (2018).

[10] D. Kundu, V. Patel, Perfect powers that are sums of squares of an arithmetic progression, arXiv:, 2018.

[11] W. McCallum, B. Poonen, The method of Chabauty and Coleman, in: Explicit Methods in Number Theory, in: Panor. Synthèses, vol. 36, Soc. Math. France, Paris, 2012, pp. 99–117.

[12] The PARI Group, PARI/GP version 2.11.0, Univ. Bordeaux, 2018, http://pari.math.u-bordeaux.fr/.

[13] V. Patel, Perfect Powers that are Sums of Consecutive Like Powers, Doctoral thesis, 2017.

[14] V. Patel, Perfect powers that are sums of consecutive squares, C. R. Math. Acad. Sci. Soc. R. Can. 40 (2) (2018) 33–38.

[15] V. Patel, S. Siksek, On powers that are sums of consecutive like powers, Res. Number Theory 3 (2) (2017) 7.

[16] Á. Pintér, A note on the equation $1^k + 2^k + \cdots + (x-1)^k = y^m$, Indag. Math. (N.S.) 8 (1997) 119–123.

Please cite this article in press as: A. Argáez-García, On perfect powers that are sums of cubes of a five term arithmetic progression, J. Number Theory (2019), https://doi.org/10.1016/j.jnt.2019.02.014

[17] Á. Pintér, On the power values of power sums, J. Number Theory 125 (2007) 412–423.

[18] Samir Siksek, Chabauty and the Mordell-Weil sieve, in: Advances on Superelliptic Curves and Their Applications, in: NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., vol. 41, IOS, Amsterdam, 2015, pp. 194–224.

[19] G. Soydan, On the Diophantine equation $(x+1)^k+(x+2)^k+..+(lx)^k=y^n$, Publ. Math. Debrecen 91 (2017) 369–382.

[20] M. Stoll, Implementing 2-descent for Jacobians of hyperelliptic curves, Acta Arith. 98 (3) (2001) 245–277.

[21] Z. Zhang, M. Bai, On the Diophantine equation $(x+1)^2+(x+2)^2+\cdots+(x+d)^2=y^n$, Funct. Approx. Comment. Math. 49 (2013) 73–77.

[22] Z. Zhang, On the Diophantine equation $(x-1)^k+x^k+(x+1)^k=y^n$, Publ. Math. Debrecen 85 (2014) 93–100.

[23] Z. Zhang, On the Diophantine equation $(x-d)^4+x^4+(x+d)^4=y^n$, Int. J. Number Theory 13 (9) (2017) 2229–2243, https://doi.org/10.1142/S1793042117501214.

Please cite this article in press as: A. Argáez-García, On perfect powers that are sums of cubes of a five term arithmetic progression, J. Number Theory (2019), https://doi.org/10.1016/j.jnt.2019.02.014