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J. Ma, J. Wu, F. Zhao

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ON A GENERALISATION OF BORDELLÈS-DAI-HEYMAN-PAN-SHPARLINSKI'S CONJECTURE

J. MA, J. WU & F. ZHAO

Abstract. Let f be an arithmetic function satisfying some simple conditions. The aim of this paper is to establish an asymptotical formula for the quantity

$$S_f(x) := \sum_{n\leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big)$$

for $x \to \infty$, where $[t]$ is the integral part of the real number t. This generalises some recent results of Bordellès, Dai, Heyman, Pan & Shparlinski ($f = \varphi =$ the Euler function), and of Zhao & Wu ($f = \sigma =$ the sum-of-divisors function).

1. Introduction

As usual, we write

$$\varphi(n) := \text{the Euler function,}$$
$$\sigma(n) := \text{the sum-of-divisors function.}$$

Denote by $[t]$ the integral part of the real number t, by $\log_2$ the iterated logarithm and by γ the Euler constant, respectively. Motivated by the following well-known results

$$\sum_{n\leqslant x} \Big[\frac{x}{n}\Big] = x\log x + (2\gamma - 1)x + O(x^{1/3}), \tag{1.1}$$

$$\sum_{n\leqslant x} \varphi(n) = \frac{3}{\pi^2}x^2 + O(x(\log x)^{2/3}(\log_2 x)^{4/3}), \tag{1.2}$$

for $x \to \infty$, Bordellès, Dai, Heyman, Pan and Shparlinski [2] proposed to investigate the asymptotical behaviour of the summation function

$$S_\varphi(x) := \sum_{n\leqslant x} \varphi\Big(\Big[\frac{x}{n}\Big]\Big),$$

as $x \to \infty$. With the help of Bourgain's new exponent pair [3], they proved the following inequalities

$$\Big(\frac{2629}{4009}\cdot\frac{6}{\pi^2} + o(1)\Big)x\log x \leqslant S_\varphi(x) \leqslant \Big(\frac{2629}{4009}\cdot\frac{6}{\pi^2} + \frac{1380}{4009} + o(1)\Big)x\log x \tag{1.3}$$

and conjectured that

$$S_\varphi(x) \sim \frac{6}{\pi^2}x\log x \quad \text{as} \quad x \to \infty. \tag{1.4}$$

The bounds in (1.3) have been sharpened by Wu [13] using the van der Corput inequality [5] simply. Very recently Zhai [15] resolved the conjecture (1.4) by combining the Vinogradov

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method with an idea of Goswami [4]. More precisely, he established the asymptotic formula

$$S_\varphi(x) = \frac{6}{\pi^2} x \log x + O\big(x(\log x)^{2/3}(\log_2 x)^{1/3}\big) \tag{1.5}$$

and showed that the error term in (1.5) is $\Omega(x)$. Denote by $\mu(n)$ the Möbius function. Define $\mathrm{id}(n) = n$ and $\mathbb{1}(n) = 1$ for all integers $n \geqslant 1$. Then $\varphi = \mathrm{id} * \mu$ and $\sigma = \mathrm{id} * \mathbb{1}$. In Zhai's proof of (1.5), the well-known inequality

$$\sum_{n \leqslant x} \mu(n) \ll x \exp\big\{-c\sqrt{\log x}\big\} \qquad (x \geqslant 1) \tag{1.6}$$

plays a key role, where $c > 0$ is a positive constant. Clearly a such bound is not true for $\mathbb{1}$. Since $\varphi(n)$ and $\sigma(n)$ often have similar properties, it seems interesting to study the analogy of (1.5) for $\sigma(n)$. By refining Zhai's approach, Wu and Zhao [16] succeeded to prove that

$$S_\sigma(x) := \sum_{n \leqslant x} \sigma\Big(\Big[\frac{x}{n}\Big]\Big) = \frac{\pi^2}{6} x \log x + O\big(x(\log x)^{2/3}(\log_2 x)^{4/3}\big) \tag{1.7}$$

and that the error term in (1.7) also is $\Omega(x)$ as $x \to \infty$.

In this paper, we would like to consider a general case of (1.5) and (1.7), and to give a uniform treatment. Inspiring some ideas from Liu and Wu [8], let r_1, r_2, r_3 be three increasing functions defined on $[1, \infty)$ such that

$$1 \leqslant r_j(x) \ll x^{\eta_j} \quad (j = 1, 2, 3), \qquad r_3(x) \to \infty \tag{1.8}$$

for $x \geqslant 1$, where $\eta_j \in (0, 1)$ are constants. Let f be an arithmetic function and let g be determined by the relation $f = \mathrm{id} * g$. We propose the following conditions on f:

$$|f(n)| \ll n r_1(n) \quad (n \geqslant 1), \tag{1.9}$$

$$\sum_{n \leqslant x} |g(n)| \ll x r_2(x) \quad (x \geqslant 1), \tag{1.10}$$

$$\sum_{n \leqslant x} g(n) = D_g x + O(x/r_3(x)) \quad (x \geqslant 1), \tag{1.11}$$

where D_g is a constant (eventually equal to 0).

Our main result is as follows.

Theorem 1.1. (i) *Let f be an arithmetic function satisfying the conditions* (1.9), (1.10) *and* (1.11). *Then for any constant $A > 0$, we have*

$$S_f(x) := \sum_{n \leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big) = C_f x \log x + O_A(x\mathcal{R}(x, z)), \tag{1.12}$$

uniformly for $x \geqslant 3$ and $1 \leqslant z \leqslant \exp\{A^{1/3}(\log x)^{2/3}(\log_2 x)^{1/3}\}$, where $C_f := \sum_{n=1}^\infty \frac{g(n)}{n^2}$ and

$$\mathcal{R}(x, z) := (\log x)^{2/3}(\log_2 x)^{1/3} r_1(x) + \frac{r_2(x/z)}{(\log x)^A} + \frac{r_2(x/z)\log x}{z} + \frac{z \log x}{r_3(\sqrt{x}/z)}. \tag{1.13}$$

Here the implied constant depends on A only.

(ii) *Let f be an arithmetic function satisfying the condition* (1.11) *and there is a positive constant $c < 1$ such that one of the following two conditions*

$$|f(p-1)| < cf(1)p \qquad \text{or} \qquad |f(p-1)| > c^{-1}f(1)p > 0$$

holds for an infinity of primes p. Then the error term of (1.12) *is $\Omega(x)$.*

Remark 1. When the sizes of $r_j(x)$ $(1 \leqslant j \leqslant 3)$ are conveneble, (1.12) of Theorem 1.1 gives, with a suitable choice of parameter z, an asymptotic formula of $S_f(x)$. Otherwise we have an upper bound.

As applications of Theorem 1.1, we consider four special arithmetic functions:

– the Euler function $\varphi(n)$;
– the alternating sum-of-divisors function $\beta(n)$;
– the sum-of-divisors function $\sigma(n)$;
– the Dedekind function $\Psi(n)$.

Let $\Omega(n)$ be the number of all prime factors of n, then the Liouville function is defined by $\lambda(n) := (-1)^{\Omega(n)}$. We have the following relations:

$$\varphi = \mathrm{id} * \mu, \qquad \beta = \mathrm{id} * \lambda, \qquad \sigma = \mathrm{id} * \mathbb{1}, \qquad \Psi = \mathrm{id} * \mu^2. \tag{1.14}$$

We have the following corollaries.

Corollary 1.2. (i) *The asymptotic formula* (1.5) *is true. The error term of* (1.5) *is* $\Omega(x)$.
(ii) *For* $x \to \infty$, *we have*

$$\sum_{n \leqslant x} \beta\Big(\Big[\frac{x}{n}\Big]\Big) = \frac{\pi^2}{15} x \log x + O\big(x(\log x)^{2/3}(\log_2 x)^{1/3}\big). \tag{1.15}$$

The error term of (1.15) *is* $\Omega(x)$.

Corollary 1.3. (i) *The asymptotic formula* (1.7) *is true. The error term of* (1.7) *is* $\Omega(x)$.
(ii) *For* $x \to \infty$, *we have*

$$\sum_{n \leqslant x} \Psi\Big(\Big[\frac{x}{n}\Big]\Big) = \frac{15}{\pi^2} x \log x + O\big(x(\log x)^{2/3}(\log_2 x)^{4/3}\big). \tag{1.16}$$

The error term of (1.16) *is* $\Omega(x)$.

Remark 2. (i) Different from [15], our proof of (1.5) does not need Theorem 1 of Liu [7]:

$$\sum_{n \leqslant x} \varphi(n) = \frac{3}{\pi^2} x^2 + O\big(x(\log x)^{2/3}(\log_2 x)^{1/3}\big),$$

which is a slight improvement of Walfisz' asymptotical formula (1.2). (see [12, Chapter 7])
(ii) Some related works can be found in [1, 9, 10, 14, 17].

2. Some lemmas

2.1. Two preliminary lemmas.

This subsection is devoted to cite two lemmas. The first one is [16, Lemma 2.3] (see also [11, Lemma 2.6]), which is a consequence of Karatsuba's estimate for trigonometric sums by Vinogradov's method [6]. This will play a key role in the proof of Theorem 1.1.

Lemma 2.1. *Define* $\psi(t) := t - [t] - \frac{1}{2}$ *for* $t \in \mathbb{R}$. *There are two absolute positive constants* c_1 *and* c_2 *such that the inequality*

$$\sum_{N \leqslant n < N'} \frac{1}{n} \psi\Big(\frac{x}{n}\Big) \ll \mathrm{e}^{-c_1(\log N)^3/(\log x)^2} (\log N)^3/(\log x)^2$$

holds uniformly for $x \geqslant 10$, $\exp\{c_2(\log x)^{2/3}\} \leqslant N \leqslant x^{2/3}$ *and* $N < N' \leqslant 2N$.

The second lemma is [16, Lemma 2.4].

Lemma 2.2. [16, Lemma 2.4] *Let $\psi(t)$ be defined as in Lemma 2.1 and $F(t) := (1/t)\psi(x/t)$. Denote by $V_F[z_1, z_2]$ the total variation of F on $[z_1, z_2]$. Then we have*

$$V_F[z_1, z_2] \ll x/z_1^2 + 1/z_1$$

uniformly for $2 \leqslant z_1 < z_2 \leqslant x$, where the implied constant is absloute.

2.2. **An expression on the mean value of f.**

Lemma 2.3. *Assume that the arithmetic function $f(n)$ satisfies the conditions* (1.11) *and* (1.10)*. Then we have*

$$\sum_{n\leqslant x} f(n) = \frac{1}{2}C_f x^2 - D_g x\frac{(z-[z])^2+[z]}{2z} + O\Big(\frac{xz}{r_3(x/z)} + \frac{xr_2(x/z)}{z}\Big) - \Delta_g(x,z) \tag{2.1}$$

uniformly for $x \geqslant 2$ and $1 \leqslant z \leqslant x^{1/3}$, where

$$\Delta_g(x,z) := x\sum_{d\leqslant x/z}\frac{g(d)}{d}\psi\Big(\frac{x}{d}\Big). \tag{2.2}$$

Further for $x \geqslant 2$ we have

$$\sum_{n\leqslant x} f(n) = \frac{1}{2}C_f x^2 + O(x(\log x)r_2(x)). \tag{2.3}$$

Proof. Since $f(n) = \sum_{dm=n} g(d)m$, we can apply the hyperbole principe to write

$$\sum_{n\leqslant x} f(n) = \sum_{dm\leqslant x} g(d)m = S_1 + S_2 - S_3, \tag{2.4}$$

where

$$S_1 := \sum_{d\leqslant x/z}\sum_{m\leqslant x/d} g(d)m, \quad S_2 := \sum_{m\leqslant z}\sum_{d\leqslant x/m} g(d)m, \quad S_3 := \sum_{d\leqslant x/z}\sum_{m\leqslant z} g(d)m.$$

Firstly we have

$$\begin{aligned} S_1 &= \frac{1}{2}\sum_{d\leqslant x/z} g(d)\Big[\frac{x}{d}\Big]\Big(\Big[\frac{x}{d}\Big]+1\Big) \\ &= \frac{1}{2}\sum_{d\leqslant x/z} g(d)\Big(\frac{x}{d} - \psi\Big(\frac{x}{d}\Big) - \frac{1}{2}\Big)\Big(\frac{x}{d} - \psi\Big(\frac{x}{d}\Big) + \frac{1}{2}\Big) \\ &= \frac{1}{2}\sum_{d\leqslant x/z} g(d)\Big\{\Big(\frac{x}{d}\Big)^2 - 2\frac{x}{d}\psi\Big(\frac{x}{d}\Big) + O(1)\Big\} \\ &= \frac{x^2}{2}\sum_{d\leqslant x/z}\frac{g(d)}{d^2} - \Delta_g(x,z) + O\Big(\frac{xr_2(x/z)}{z}\Big) \end{aligned}$$

where we have used the condition (1.10) for bounding the contribution of the term $O(1)$. On the other hand, by (1.11), a simple partial integration leads to

$$\sum_{d\leqslant x/z}\frac{g(d)}{d^2} = C_f - \int_{x/z}^{\infty} t^{-2}\,\mathrm{d}(D_g t + O(t/r_3(t))) = C_f - D_g\frac{z}{x} + O\Big(\frac{z}{xr_3(x/z)}\Big).$$

Inserting this into the preceding formula, we find that

$$S_1 = \frac{1}{2}C_f x^2 - \frac{1}{2}D_g xz + O\Big(\frac{xz}{r_3(x/z)} + \frac{xr_2(x/z)}{z}\Big) - \Delta_g(x,z). \tag{2.5}$$

In view of the condition (1.11), it follows that

$$S_2 = \sum_{m\leqslant z} m\Big\{D_g\frac{x}{m} + O\Big(\frac{x}{mr_3(x/m)}\Big)\Big\} = D_g x[z] + O\Big(\frac{xz}{r_3(x/z)}\Big), \tag{2.6}$$

$$S_3 = \Big\{D_g\frac{x}{z} + O\Big(\frac{x}{zr_3(x/z)}\Big)\Big\}\frac{[z]([z]+1)}{2} = D_g x\frac{[z]([z]+1)}{2z} + O\Big(\frac{xz}{r_3(x/z)}\Big). \tag{2.7}$$

Now (2.1) follows from (2.4), (2.5), (2.6) and (2.7).

Finally, taking $z = 1$ in (2.1) yields

$$\sum_{n\leqslant x} f(n) = \frac{1}{2}C_f x^2 + O(xr_2(x)) - \Delta_g(x,1).$$

On the other hand, using the hypothesis (1.10), a simple partial integration leads to

$$\Delta_g(x,1) \ll x\sum_{d\leqslant x}|g(d)|/d \ll x(\log x)r_2(x).$$

Inserting this into the preceding formula, we obtain (2.3). □

2.3. **Bound on an average of Δ_g.**

Lemma 2.4. *Assume that the arithmetic function $f(n)$ satisfies the hypothesis* (1.10)*. Let $A > 0$ be a constant and let $N_0 := \exp\{((A+3)/c_1)^{1/3}(\log x)^{2/3}(\log_2 x)^{1/3}\}$, where c_1 is given as in Lemma 2.1. Let $\Delta_g(x,z)$ be defined by* (2.2)*. Then we have*

$$\Big|\sum_{N_0<n\leqslant\sqrt{x}} \Delta_g\Big(\frac{x}{n}, z\Big)\Big| + \Big|\sum_{N_0<n\leqslant\sqrt{x}} \Delta_g\Big(\frac{x}{n} - 1, z\Big)\Big| \ll_A \frac{xr_2(x/z)}{(\log x)^A} + \frac{xr_2(x/z)\log x}{z} \tag{2.8}$$

uniformly for $x \geqslant 10$ and $2 \leqslant z \leqslant N_0^{1/2}$.

Proof. Denote by $\Delta_{g,1}(x,z)$ and $\Delta_{g,2}(x,z)$ two sums on the left-hand side of (2.8), respectively. By (2.2) of Lemma 2.3, we can write

$$\begin{aligned}\Delta_{g,1}(x,z) &= x\sum_{N_0<n\leqslant\sqrt{x}}\sum_{d\leqslant x/(nz)}\frac{g(d)}{dn}\psi\Big(\frac{x}{dn}\Big)\\ &= x\sum_{d\leqslant x/(N_0 z)}\frac{g(d)}{d}\sum_{N_0<n\leqslant\min(\sqrt{x},x/(dz))}\frac{1}{n}\psi\Big(\frac{x}{dn}\Big)\\ &= x\Delta_{g,1}^{\dagger}(x,z) + x\Delta_{g,1}^{\sharp}(x,z),\end{aligned}$$

where

$$\Delta_{g,1}^{\dagger}(x,z) := \sum_{d\leqslant x/(N_0 z)}\frac{g(d)}{d}\sum_{N_0<n\leqslant(x/d)^{2/3}}\frac{1}{n}\psi\Big(\frac{x}{dn}\Big),$$

$$\Delta_{g,1}^{\sharp}(x,z) := \sum_{d\leqslant x/(N_0 z)}\frac{g(d)}{d}\sum_{(x/d)^{2/3}<n\leqslant\min\{\sqrt{x},x/(dz)\}}\frac{1}{n}\psi\Big(\frac{x}{dn}\Big).$$

For $0 \leqslant k \leqslant (\log((x/d)^{2/3}/N_0))/\log 2$, let $N_k := 2^k N_0$ and define

$$\mathfrak{S}_k(d) := \sum_{N_k < n \leqslant 2N_k} \frac{1}{n}\psi\Big(\frac{x}{dn}\Big).$$

Noticing that $N_0 \leqslant N_k \leqslant (x/d)^{2/3}$, we can apply Lemma 2.1 to derive that

$$\mathfrak{S}_k(d) \ll \mathrm{e}^{-\vartheta((\log N_k)^3/(\log(x/d))^2)}$$

with $\vartheta(t) := c_1 t - \log t$. It is clear that $\vartheta(t)$ is increasing on $[c_1, \infty)$. On the other hand, for $k \geqslant 0$ and $d \leqslant x/(N_0 z)$ we have

$$(\log N_k)^3/(\log(x/d))^2 \geqslant (\log N_0)^3/(\log x)^2 = ((A+3)/c_1)\log_2 x.$$

Thus

$$\begin{aligned}\vartheta((\log N_k)^3/(\log(x/d))^2) &\geqslant \vartheta(((A+3)/c_1)\log_2 x)\\ &= (A+3)\log_2 x - \log(((A+3)/c_1)\log_2 x)\\ &\geqslant (A+2)\log_2 x,\end{aligned}$$

which implies that $\mathfrak{S}_k(d) \ll (\log x)^{-A-2}$. Inserting this into the expression of $\Delta^{\dagger}_{g,1}(x,z)$ and using the condition (1.10), we find that

$$\begin{aligned}\Delta^{\dagger}_{g,1}(x,z) &\ll \sum_{d\leqslant x/(N_0 z)} \frac{|g(d)|}{d} \sum_{2^k N_0 \leqslant (x/d)^{2/3}} |\mathfrak{S}_k(d)|\\ &\ll \frac{1}{(\log x)^{A+1}} \sum_{d\leqslant x/z} \frac{|g(d)|}{d} \ll \frac{r_2(x/z)}{(\log x)^A}.\end{aligned} \tag{2.9}$$

Next we bound $\Delta^{\sharp}_{g,1}(x,z)$. It is well known that, if $F(t)$ is a real function with bounded variation on each interval $[n, n+1]$, we have

$$\sum_{N_1 < n \leqslant N_2} F(n) = \int_{N_1}^{N_2} F(t)\,\mathrm{d}t + \tfrac{1}{2}\big(F(N_1) + F(N_2)\big) + O\big(V_F[N_1, N_2]\big),$$

where $N_1, N_2 \in \mathbb{Z}^+$ and $V_F[N_1, N_2]$ denotes the total variation of F on $[N_1, N_2]$. We apply this formula to

$$F(t) = \frac{1}{t}\psi\Big(\frac{(x/d)}{t}\Big), \quad N_1 = [(x/d)^{2/3}], \quad N_2 = [\min\{\sqrt{x},\, x/(dz)\}].$$

After Lemma 2.2, we have $V_F[N_1, N_2] \ll (x/d)^{-1/3}$. Putting $u = (x/d)/t$ we obtain

$$\begin{aligned}\sum_{(x/d)^{2/3} < n \leqslant \min\{\sqrt{x}, x/(dz)\}} \frac{1}{n}\psi\Big(\frac{x}{dn}\Big) &= \int_{\max(\sqrt{x}/d,\, z)}^{(x/d)^{1/3}} \frac{\psi(u)}{u}\,\mathrm{d}u + O\big((x/d)^{-1/3}\big)\\ &\ll z^{-1} + (x/d)^{-1/3}.\end{aligned} \tag{2.10}$$

Using (2.10) and (1.10), a simple partial integration allows us to derive that

$$\begin{aligned}\Delta^{\sharp}_{g,1}(x,z) &\ll \sum_{d\leqslant x/z} \frac{|g(d)|}{d}\big(z^{-1} + (x/d)^{-1/3}\big)\\ &\ll \frac{r_2(x/z)\log x}{z} + \frac{r_2(x/z)\log x}{(N_0 z)^{1/3}} \ll \frac{r_2(x/z)\log x}{z},\end{aligned} \tag{2.11}$$

since $z \leqslant N_0^{1/2}$. Combining (2.9) and (2.11), it follows that

$$\big|\Delta_{g,1}(x,z)\big| \ll \frac{xr_2(x/z)}{(\log x)^A} + \frac{xr_2(x/z)\log x}{z}.$$

Similarly we can prove the same bound for $\big|\Delta_{g,2}(x,z)\big|$. This completes the proof. □

3. Proof of Theorem 1.1

Putting $d = [x/n]$, we have $x/n - 1 < d \leqslant x/n$ and $x/(d+1) < n \leqslant x/d$. Following Goswami [4], we write

$$\begin{aligned} S_f(x) &= \sum_{d\leqslant x} f(d) \sum_{x/(d+1)<n\leqslant x/d} 1 \\ &= \sum_{dn\leqslant x} f(d) - \sum_{(d+1)n\leqslant x} f(d) \\ &= \sum_{dn\leqslant x} (f(d) - f(d-1)), \end{aligned} \tag{3.1}$$

where we have set that $f(0) = 0$. By the hyperbole principe, we can decompose $S_f(x)$ into three parts:

$$S_f(x) = \mathcal{S}_1(x) + \mathcal{S}_2(x) - \mathcal{S}_3(x), \tag{3.2}$$

where

$$\begin{aligned} \mathcal{S}_1(x) &:= \sum_{d\leqslant\sqrt{x},\, dn\leqslant x} (f(d) - f(d-1)), \\ \mathcal{S}_2(x) &:= \sum_{n\leqslant\sqrt{x},\, dn\leqslant x} (f(d) - f(d-1)), \\ \mathcal{S}_3(x) &:= \sum_{d\leqslant\sqrt{x},\, n\leqslant\sqrt{x}} (f(d) - f(d-1)). \end{aligned}$$

In view of the hypothesis (1.9), we can derive that

$$\mathcal{S}_3(x) = [\sqrt{x}]f([\sqrt{x}]) \ll xr_1(x). \tag{3.3}$$

For evaluating $\mathcal{S}_1$, we write

$$\begin{aligned} \mathcal{S}_1(x) &= \sum_{d\leqslant\sqrt{x}} (f(d) - f(d-1))\Big[\frac{x}{d}\Big] \\ &= x\sum_{d\leqslant\sqrt{x}} \frac{f(d) - f(d-1)}{d} + O\Big(\sum_{d\leqslant\sqrt{x}} |f(d) - f(d-1)|\Big). \end{aligned} \tag{3.4}$$

With the help of the hypothesis (1.9) and (2.3) of Lemma 2.3, it follows that

$$\begin{aligned} \sum_{d\leqslant\sqrt{x}} \frac{f(d) - f(d-1)}{d} &= \sum_{d\leqslant\sqrt{x}} \frac{f(d)}{d^2} - \sum_{d\leqslant\sqrt{x}} \frac{f(d)}{d^2(d+1)} \\ &= \int_{1-}^{\sqrt{x}} t^{-2}\,\mathrm{d}\big(\tfrac{1}{2}C_f t^2 + O(tr_2(t)\log t)\big) + O(1) \\ &= \frac{1}{2}C_f \log x + O(1). \end{aligned}$$

By the hypothesis (1.9), we have

$$\sum_{d\leqslant\sqrt{x}}|f(d)-f(d-1)|\leqslant 2\sum_{d\leqslant\sqrt{x}}|f(d)|\ll\sum_{d\leqslant\sqrt{x}}dr_1(d)\ll xr_1(x).$$

Inserting these estimates into (3.4), we find that

$$\mathcal{S}_1(x)=\frac{1}{2}C_f x\log x+O(xr_1(x)). \tag{3.5}$$

Finally we evaluate $\mathcal{S}_2$. Let $N_0:=\exp\{((A+3)/c_1)^{1/3}(\log x)^{2/3}(\log_2 x)^{1/3}\}$, where c_1 is the constant given as in Lemma 2.1. We write

$$\mathcal{S}_2(x)=\mathcal{S}_2^{\dagger}(x)+\mathcal{S}_2^{\sharp}(x), \tag{3.6}$$

where

$$\mathcal{S}_2^{\dagger}(x):=\sum_{n\leqslant N_0,\,dn\leqslant x}(f(d)-f(d-1)),$$
$$\mathcal{S}_2^{\sharp}(x):=\sum_{N_0<n\leqslant\sqrt{x},\,dn\leqslant x}(f(d)-f(d-1)).$$

Using the hypothesis (1.9), we have

$$\mathcal{S}_2^{\dagger}(x)=\sum_{n\leqslant N_0}f([x/n])\ll x\sum_{n\leqslant N_0}r_1(x/n)/n\ll x(\log x)^{2/3}(\log_2 x)^{1/3}r_1(x).$$

On the other hand, (2.1) of Lemma 2.3 allows us to derive that

$$\sum_{d\leqslant x}f(d)-\sum_{d\leqslant x-1}f(d)=C_f x+O\Big(\frac{xz}{r_3(x/z)}+\frac{xr_2(x/z)}{z}\Big)-\Delta_g(x,z)+\Delta_g(x-1,z).$$

Thus

$$\begin{aligned}\mathcal{S}_2^{\sharp}(x)&=\sum_{N_0<n\leqslant\sqrt{x}}\Big\{C_f\frac{x}{n}+O\Big(\frac{xz}{nr_3(\sqrt{x}/z)}+\frac{xr_2(x/z)}{nz}\Big)-\Delta_g\Big(\frac{x}{n},z\Big)+\Delta_g\Big(\frac{x}{n}-1,z\Big)\Big\}\\&=\frac{1}{2}C_f x\log x+O\Big(x(\log x)^{2/3}(\log_2 x)^{1/3}+\frac{xz\log x}{r_3(\sqrt{x}/z)}+\frac{xr_2(x/z)\log x}{z}\Big)\\&\quad-\Delta_{g,1}(x,z)+\Delta_{g,2}(x,z),\end{aligned}$$

where

$$\Delta_{g,1}(x,z):=\sum_{N_0<n\leqslant\sqrt{x}}\Delta_g\Big(\frac{x}{n},z\Big)\ll\frac{xr_2(x/z)}{(\log x)^A}+\frac{x(\log x)r_2(x/z)}{z},$$
$$\Delta_{g,2}(x,z):=\sum_{N_0<n\leqslant\sqrt{x}}\Delta_g\Big(\frac{x}{n}-1,z\Big)\ll\frac{xr_2(x/z)}{(\log x)^A}+\frac{x(\log x)r_2(x/z)}{z},$$

thanks to Lemma 2.4. Inserting these estimates into (3.6), we find that

$$\mathcal{S}_2(x)=\frac{1}{2}C_f x\log x+O(x\mathcal{R}(x,z)), \tag{3.7}$$

where $\mathcal{R}(x,z)$ is defined as in (1.13)

Now the required result (1.12) follows from (3.2), (3.3), (3.5) and (3.7).

Finally we prove the second assertion. Let $E(x)$ be the error term of (1.12), i.e.

$$S_f(x)=C_f x\log x+E(x),$$

and define $E^*(x) := \max\{|E(x)|, |E(x-1)|\}$. Firstly we suppose that

$$|f(p-1)| < cf(1)p \tag{3.8}$$

holds for an infinity of primes p. In view of (3.1), for each prime p we can write

$$\begin{aligned}\sum_{d\mid p}(f(d)-f(d-1)) &= S_f(p)-S_f(p-1)\\ &= C_f p\log p - C_f(p-1)\log(p-1)+E(p)-E(p-1)\\ &\leqslant 2E^*(p)+O(\log p).\end{aligned} \tag{3.9}$$

On the other hand, our hypothesis (3.8) and (1.11) allow us to deduce that

$$\begin{aligned}\sum_{d\mid p}(f(d)-f(d-1)) &= f(p)-f(p-1)+f(1)\\ &= g(1)p+g(p)f(1)-f(p-1)+f(1)\\ &\geqslant (1-c)f(1)p+O(p/r_3(p))\\ &\geqslant \tfrac{1}{2}(1-c)f(1)p\end{aligned} \tag{3.10}$$

for an infinity of primes p. Combining (3.9) and (3.10), we find that

$$E^*(p) \geqslant \tfrac{1}{5}(1-c)f(1)p$$

for an infinity of primes p.

Next we suppose that

$$|f(p-1)| > c^{-1}f(1)p > 0 \tag{3.11}$$

holds for an infinity of primes p. Similar to (3.9), for each prime p we can write

$$\sum_{d\mid p}(f(d)-f(d-1)) \geqslant -2E^*(p)+O(\log p). \tag{3.12}$$

Similar to (3.10), our hypothesis (3.11) and (1.11) allow us to deduce that

$$\sum_{d\mid p}(f(d)-f(d-1)) \leqslant -\tfrac{1}{2}(c^{-1}-1)f(1)p \tag{3.13}$$

for an infinity of primes p. Combining (3.12) and (3.13), we find that

$$E^*(p) \geqslant \tfrac{1}{5}(c^{-1}-1)f(1)p > 0$$

for an infinity of primes p. □

4. Proof of Corollary 1.2

4.1. **Proof of** (1.5).

Since $\varphi = \mathrm{id} * \mu$, we have $g = \mu$ and the following well-known bound

$$\sum_{n\leqslant x}\mu(n) \ll \frac{x}{(\log x)^2} \qquad (x \geqslant 2).$$

Thus $\varphi(n)$ verifies the conditions (1.9), (1.10) and (1.11) with $D_\mu = 0$ and

$$r_1(x) = 1, \qquad r_2(x) = 1, \qquad r_3(x) = \log^2(3x),$$

Zhai's (1.5) follows from Theorem 1.1 thanks to the choice of $z = \log^{1/3}(3x)$.

For each odd prime p, we can write $p-1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\varphi(p-1) = \varphi(2^\nu)\varphi(m) \leqslant 2^{\nu-1}m < \tfrac{1}{2}p$$

for all odd primes p and Theorem 1.1(ii) implies that the error term of (1.5) is $\Omega(x)$.

4.2. **Proof of** (1.15)**.**

In this case, we have $g = \lambda$ and the following well-known bound

$$\sum_{n\leqslant x} \lambda(n) \ll \frac{x}{(\log x)^2} \qquad (x \geqslant 2).$$

Thus the function $\beta(n)$ verifies the conditions (1.9), (1.10) and (1.11) with $D_\beta = 0$ and

$$r_1(x) = 1, \qquad r_2(x) = 1, \qquad r_3(x) = \log^2(3x).$$

Now (1.15) follows from Theorem 1.1 thanks to the choice of $z = \log^{1/3}(3x)$.

For all primes p and integers $\nu \geqslant 1$, we have

$$\beta(p^\nu) = \sum_{0\leqslant j\leqslant \nu} (-1)^{\nu-j} p^j = \frac{p^{\nu+1} + (-1)^\nu}{p+1} \leqslant \begin{cases} \frac{3}{4} 2^\nu & \text{if } p = 2, \\ p^\nu & \text{otherwise.} \end{cases}$$

For each odd prime p, we can write $p - 1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\beta(p-1) = \beta(2^\nu)\beta(m) \leqslant \tfrac{3}{4}(p-1) \leqslant \tfrac{3}{4}p$$

for all odd primes p and Theorem 1.1(ii) implies that the error term of (1.5) is $\Omega(x)$.

5. Proof of Corollary 1.3

5.1. **Proof of** (1.7)**.**

In this case, we have $g = \mathbb{1}$ and

$$\sum_{n\leqslant x} \mathbb{1}(n) = x + O(1) \qquad (x \geqslant 2).$$

Thus the function $\sigma(n)$ verifies the conditions (1.9), (1.10) and (1.11) with $D_{\mathbb{1}} = 1$ and

$$r_1(x) = \log_2(3x) + 1, \qquad r_3(x) = \sqrt{x}, \qquad r_2(x) = 1.$$

Wu-Zhao's (1.7) follows from Theorem 1.1 thanks to the choice of $z = \log^{1/3}(3x)$.

For all odd primes p, we have

$$\sigma(p-1) \geqslant (p-1) + \tfrac{1}{2}(p-1) + 1 > \tfrac{5}{4}p.$$

Thus Theorem 1.1(ii) implies that the error term of (1.7) is $\Omega(x)$.

5.2. **Proof of** (1.16)**.**

In this case, we have $g = \mu^2$ and

$$\sum_{n\leqslant x} \mu(n)^2 = (6/\pi^2)x + O(\sqrt{x}) \qquad (x \geqslant 2).$$

Thus the function $\Psi(n)$ verifies the conditions (1.9), (1.10) and (1.11) with $D_{\mu^2} = 6/\pi^2$ and

$$r_1(x) = \log_2(3x) + 1, \qquad r_2(x) = 1, \qquad r_3(x) = \sqrt{x}.$$

Now (1.16) follows from Theorem 1.1 thanks to the choice of $z = \log^{1/3}(3x)$.

For all primes p and integers $\nu \geqslant 1$, we have

$$\Psi(p^\nu) = p^\nu + p^{\nu-1} \geqslant \begin{cases} \frac{3}{2} 2^\nu & \text{if } p = 2, \\ p^\nu & \text{otherwise.} \end{cases}$$

For each odd prime p, we can write $p-1 = 2^\nu m$ with $2 \nmid m$. Thus

$$\Psi(p-1) = \gamma(2^\nu)\gamma(m) \geqslant \tfrac{3}{2}(p-1) \geqslant \tfrac{5}{4}p$$

for all odd primes $p \geqslant 7$ and Theorem 1.1(ii) implies that the error term of (1.16) is $\Omega(x)$.

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Jing Ma, School of Mathematics, Jilin University, Changchun 130012, P. R. China
Email address: `jma@jlu.edu.cn`

Jie Wu, CNRS LAMA 8050, Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est Créteil, 94010 Créteil Cedex, France
Email address: `jie.wu@math.cnrs.fr`

Feng Zhao, Department of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Jinshui E Road, Zhengzhou, 450046 Henan, P. R. China
Email address: `zhaofeng@ncwu.edu.cn`