



Symmetric recurrence relations and binomial transforms

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ABSTRACT

We consider a family of sums which satisfy symmetric recurrence relations. A sufficient and necessary condition for the existence of such recurrence relations is given. Let us call a pair of sequence (a_n, b_n) a binomial pair if a_n is the binomial transform of b_n . We give some ways of constructing new binomial pairs from old ones. We further generalize the binomial transform by adding a parameter and show that the generalized binomial transform is an involution.

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1. Introduction

Chen and Sun [2] presented an interesting approach to proving Bernoulli number identities by using the extended Zeilberger's algorithm. We observe that many identities in [2] are of the form

$$\sum_{k=0}^m f(m, k) a_{n+k} = \sum_{k=0}^n f(n, k) b_{m+k}. \quad (1.1)$$

In this paper, we present a way to prove and construct this kind of identities. We further give a generalization of this kind of identities.

Our approach is based on the following observations. To prove identity (1.1), it suffices to prove the following two assertions.

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(1) The sum

$$S(m, n) = \sum_{k=0}^m f(m, k) a_{n+k} \quad (1.2)$$

satisfies a symmetric recurrence relation

$$\alpha S(m, n) + S(m+1, n) + S(m, n+1) = 0, \quad (1.3)$$

where α is a constant independent of m, n .

(2) The identity holds for $m = 0$.

In Section 2, we characterize those coefficients $f(m, k)$ such that (1.3) holds for an arbitrary sequence $\{x_k\}_{k \geq 0}$. More precisely, we have

$$f(m, k) = f(0, 0)(-1)^m \binom{m}{k} \alpha^{m-k}.$$

With this fact, assertion (2) becomes the identity

$$a_n = (-1)^n \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} b_k. \quad (1.4)$$

When $\alpha = -1$, the transform from $\{b_n\}_{n \geq 0}$ to $\{a_n\}_{n \geq 0}$ is called the binomial transform by Knuth [4]. For this reason, we call a pair of sequences (a_n, b_n) satisfying (1.4) a *binomial pair with parameter α* and we say a_n is the *binomial transform of b_n with parameter α* . In Section 3, we give some ways to construct new binomial pairs from the old ones. These results enable us to derive many identities of form (1.1). In particular, we obtain a simpler form of the results of Chen [1].

We then give a generalization of the above results. Let t be a non-negative integer. We show that

$$(-1)^m \sum_{k=0}^{m+n} \binom{m+tk}{m+n-k} \alpha^{m+n-k} a_k = (-1)^n \sum_{k=0}^{m+n} \binom{n+tk}{m+n-k} \alpha^{m+n-k} b_k$$

if and only if

$$a_n = (-1)^n \sum_{k=0}^n \frac{(2t+1)k+1}{tn+(t+1)k+1} \binom{(t+1)n+tk}{n-k} \alpha^{n-k} b_k. \quad (1.5)$$

We also show that the transform (1.5) from the sequence $\{b_n\}_{n \geq 0}$ to the sequence $\{a_n\}_{n \geq 0}$ is an involution.

2. Symmetric recurrence relations

In this section, we characterize the coefficients $f(m, k)$ such that the sum $S(m, n)$ given by (1.2) satisfies the symmetric recurrence relation (1.3). The result explains in some sense why the identities are of the form

$$(-1)^m \sum_{k=0}^m \binom{m}{k} \alpha^{m-k} a_{n+k} = (-1)^n \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} b_{m+k}.$$

Theorem 2.1. Let α be a constant. The following two statements are equivalent.

(1) For any sequence $\{a_k\}_{k \geq 0}$, the sum

$$S(m, n) = \sum_{k=0}^m f(m, k) a_{n+k}$$

satisfies

$$\alpha S(m, n) + S(m+1, n) + S(m, n+1) = 0. \quad (2.1)$$

(2) We have

$$f(m, k) = f(0, 0)(-1)^m \binom{m}{k} \alpha^{m-k}. \quad (2.2)$$

Proof. Noting that $S(m, n)$ depends only on the values of $f(m, k)$ with $m \geq k \geq 0$, we may set $f(m, k) = 0$ if $k > m$ or $k < 0$. By the definition of $S(m, n)$, we have

$$\begin{aligned} & \alpha S(m, n) + S(m+1, n) + S(m, n+1) \\ &= \sum_{k=0}^{m+1} (\alpha f(m, k) + f(m+1, k) + f(m, k-1)) x_{n+k}. \end{aligned}$$

Therefore, (2.1) holds for an arbitrary sequence $\{a_k\}_{k \geq 0}$ if and only if

$$\alpha f(m, k) + f(m+1, k) + f(m, k-1) = 0. \quad (2.3)$$

It is easy to check that (2.3) holds for

$$f(m, k) = f(0, 0)(-1)^m \binom{m}{k} \alpha^{m-k}.$$

On the other hand, let

$$F_m(x) = \sum_{k=0}^{\infty} f(m, k) x^k$$

be the generating function of $f(m, k)$. Then (2.3) implies that

$$F_{m+1}(x) + \alpha F_m(x) + x F_m(x) = 0.$$

Hence

$$F_m(x) = (-1)^m (x + \alpha)^m F_0(x) = f(0, 0)(-1)^m \sum_{k=0}^m \binom{m}{k} x^k \alpha^{m-k},$$

implying (2.2). $\square$

By the symmetry of the recurrence relation, we immediately derive by induction that

Corollary 2.2. *Let α be a constant. Then the identity*

$$(-1)^m \sum_{k=0}^m \binom{m}{k} \alpha^{m-k} a_{n+k} = (-1)^n \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} b_{m+k}$$

holds for any non-negative integers m and n if and only if

$$a_n = (-1)^n \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} b_k.$$

3. Binomial pairs

Let $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ be two sequences. Recall that (a_n, b_n) is called a binomial pair with parameter α if (1.4) holds. In this section, we will give some ways to construct binomial pairs.

Prodinger [6] used the exponential generating function to study the binomial transform of the Fibonacci numbers. Our approach is also based on the exponential generating functions. Let

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad g(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}, \quad (3.1)$$

be the exponential generating functions of $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$, respectively. By standard computation, we see that relation (1.4) is equivalent to

$$f(x) = g(-x)e^{-\alpha x}. \quad (3.2)$$

From (3.2), we obtain some properties on binomial pairs.

Theorem 3.1. *Let (a_n, b_n) be a binomial pair with parameter α . Then*

- (a) (b_n, a_n) is a binomial pair with parameter α .
- (b) Let s be a non-negative integer and

$$x_n = \begin{cases} \frac{n!}{(n-s)!} a_{n-s}, & n \geq s, \\ 0, & n < s, \end{cases} \quad y_n = \begin{cases} (-1)^s \frac{n!}{(n-s)!} b_{n-s}, & n \geq s, \\ 0, & n < s. \end{cases}$$

Then (x_n, y_n) is a binomial pair with parameter α .

- (c) Let s be a non-negative integer and

$$x_n = \frac{n!}{(n+s)!} a_{n+s} - (-1)^{s+n} \sum_{k=0}^{s-1} \frac{n!}{k!(n+s-k)!} \alpha^{n+s-k} b_k,$$

$$y_n = (-1)^s \frac{n!}{(n+s)!} b_{n+s}.$$

Then (x_n, y_n) is a binomial pair with parameter α .

(d) Let

$$A_n(t) = \sum_{k=0}^n \binom{n}{k} a_k t^{n-k}, \quad B_n(t) = \sum_{k=0}^n \binom{n}{k} b_k t^{n-k}.$$

Then $(A_n(\beta), B_n(\gamma))$ is a binomial pair with parameter $\alpha - \beta - \gamma$.

(e) $(a_{n+1}, -b_{n+1} - \alpha b_n)$ is a binomial pair with parameter α .

Proof.

(a) Noting that

$$f(x) = g(-x)e^{-\alpha x} \iff g(x) = f(-x)e^{-\alpha x},$$

we immediately obtain (a).

(b) Let $\tilde{f}(x) = x^s f(x)$ and $\tilde{g}(x) = (-x)^s g(x)$. We have

$$\tilde{f}(x) = x^s f(x) = x^s g(-x)e^{-\alpha x} = \tilde{g}(-x)e^{-\alpha x}.$$

By taking the coefficients of $\frac{x^n}{n!}$ in $\tilde{f}(x)$ and $\tilde{g}(x)$, we see that (b) holds.

(c) Let

$$g_s(x) = \sum_{k=0}^{s-1} b_k \frac{x^k}{k!} \quad \text{and} \quad \tilde{g}(x) = \frac{g(x) - g_s(x)}{(-x)^s}.$$

Let further $\tilde{f}(x) = \tilde{g}(-x)e^{-\alpha x}$. Since

$$\tilde{f}(x) = \frac{g(-x) - g_s(-x)}{x^s} e^{-\alpha x} = \frac{f(x)}{x^s} - \frac{g_s(-x)e^{-\alpha x}}{x^s},$$

we see that $\tilde{f}(x)$ is a power series whose coefficient of x^n equals

$$\frac{a_{n+s}}{(n+s)!} - (-1)^{s+n} \sum_{k=0}^{s-1} \frac{\alpha^{n+s-k}}{k!(n+s-k)!} b_k.$$

Noting further that the coefficient of x^n in $\tilde{g}(x)$ is $(-1)^s \frac{b_{n+s}}{(n+s)!}$, we thus obtain (c).

(d) We have

$$e^{\beta x} f(x) = e^{\beta x} g(-x)e^{-\alpha x} = e^{-\gamma x} g(-x)e^{-(\alpha-\beta-\gamma)x}.$$

Noting that $e^{\beta x} f(x)$ and $e^{\gamma x} g(x)$ are the exponential generating functions of the sequences $A_n(\beta)$ and $B_n(\gamma)$ respectively, we get (d).

(e) Since

$$f'(x) = e^{-\alpha x} (-g'(-x) - \alpha g(-x)),$$

we see that (e) holds. $\square$

Example. It is known (see, for example [7]) that (B_n, B_n) is a binomial pair with parameter 1, where B_n is the n -th Bernoulli number. Then by Theorem 3.1, we derive that

$$(nB_{n-1}, -nB_{n-1}), \quad \left(\frac{B_{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1}, -\frac{B_{n+1}}{n+1} \right), \quad (B_{n+1}, -B_{n+1} - B_n)$$

are binomial pairs with parameter 1 and $(B_n(s), B_n(t))$ is a binomial pair with parameter $1 - s - t$, where

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} x^{n-k} B_k$$

is the Bernoulli polynomial. Applying Theorem 3.1(b) further to the binomial pair $(B_{n+1}, -B_{n+1} - B_n)$, we derive that $(nB_n, n(B_n + B_{n-1}))$ is a binomial pair. Noting further that $(nB_{n-1}, -nB_{n-1})$ is a binomial pair, we derive that $n(2B_n + B_{n-1})$ is invariant under the binomial transform with parameter 1.

Combining Corollary 2.2 and Theorem 3.1, we obtain

Corollary 3.2. Let s be an integer and (a_n, b_n) be a binomial pair with parameter α . Then we have

$$\begin{aligned} & (-1)^m \sum_{k=0}^m \binom{m}{k} \alpha^{m-k} \frac{(n+k)!}{(n+s+k)!} a_{n+s+k} \\ &= (-1)^{n+s} \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} \frac{(m+k)!}{(m+s+k)!} b_{m+s+k} \\ & \quad + (-1)^{m+n+s} \sum_{k=0}^{s-1} \frac{n!(s-k)_m}{k!(m+n+s-k)!} \alpha^{m+n+s-k} b_k \end{aligned} \quad (3.3)$$

$$\begin{aligned} &= (-1)^{n+s} \sum_{k=0}^n \binom{n}{k} \alpha^{n-k} \frac{(m+k)!}{(m+s+k)!} b_{m+s+k} \\ & \quad - (-1)^{m+n} \sum_{k=0}^{s-1} \frac{m!(s-k)_n}{k!(m+n+s-k)!} \alpha^{m+n+s-k} a_k, \end{aligned} \quad (3.4)$$

where $(x)_m = x(x+1) \cdots (x+m-1)$ is the raising factorial. In the above sums, we regard the summand containing a_k or b_k with negative index k as zero.

Proof. When $s \leq 0$, by Theorem 3.1(b) and Corollary 2.2, we immediately get (3.3) and (3.4).

If $s > 0$, by Theorem 3.1(c) we have an extra term

$$(-1)^m \sum_{l=0}^m \binom{m}{l} \alpha^{m-l} (-1)^{s+n+l} \sum_{k=0}^{s-1} \frac{(n+l)!}{k!(n+l+s-k)!} \alpha^{n+l+s-k} b_k.$$

Denote the sum by $\sum_{l=0}^m \sum_{k=0}^{s-1} F(m, n, s, l, k)$. By Zeilberger's algorithm [5], we find that

$$\sum_{l=0}^{m+1} F(m+1, n, s, l, k) = -\alpha \frac{m+s-k}{m+n+1+s-k} \sum_{l=0}^m F(m, n, s, l, k).$$

Therefore,

$$\begin{aligned}\sum_{l=0}^m F(m, n, s, l, k) &= (-\alpha)^m \frac{(s-k)_m}{(n+1+s-k)_m} F(0, n, s, 0, k) \\ &= (-1)^{m+n+s} \frac{n!(s-k)_m}{k!(m+n+s-k)!} \alpha^{m+n+s-k} b_k.\end{aligned}$$

This completes the proof of (3.3). Now applying (3.3) for the binomial pair (b_n, a_n) and interchanging m and n , we arrive at (3.4). $\square$

Remark 1. As pointed by one of the referees, (a_n, b_n) is a binomial pair with parameter α if and only if $(\alpha^{-n}a_n, \alpha^{-n}b_n)$ is a binomial pair with parameter 1. Therefore, Corollary 3.2 is equivalent to its special case $\alpha = 1$.

Remark 2. Taking $\alpha = 1$ and substituting b_n with $(-1)^n \sum_{k=0}^n \binom{n}{k} a_k$, we see that the last term of (3.3) can be written as

$$\sum_{k=0}^{s-1} \sum_{l=k}^{s-1} \frac{(-1)^{m+n+s+l} n!(s-l)_m}{k!(l-k)!(m+n+s-l)!} a_k.$$

The identity (3.3) written in this form is equivalent to the result of Chen [1, Theorem 1.1]. Furthermore, by (3.4) we see that the last term in Theorem 1.1 of Chen can be simplified to

$$(-1)^{n+1} \sum_{j=0}^{s-1} \frac{m!(s-j)_n}{j!(m+n+s-j)!} a_j.$$

4. A generalization

In this section, we give a generalization of the binomial transform by introducing a new parameter t . The original binomial transform corresponds to the case $t = 0$.

Our main result can be stated as follows.

Theorem 4.1. Let t be a non-negative integer and $\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}$ be two sequences. The following three statements are equivalent.

(a) For any non-negative integers m, n , we have

$$(-1)^m \sum_{k=0}^{m+n} \binom{m+tk}{m+n-k} \alpha^{m+n-k} a_k = (-1)^n \sum_{k=0}^{m+n} \binom{n+tk}{m+n-k} \alpha^{m+n-k} b_k. \quad (4.1)$$

(b) For any non-negative integer n , we have

$$\sum_{k=0}^n \binom{tk}{n-k} \alpha^{n-k} a_k = (-1)^n \sum_{k=0}^n \binom{n+tk}{n-k} \alpha^{n-k} b_k. \quad (4.2)$$

(c) For any non-negative integer n , we have

$$a_n = (-1)^n \sum_{k=0}^n \frac{(2t+1)k+1}{tn+(t+1)k+1} \binom{(t+1)n+tk}{n-k} \alpha^{n-k} b_k. \quad (4.3)$$

Proof. Let

$$f(m, n, k, t) = (-1)^m \binom{m+tk}{m+n-k} \alpha^{m+n-k}.$$

We can verify that

$$\alpha f(m, n, k, t) + f(m+1, n, k, t) + f(m, n+1, k, t) = 0.$$

Therefore, for any sequence $\{x_k\}_{k \geq 0}$, the sum

$$S_t(m, n) = \sum_{k=0}^{m+n} f(m, n, k, t) x_k = (-1)^m \sum_{k=0}^{m+n} \binom{m+tk}{m+n-k} \alpha^{m+n-k} x_k$$

satisfies the symmetric recurrence relation

$$\alpha S_t(m, n) + S_t(m+1, n) + S_t(m, n+1) = 0.$$

Hence by induction on m , (4.1) holds if and only if (4.2) holds.

We next show that (4.3) implies (4.2). It is known that [3, (5.62)]

$$\sum_{i=0}^m \frac{u}{wi+u} \binom{wi+u}{i} \binom{w(m-i)+v}{m-i} = \binom{wm+u+v}{m}, \quad (4.4)$$

where $u, v, w \in \mathbb{C}$ are arbitrary complex numbers and the binomial coefficients are defined by

$$\binom{x}{k} = \frac{x(x-1) \cdots (x-k+1)}{k!}, \quad x \in \mathbb{C}.$$

Substituting

$$i = l - k, \quad m = n - k, \quad w = -t, \quad u = -(2t+1)k - 1, \quad v = tn$$

into (4.4), we obtain

$$\sum_{l=k}^n \frac{(2t+1)k+1}{tl+(t+1)k+1} \binom{-tl-(t+1)k-1}{l-k} \binom{tl}{n-l} = \binom{-(t+1)k-1}{n-k}.$$

Since $\binom{x}{k} = (-1)^k \binom{-x+k-1}{k}$, the above identity can be rewritten as

$$\sum_{l=k}^n (-1)^l \frac{(2t+1)k+1}{tl+(t+1)k+1} \binom{(t+1)l+tk}{l-k} \binom{tl}{n-l} = (-1)^n \binom{n+tk}{n-k}.$$

By relation (4.3), we derive that

$$\begin{aligned} \sum_{l=0}^n \binom{tl}{n-l} \alpha^{n-l} a_l &= \sum_{k=0}^n \alpha^{n-k} b_k \sum_{l=k}^n \binom{tl}{n-l} (-1)^l \frac{(2t+1)k+1}{tl+(t+1)k+1} \binom{(t+1)l+tk}{l-k} \\ &= (-1)^n \sum_{k=0}^n \binom{n+tk}{n-k} \alpha^{n-k} b_k. \end{aligned}$$

It remains to show that (4.2) also implies (4.3). Let $\{b_n\}_{n \geq 0}$ be an arbitrary sequence. Since the coefficient of a_n in the left hand side of (4.2) is 1, there exists a unique sequence $\{a_n^*\}_{n \geq 0}$ such that (4.2) holds. But we have proven that the sequence $\{a_n\}_{n \geq 0}$ given by (4.3) satisfies (4.2). Thus we must have $a_n^* = a_n$, completing the proof. $\square$

Here are some examples of the applications of Theorem 4.1.

Take $b_0 = 1$ and $b_k = 0$ for $k \geq 1$. Then by (4.3) we have

$$a_n = (-1)^n \frac{1}{tn+1} \binom{(t+1)n}{n} \alpha^n.$$

We thus derive that

$$(-1)^m \sum_{k=0}^{m+n} \binom{m+tk}{m+n-k} \frac{(-1)^k}{tk+1} \binom{(t+1)k}{k} = (-1)^n \binom{n}{m+n}. \quad (4.5)$$

Take $t = \alpha = 1$ and $b_k = \binom{-1/2}{k}$. By Zeilberger's algorithm, we find that

$$a_n = b_n = \binom{-\frac{1}{2}}{n}.$$

Therefore,

$$(-1)^m \sum_{k=0}^{m+n} \binom{m+k}{m+n-k} \binom{-\frac{1}{2}}{k} = (-1)^n \sum_{k=0}^{m+n} \binom{n+k}{m+n-k} \binom{-\frac{1}{2}}{k}. \quad (4.6)$$

Take $t = \alpha = 1$ and $b_k = \binom{1/2}{k} 4^k$. By Zeilberger's algorithm, we find that

$$a_n = (-1)^n \frac{7n+2}{(n+1)(n+2)} \binom{2n}{n}.$$

Therefore,

$$\begin{aligned} (-1)^m \sum_{k=0}^{m+n} \binom{m+k}{m+n-k} (-1)^k \frac{7k+2}{(k+1)(k+2)} \binom{2k}{k} \\ = (-1)^n \sum_{k=0}^{m+n} \binom{n+k}{m+n-k} \binom{\frac{1}{2}}{k} 4^k. \end{aligned} \quad (4.7)$$

Now we consider the sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ which satisfy the relation (4.3). We call (a_n, b_n) a *generalized binomial pair* with parameters (α, t) and we say a_n is the *generalized binomial transform* of b_n with parameters (α, t) .

As a consequence of Theorem 4.1, we find that the generalized binomial transform is an involution.

Corollary 4.2. Suppose that (a_n, b_n) is a generalized binomial pair. Then (b_n, a_n) is also a generalized binomial pair with the same parameters.

Proof. Suppose that (a_n, b_n) is a generalized binomial pair with parameters (α, t) . By Theorem 4.1, we have (4.1). Taking $n = 0$ in (4.1), we get

$$(-1)^m \sum_{k=0}^m \binom{m+tk}{m-k} \alpha^{m-k} a_k = \sum_{k=0}^m \binom{tk}{m-k} \alpha^{m-k} b_k,$$

which is equivalent to (4.2) by exchanging a_n and b_n . Since (4.2) is equivalent to (4.3), we derive that (b_n, a_n) is a generalized binomial pair with the same parameters. $\square$

An equivalent statement of Corollary 4.2 is as follows. The infinite matrix

$$A_{\alpha,t} = \left((-1)^n \frac{(2t+1)k+1}{tn+(t+1)k+1} \binom{(t+1)n+tk}{n-k} \alpha^{n-k} \right)_{n,k \geq 0} \quad (4.8)$$

is an involutory matrix, i.e., $A_{\alpha,t}^2 = I$, the identity matrix. We identify a column vector $\mathbf{a} = (a_0, a_1, \dots)^t$ with the sequence $\{a_n\}_{n \geq 0}$. Then a sequence $\{a_n\}_{n \geq 0}$ is invariant under the generalized binomial transform if and only if $A_{\alpha,t} \mathbf{a} = \mathbf{a}$. By Wang's result [8, Theorem 2.2 and Remark 2.3], we derive that

Corollary 4.3. A sequence $\{a_n\}_{n \geq 0}$ is invariant under the generalized binomial transform if and only if $\mathbf{a} = (A_{\alpha,t} + I) \mathbf{v}$ for some vector $\mathbf{v}$. In other words, the invariant space $\{\mathbf{a} : A_{\alpha,t} \mathbf{a} = \mathbf{a}\}$ is generated by the column vectors of $A_{\alpha,t} + I$.

Let $\delta_{n,k}$ be the Kronecker delta. By Corollary 4.3, we obtain that for any non-negative integer k , the sequence $\{a_n\}_{n \geq 0}$ given by

$$a_n = (-1)^n \frac{(2t+1)k+1}{tn+(t+1)k+1} \binom{(t+1)n+tk}{n-k} \alpha^{n-k} + \delta_{n,k}$$

is invariant under the generalized binomial transform.

We remark that the generalized binomial pair does not have similar properties to items (b)–(d) of Theorem 3.1.

Let s be a non-negative integer. An analogue of Theorem 3.1(b) can be stated as follows. There exist two sequences $\{\lambda_n\}_{n \geq 0}$ and $\{\mu_n\}_{n \geq 0}$ such that for any generalized binomial pair (a_n, b_n) , the pair $(\lambda_n a_{n-s}, \mu_n b_{n-s})$ is also a generalized binomial pair. Let us consider the special case of $t = 1$ and $s = \alpha = 1$. By the definition of generalized binomial pairs, we have

$$b_0 = a_0, \quad b_1 = -a_0 - a_1, \quad b_2 = 2a_0 + 4a_1 + a_2,$$

and

$$\mu_1 b_0 = -\lambda_1 a_0, \quad \mu_2 b_1 = 4\lambda_1 a_0 + \lambda_2 a_1, \quad \mu_3 b_2 = -14\lambda_1 a_0 - 7\lambda_2 a_1 - \lambda_3 a_2.$$

Since $\{a_n\}_{n \geq 0}$ can be chosen arbitrarily, we have $\lambda_2 = 4\lambda_1$ and $\lambda_3 = 7\lambda_1$. But this result contradicts to the condition that

$$b_3 = -5a_0 - 14a_1 - 7a_2 - a_3, \quad \mu_4 b_3 = 48\lambda_1 a_0 + 35\lambda_2 a_1 + 10\lambda_3 a_2 + \lambda_4 a_3.$$

A similar discussion shows that we do not have an analogue of Theorem 3.1(c).

Theorem 3.1(d) can be explained in terms of transform matrices as follows. Let

$$A_\alpha = \left((-1)^n \binom{n}{k} \alpha^{n-k} \right)_{n,k \geq 0} \quad \text{and} \quad B_\alpha = \left(\binom{n}{k} \alpha^{n-k} \right)_{n,k \geq 0}.$$

We have

$$B_\gamma A_\alpha B_{-\beta} = A_{\alpha-\beta-\gamma}.$$

Therefore, if $\mathbf{y} = A_\alpha \mathbf{x}$, then

$$B_\gamma \mathbf{y} = B_\gamma A_\alpha B_{-\beta} (B_\beta \mathbf{x}) = A_{\alpha-\beta-\gamma} (B_\beta \mathbf{x}),$$

which is equivalent to Theorem 3.1(d). Now let us replace A_α by the matrix $A_{\alpha,t}$ given by (4.8) and replace B_β by the matrix

$$B_{\alpha,t} = \left(\frac{(2t+1)k+1}{tn+(t+1)k+1} \binom{(t+1)n+tk}{n-k} \alpha^{n-k} \right)_{n,k \geq 0}.$$

It can be checked that, in general, $B_{\gamma,t} A_{\alpha,t} B_{-\beta,t} \neq A_{\alpha',t}$ for any α' .

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