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Asymptotic expansions for the psi function and the Euler-Mascheroni constant

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Abstract

Let $r \neq 0$ and $s \neq 0$ be two given real numbers. Chen [C.-P. Chen, Inequalities and asymptotic expansions for the psi function and the Euler-Mascheroni constant, J. Number Theory 163 (2016), 596-607.] obtained recursive relation for determining the coefficients $a_j(r,s)$ such that

$$\psi(x+1) \sim \ln x + \left(1 - \frac{1}{r}\right)\frac{1}{x} + \frac{1}{s}\ln\left(1 + \sum_{j=1}^{\infty}\frac{a_j(r,s)}{x^j}\right), \quad x \to \infty,$$

where ψ denotes the psi function. As a consequence, the asymptotic expansion for the Euler-Mascheroni constant was derived. In this paper, we provide an explicit formula for these coefficients in terms of the cycle indicator polynomial of symmetric group which is an important tool in enumerative combinatorics. Also using this tool, we directly obtain an alternative form of the recursive relation for determining the coefficients $a_j(r,s)$. Furthermore we describe their asymptotic behavior for the special case $r = 2$.

Keywords: Psi function; Euler-Mascheroni constant; Asymptotic expansion; Cycle indicator polynomial

Mathematics Subject Classification: 11Y60; 41A60

1. Introduction

It is well known that the Euler gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \quad x > 0$$

is one of the most important functions in mathematical analysis, especially in the area of special functions. It has a lot of applications in various diverse areas and it has been staying in the middle of attention of many authors in years. Many researches are

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devoted to establishing approximation formulas for the gamma function and the related psi function. A formula for approximation of $\Gamma(x)$ for large value of x is of special attraction. It is stated as follows

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \quad x \to \infty.$$

This formula was improved by an asymptotic series which is often called the Stirling series

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left\{\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right\}, \quad x \to \infty,$$

where B_i denotes the ith Bernoulli number defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{i=1}^{\infty} \frac{B_i}{i!} x^i.$$

The first Bernoulli numbers are $B_1 = 1/2$, $B_2 = 1/6$, $B_4 = -1/30$ with $B_{2i+1} = 0$, for each integer $i \geq 1$. The Laplace formula [1] for asymptotic expansion of the gamma function is

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots\right\}, \quad x \to \infty.$$

In [7], Chen and Lin proved that the gamma function has the following asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \sum_{j=1}^{\infty} \frac{b_j}{x^j}\right)^{x^l/r}, \quad x \to \infty,$$

where the coefficients b_j are given by

$$b_j = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{B_2}{1\cdot 2}\right)^{k_1}\left(\frac{B_3}{2\cdot 3}\right)^{k_2}\cdots\left(\frac{B_{j+1}}{j\cdot(j+1)}\right)^{k_j},$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+l)k_1 + (2+l)k_2 + \cdots + (j+l)k_j = j.$$

When $l = 0$, this result reduces to the main theorem in [6]. For more works on asymptotic expansions and approximations of the gamma function, one is referred to [4, 5, 8, 12–15, 17–22, 24–35, 38, 40] and references therein.

The logarithmic derivative of the gamma function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the psi function which is connected to the Euler-Mascheroni constant and harmonic numbers through the well-known relation

$$\psi(n+1) = -\gamma + H_n.$$

In [9], Chen gave the asymptotic expansion of the psi function by

$$\psi(x+1) \sim \ln x + \left(1 - \frac{1}{r}\right)\frac{1}{x} + \frac{1}{s}\ln\left(1 + \sum_{j=1}^{\infty}\frac{a_j(r,s)}{x^j}\right), \quad x \to \infty, \tag{1.1}$$

where $r \neq 0$ and $s \neq 0$ are any given real numbers. The coefficients $a_j \equiv a_j(r,s)$ in (1.1) are given by the recursive relation

$$a_j = b_j + \frac{1}{j}\sum_{k=1}^{j-1} k b_k a_{j-k}, \quad j \geq 1 \tag{1.2}$$

and

$$b_1 = \frac{s(2-r)}{2r}, \quad b_{2k+1} = 0, \quad b_{2k} = \frac{-sB_{2k}}{2k}, \quad k \geq 1. \tag{1.3}$$

Based on the above complete asymptotic expansion, the asymptotic formula for the Euler-Mascheroni constant was proposed:

$$\gamma \sim H_n - \left(1 - \frac{1}{r}\right)\frac{1}{n} - \ln n - \frac{1}{s}\ln\left(1 + \sum_{j=1}^{\infty}\frac{a_j(r,s)}{n^j}\right), \quad n \to \infty.$$

This formula unifies the following three approximation formulas due to Mortici [23] and develops them to complete asymptotic expansions:

$$\gamma = H_n - \left(1 - \frac{1}{6-2\sqrt{6}}\right)\frac{1}{n} - \ln n - \ln\left(1 + \frac{1}{\sqrt{6}n}\right) + O\left(\frac{1}{n^3}\right), \quad n \to \infty,$$

$$\gamma = H_n - \left(1 - \frac{1}{6+2\sqrt{6}}\right)\frac{1}{n} - \ln n - \ln\left(1 - \frac{1}{\sqrt{6}n}\right) + O\left(\frac{1}{n^3}\right), \quad n \to \infty,$$

and

$$\gamma = H_n - \left(1 - \frac{1}{2}\right)\frac{1}{n} - \ln n - \frac{1}{2}\ln\left(1 - \frac{1}{6n^2}\right) + O\left(\frac{1}{n^4}\right), \quad n \to \infty.$$

Motivated by these interesting works, in this paper we provide an explicit formula for the coefficients $a_j(r,s)$ in terms of the cycle indicator polynomial of symmetric group which is an important tool in enumerative combinatorics. Also by means of this useful tool, we directly obtain an alternative form of the recursive relation for determining the coefficients $a_j(r,s)$ established by Chen [9]. Furthermore we describe their asymptotic behavior of $a_j(r,s)$ for the special case $r = 2$.

2. Main results

Firstly, let us recall the notions of groups of permutations and cycle indicator polynomial. For more details one can refer to [10]. A group $\mathfrak{G}$ of permutation of a finite set N be a subgroup of the group $\mathfrak{S}(N)$ of all permutations of N, and we denote $\mathfrak{G} \leq \mathfrak{S}(N)$. $|\mathfrak{G}|$ is called the order of $\mathfrak{G}$, and $|N|$ its degree.

Let $[n] = \{1, 2, \ldots, n\}$ and $\mathbb{N}$ be a set of non-negative integers. For every permutation $\sigma \in \mathfrak{G}(N)$, $|N| = n$, denote $c_i(\sigma)$ the number of orbits of length i of σ, $i \in [n]$. We define the cycle indicator polynomial $Z(x_1, x_2, \ldots, x_n) := Z(\mathfrak{G}; x_1, x_2, \ldots, x_n)$ of a group of permutations $\mathfrak{G}$ of N:

$$Z(x_1, x_2, \ldots, x_n) = \frac{1}{|\mathfrak{G}|} \sum_{\sigma \in \mathfrak{G}} x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} \cdots x_n^{c_n(\sigma)}.$$

If $\mathfrak{G} = \mathfrak{S}(N)$ (the symmetric group of degree n), the cycle indicator polynomial denoted by $Z_n(x_1, x_2, \ldots, x_n) := Z_n(\mathfrak{G}; x_1, x_2, \ldots, x_n)$ is explicitly expressed by

$$Z_n(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{c \in \varpi_n} (c; n)^* x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n},$$

where

$$\varpi_n = \left\{ c := (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n \,\middle|\, \sum_{k=1}^{n} k c_k = n \right\} \tag{2.1}$$

is a set in which an element corresponds to a way of partition of n, and

$$(c; n)^* = \frac{n!}{c_1! c_2! \cdots c_n! 1^{c_1} 2^{c_2} \cdots n^{c_n}}$$

is the number of permutations of type $[\![c]\!] = [\![c_1, c_2, \ldots, c_n]\!]$.

From the definition of the cycle indicator polynomial we can directly calculate the first few cases:

$$\begin{aligned}
&Z_0 = 1, \\
&Z_1(x_1) = x_1, \\
&Z_2(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2), \\
&Z_3(x_1, x_2, x_3) = \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3), \\
&Z_4(x_1, x_2, x_3, x_4) = \frac{1}{24}(x_1^4 + 6x_1^2x_2 + 3x_2^2 + 8x_1x_3 + 6x_4), \\
&Z_5(x_1, x_2, x_3, x_4, x_5) = \frac{1}{120}(x_1^5 + 10x_1^3x_2 + 15x_1x_2^2 + 20x_1^2x_3 + 20x_2x_3 + 30x_1x_4 + 24x_5), \\
&Z_6(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{720}(x_1^6 + 15x_1^4x_2 + 45x_1^2x_2^2 + 40x_1^3x_3 \\
&\qquad + 15x_2^3 + 120x_1x_2x_3 + 90x_1^2x_4 + 40x_3^2 + 90x_2x_4 + 144x_1x_5 + 120x_6).
\end{aligned}$$

The ordinary generating function of the cycle indicator polynomial is

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m}\right) = 1 + \sum_{n=1}^{\infty} Z_n(x_1, x_2, \ldots, x_n)t^n. \tag{2.2}$$

By (2.2) the following recurrence relation is obvious:

$$Z_0 = 1, \quad nZ_n(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} x_k Z_{n-k}(x_1, x_2, \ldots, x_{n-k}),\ n \geq 1. \tag{2.3}$$

It is worth noticing that the cycle indicator polynomials are well connected with the well-known Bell polynomials [2, 10] by

$$Z_n(x_1, x_2, \ldots, x_n) = \frac{1}{n!} Y_n(0!x_1, 1!x_2, \ldots, (n-1)!x_n),\ n = 1, 2, \ldots.$$

Using the cycle indicator polynomials, we obtain the explicit expressions of the coefficients $a_j(r, s)$ or a_j:

Theorem 2.1. *Let $r \neq 0$ and $s \neq 0$ be two given real numbers. The psi function has the following asymptotic expansion:*

$$\psi(x+1) \sim \ln x + \left(1 - \frac{1}{r}\right)\frac{1}{x} + \frac{1}{s}\ln\left(1 + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right), \quad x \to \infty \tag{2.4}$$

with the coefficients a_j explicitly given by

$$a_j = \begin{cases} \sum_{k=0}^{m-1} \frac{1}{(2m-1-2k)!} Z_k\left(-\frac{sB_2}{2}, -\frac{sB_4}{2}, \ldots, -\frac{sB_{2k}}{2}\right)\left(\frac{s(2-r)}{2r}\right)^{2m-1-2k}, & j = 2m-1, \\ \sum_{k=0}^{m} \frac{1}{(2m-2k)!} Z_k\left(-\frac{sB_2}{2}, -\frac{sB_4}{2}, \ldots, -\frac{sB_{2k}}{2}\right)\left(\frac{s(2-r)}{2r}\right)^{2m-2k}, & j = 2m, \end{cases} \tag{2.5}$$

where $m \geq 1$.

Proof. The psi function has asymptotic expansion [1, 16]:

$$\psi(x+1) - \ln x \sim \frac{1}{2x} - \sum_{j=1}^{\infty} \frac{B_{2j}}{2jx^{2j}}, \quad x \to \infty, \tag{2.6}$$

where B_n are the Bernoulli numbers. Comparing with (2.4) and (2.6) we have

$$\ln\left(1 + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right) \sim \frac{s(2-r)}{2r}\frac{1}{x} - \sum_{j=1}^{\infty} \frac{sB_{2j}}{2j}\frac{1}{x^{2j}}, \quad x \to \infty,$$

which is equivalent to

$$1 + \sum_{j=1}^{\infty} \frac{a_j}{x^j} \sim \exp\left(\frac{s(2-r)}{2r}\frac{1}{x}\right)\exp\left(-\sum_{j=1}^{\infty} \frac{sB_{2j}}{2j}\frac{1}{x^{2j}}\right), \quad x \to \infty. \tag{2.7}$$

By (2.2) we have

$$\exp\left(\frac{s(2-r)}{2r}\frac{1}{x}\right)\exp\left(-\sum_{j=1}^{\infty}\frac{sB_{2j}}{2j}\frac{1}{x^{2j}}\right)$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{s(2-r)}{2r}\right)^{k}\frac{1}{x^{k}}\sum_{k=0}^{\infty}Z_k\left(-\frac{sB_2}{2},-\frac{sB_4}{2},\dots,-\frac{sB_{2k}}{2}\right)\frac{1}{x^{2k}}.$$

Combining with (2.7) we easily obtain (2.5). □

According to (2.2), it is not difficult to verify that

Lemma 2.1. *For $m \geq 1$, we have*

$$Z_{2m-1}(x_1,x_2,0,x_4,0,\dots,x_{2m-2},0)=\sum_{k=0}^{m-1}Z_k\left(\frac{x_2}{2},\frac{x_4}{2},\dots,\frac{x_{2k}}{2}\right)\frac{x_1^{2m-1-2k}}{(2m-1-2k)!},$$
$$Z_{2m}(x_1,x_2,0,x_4,0,\dots,x_{2m-2},0,x_{2m})=\sum_{k=0}^{m}Z_k\left(\frac{x_2}{2},\frac{x_4}{2},\dots,\frac{x_{2k}}{2}\right)\frac{x_1^{2m-2k}}{(2m-2k)!}.$$

Thus, according to this lemma we can obtain an alternative form of the expressions of the a_j's.

Theorem 2.2. *Let $r \neq 0$ and $s \neq 0$ be two given real numbers. The psi function has the following asymptotic expansion:*

$$\psi(x+1)\sim\ln x+\left(1-\frac{1}{r}\right)\frac{1}{x}+\frac{1}{s}\ln\left(1+\sum_{j=1}^{\infty}\frac{a_j}{x^j}\right),\quad x\to\infty,$$

with the coefficients a_j explicitly given by

$$a_j=\begin{cases} Z_{2m-1}\left(\frac{s(2-r)}{2r},-\frac{sB_2}{2},0,-\frac{sB_4}{2},0,\dots,-\frac{sB_{2m-2}}{2},0\right), & j=2m-1,\\ Z_{2m}\left(\frac{s(2-r)}{2r},-\frac{sB_2}{2},0,-\frac{sB_4}{2},0,\dots,-\frac{sB_{2m-2}}{2},0,-\frac{sB_{2m}}{2}\right), & j=2m,\end{cases}\tag{2.8}$$

where $m \geq 1$.

Using the recurrence relation of the cycle indicator polynomials, i.e., (2.3), we can calculate the coefficients a_j recursively.

Theorem 2.3. *Let $r \neq 0$ and $s \neq 0$ be two given real numbers. Let $a_0 = 1$ and $a_1 = s(2-r)/(2r)$. The psi function has the following asymptotic expansion:*

$$\psi(x+1)\sim\ln x+\left(1-\frac{1}{r}\right)\frac{1}{x}+\frac{1}{s}\ln\left(1+\sum_{j=1}^{\infty}\frac{a_j}{x^j}\right),\quad x\to\infty,$$

with the coefficients a_j recursively given by

$$a_{2m-1} = \frac{1}{2m-1}\left(\frac{s(2-r)}{2r}a_{2m-2} - \frac{s}{2}\sum_{k=1}^{m-1} B_{2m-2k}a_{2k-1}\right), \tag{2.9}$$

$$a_{2m} = \frac{1}{2m}\left(\frac{s(2-r)}{2r}a_{2m-1} - \frac{s}{2}\sum_{k=0}^{m-1} B_{2m-2k}a_{2k}\right), \tag{2.10}$$

where $m \geq 1$.

Proof. Taking

$$x_1 = \frac{s(2-r)}{2r}, \quad x_{2j} = -\frac{sB_{2j}}{2}, \quad x_{2j+1} = 0, \quad 1 \leq j \leq m-1$$

in (2.3) and combining with (2.8) implies that (2.9) is true. The proof of (2.10) is similar. □

Remark 2.1. *In fact, the recursive relation* (1.2) *can be divided into* (2.9) *and* (2.10) *because $b_{2k+1} = 0,\ k \geq 1$. This means that by means of the cycle indicator polynomials we rediscover the recurrence relation of the a_j's.*

Applying Theorems 2.1, 2.2 and 2.3 to the asymptotic expansion of the Euler-Mascheroni constant γ, we have similar results.

Corollary 2.1. *Let $r \neq 0$ and $s \neq 0$ be two given real numbers. The Euler-Mascheroni constant γ has the following asymptotic expansion:*

$$\gamma \sim H_n - \left(1 - \frac{1}{r}\right)\frac{1}{n} - \frac{1}{s}\ln\left(n^s\left(1 + \sum_{j=1}^{\infty}\frac{a_j}{n^j}\right)\right), \quad n \to \infty,$$

where the coefficients a_j are explicitly given by (2.5) *or* (2.8).

Corollary 2.2. *Let $r \neq 0$ and $s \neq 0$ be two given real numbers. Let $a_0 = 1$ and $a_1 = s(2-r)/(2r)$. The Euler-Mascheroni constant γ has the following asymptotic expansion:*

$$\gamma \sim H_n - \left(1 - \frac{1}{r}\right)\frac{1}{n} - \frac{1}{s}\ln\left(n^s\left(1 + \sum_{j=1}^{\infty}\frac{a_j}{n^j}\right)\right), \quad n \to \infty,$$

where the coefficients a_j are recursively given by (2.9) *and* (2.10).

The case $r = 2$ is of special interest. From Theorems 2.1, 2.2 and 2.3 we have

Corollary 2.3. *Let $s \neq 0$ be a given real numbers. The psi function has the following asymptotic expansion:*

$$\psi(x+1) \sim \ln x + \frac{1}{2x} + \frac{1}{s}\ln\left(1 + \sum_{j=1}^{\infty}\frac{a_j}{x^j}\right), \quad x \to \infty,$$

with the coefficients a_j explicitly given by

$$a_j = \begin{cases} 0, & j = 2m-1, \\ Z_m\left(-\frac{sB_2}{2}, -\frac{sB_4}{2}, \ldots, -\frac{sB_{2m}}{2}\right), & j = 2m, \end{cases} \tag{2.11}$$

where $m \geq 1$. The coefficients $a_{2m}(m \geq 1)$ can also be recursively given by

$$a_{2m} = -\frac{s}{4m}\sum_{k=0}^{m-1} B_{2m-2k}a_{2k}$$

with $a_0 = 1$.

Though (2.11) is a real explicit formula for the coefficients $a_j (j \geq 1)$, the evaluation of a_j requires listing all or almost all of the solutions of (2.1). It also seems difficult to find a simpler explicit expression for a_j in general. Instead we provide an asymptotic formula for them.

Theorem 2.4. *Let $s \neq 0$ be a given real numbers. The psi function has the following asymptotic expansion:*

$$\psi(x+1) \sim \ln x + \frac{1}{2x} + \frac{1}{s}\ln\left(1 + \sum_{j=1}^{\infty}\frac{a_j}{x^j}\right), \quad x \to \infty, \tag{2.12}$$

where we have

$$a_{2m-1} = 0, \quad m \geq 1, \tag{2.13}$$

and

$$a_{2m} \sim \frac{(-1)^m 2s(2m-1)!}{(2\pi)^{2m}}, \tag{2.14}$$

as $m \to +\infty$.

Before we give the proof of Theorem 2.4, some lemmas are presented below.

Lemma 2.2. *Let $G(x) = \sum_{n=1}^{\infty} g_n x^n$ and $F(x) = \sum_{n=1}^{\infty} f_n x^n$ be two formal power series, and let*

$$H(x) := F(G(x)) = \sum_{n=1}^{\infty} h_n x^n.$$

Suppose that $F(x)$ is analytic in a neighborhood of the origin, $g_n \neq 0$ and
(i) $g_{n-1} = o(g_n)$ as $n \to +\infty$,
(ii) $\sum_{j=1}^{n-1} |g_j g_{n-j}| = O(g_{n-1})$ as $n \to +\infty$.
Then we have $h_n \sim f_1 g_n$ as $n \to +\infty$.

This lemma is due to Bender [3]. In the original theorem of Bender, the assumptions on F are more general. However, as it was noted by Odlyzko [36], those assumptions are automatically satisfied if F is analytic at the origin. See also [11].

It is well known that the Bernoulli numbers can be expressed in terms of the Riemann zeta function as [37]

$$B_{2k} = (-1)^{k+1}\frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k).$$

From this formula, one can easily deduce Lemmas 2.3 and 2.4, see also [11, 39].

Lemma 2.3. *For any $k \geq 0$ we have*

$$\frac{2(2k)!}{(2\pi)^{2k}} < |B_{2k}| < \frac{4(2k)!}{(2\pi)^{2k}}. \tag{2.15}$$

Lemma 2.4. *As $k \to +\infty$, we have*

$$B_{2k} \sim (-1)^{k+1}\frac{2(2k)!}{(2\pi)^{2k}}. \tag{2.16}$$

Now we turn to prove Theorem 2.4. From (2.6) and (2.12) we have

$$1 + \sum_{j=1}^{\infty}\frac{a_j}{x^j} \sim \exp\left(-\sum_{j=1}^{\infty}\frac{sB_{2j}}{2j}\frac{1}{x^{2j}}\right), \quad x \to \infty. \tag{2.17}$$

which immediately leads to

$$a_{2m-1} = 0, \quad m \geq 1.$$

Therefore, (2.17) can be rewritten as

$$\sum_{j=0}^{\infty}\frac{a_{2j}}{x^{2j}} \sim \exp\left(-\sum_{j=1}^{\infty}\frac{sB_{2j}}{2j}\frac{1}{x^{2j}}\right). \tag{2.18}$$

Let

$$G(t) = -\sum_{j=1}^{\infty}\frac{sB_{2j}}{2j}t^j$$

and

$$F(t) = e^t - 1$$

in Lemma 2.2. It is clear that $F(t)$ is analytic in a neighborhood of the origin. It follows from (2.18) that

$$H(t) = F(G(t)) = \sum_{j=1}^{\infty} a_{2j}t^j.$$

Applying Lemma 2.3 to the sequence

$$g_j := \frac{-sB_{2j}}{2j},$$

we obtain

$$\frac{2|s|(2m-1)!}{(2\pi)^{2m}} < |g_m| < \frac{4|s|(2m-1)!}{(2\pi)^{2m}},$$

for every $m \geq 1$. Since $g_m \neq 0$, and

$$0 \leq \lim_{m\to+\infty} \left|\frac{g_{m-1}}{g_m}\right| < \lim_{m\to+\infty} \frac{2(2\pi)^2}{(2m-2)(2m-1)} = 0,$$

we obtain

$$g_{m-1} = o(g_m) \quad as\ m \to \infty,$$

which implies that the first condition of Lemma 2.2 holds. It follows from (2.15) that

$$\begin{aligned}\sum_{j=1}^{m-1} |g_j g_{m-j}| &< \frac{16s^2}{(2\pi)^{2m}} \sum_{j=1}^{m-1} (2j-1)!(2(m-j)-1)! \\ &< \frac{2|s||g_{m-1}|}{\pi^2} \sum_{j=1}^{m-1} \frac{(2j-1)!(2(m-j)-1)!}{(2m-3)!}.\end{aligned}$$

By simple calculation we have

$$\begin{aligned}\sum_{j=1}^{m-1} \frac{(2j-1)!(2(m-j)-1)!}{(2m-3)!} &= (2m-2)\sum_{j=1}^{m-1} \frac{1}{\binom{2m-2}{2j-1}} \\ &= (2m-2)\left\{\frac{2}{2m-2} + \sum_{j=2}^{m-2} \frac{1}{\binom{2m-2}{2j-1}}\right\}.\end{aligned}$$

If m is large enough, then there exists a constant $C_1 > 0$ such that

$$\binom{2m-2}{2j-1} \geq \frac{1}{C_1} m^3,$$

for $2 \leq j \leq m-2$. Thus,

$$\sum_{j=2}^{m-2} \frac{1}{\binom{2m-2}{2j-1}} \leq \frac{C_1}{m^2}.$$

This implies that for m large enough there exists a constant $C_2 > 0$ such that

$$\sum_{j=1}^{m-1} \frac{(2j-1)!(2(m-j)-1)!}{(2m-3)!} \leq C_2.$$

Therefore, we have

$$\sum_{j=1}^{m-1} |g_j g_{m-j}| = O(g_{m-1}) \quad as \quad m \to +\infty,$$

which implies that the second condition of Lemma 2.2 is satisfied. According to Lemmas 2.2 and 2.4, we have

$$a_{2m} \sim g_m = -\frac{sB_{2m}}{2m} \sim \frac{(-1)^m 2s(2m-1)!}{(2\pi)^{2m}},$$

as $m \to +\infty$. This completes the proof of Theorem 2.4.

According to Theorem 2.4, we immediately have the following corollary.

Corollary 2.4. *Let $s \neq 0$ be a given real numbers. The Euler-Mascheroni constant γ has the following asymptotic expansion:*

$$\gamma \sim H_n - \frac{1}{2n} - \frac{1}{s} \ln \left(n^s \left(1 + \sum_{j=1}^{\infty} \frac{a_j}{n^j} \right) \right), \quad n \to \infty,$$

where we have

$$a_{2m-1} = 0, \quad m \geq 1,$$

and

$$a_{2m} \sim \frac{(-1)^m 2s(2m-1)!}{(2\pi)^{2m}},$$

as $m \to +\infty$.

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