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The values of binary linear forms at prime arguments

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Abstract

Suppose that λ_1 and λ_2 are positive real numbers such that λ_1/λ_2 is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence and $\delta > 0$. Denote by $E(\mathcal{V}, X, \delta)$ the number of $v \in \mathcal{V}$ with $v \leq X$ such that the inequality $|\lambda_1 p_1 + \lambda_2 p_2 - v| < v^{-\delta}$ has no solution in primes p_1, p_2. We prove that for all $X \geq 1$, $E(\mathcal{V}, X, \delta) \ll X^{f(\delta)+\varepsilon}$ for any $\varepsilon > 0$ with $f(\delta) = \max(5/9 + 2\delta, 2/3 + 4\delta/3)$, which improves the earlier result.

Keywords: Diophantine inequalities, primes, Davernport–Heilbronn method

2000 MSC: 11P32, 11P05, 11P55

1. Introduction

Davenport and Heilbronn first considered the Diophantine inequalities. Given $k \geq 1$ and s nonzero real numbers $\lambda_1, \cdots, \lambda_s$ (not all in rational ratio,

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not all negative), we write

$$F(\mathbf{p}) = \sum_{j=1}^{s} \lambda_j p_j^k,$$

where $\mathbf{p} = (p_1, \cdots, p_s)$ with each p_j a prime. Various authors have considered the distribution of values of such forms, for example, see [11, 12]. Here we continue in the direction started by Brüdern, Cook and Perelli [1] and followed by Cook and Harman [4], Cai [3] and Wang [13]. We call a set of positive reals $\mathcal{V}$ a well–spaced set if there is a $c > 0$ such that

$$u, v \in \mathcal{V}, \quad u \neq v \quad \Rightarrow |u - v| > c.$$

We further assume that

$$|\{v \in \mathcal{V} : 0 \leq v \leq X\}| \gg X^{1-\varepsilon}.$$

In this paper, suppose that λ_1 and λ_2 are positive real numbers such that λ_1/λ_2 is irrational and algebraic. We consider the distribution of the values of a given binary linear form

$$\lambda_1 p_1 + \lambda_2 p_2.$$

This problem can be considered as real analogous of binary linear Goldbach problem.

Let $\mathcal{V}$ be a well–spaced sequence, and let $E(\mathcal{V}, X, \delta)$ denote the number of $v \in \mathcal{V}$ with $v \leq X$ such that the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 - v| < v^{-\delta} \tag{1.1}$$

has no solution in primes p_1, p_2.

In [1], Brüdern, Cook and Perelli first considered this problem and showed that for any $\varepsilon > 0$,

$$E(\mathcal{V}, X, \delta) \ll X^{2/3+2\delta+\varepsilon}. \tag{1.2}$$

Furthermore, they showed that $2/3+2\delta$ can be replaced by $1/2+2\delta$ under the assumption of generalized Riemann hypothesis (GRH). Subsequently, Cook and Harman [4] and Cai [3], respectively, proved that $\min(2/3 + 2\delta, 4/5 + \delta)$ is admissible. Recently, Wang [13] established that $\max(3/5+2\delta, 2/3+4\delta/3)$ is also admissible.

Using the vector sieve of [8], we prove the following result.

Theorem 1.1. *Suppose that λ_1 and λ_2 are positive real numbers such that λ_1/λ_2 is irrational and algebraic. Let $\mathcal{V}$ be a well–spaced sequence. Let $\delta > 0$. Then for all $X \geq 1$, we have*

$$E(\mathcal{V}, X, \delta) \ll X^{f(\delta)+\varepsilon}, \tag{1.3}$$

for any $\varepsilon > 0$ with

$$f(\delta) = \max(5/9 + 2\delta, 2/3 + 4\delta/3). \tag{1.4}$$

Note that the bound in Theorem 1.1 is non-trivial for $\delta < 2/9$ and contributes an improvement for $1/6 \leq \delta < 2/9$. Our improvement comes from using Matomäki's combination of the vector sieve, the Harman sieve, some results on averages of bilinear exponential sums and some extra technical work which result in enlarging the major arc. It would be worth emphasizing that Matomäki's result can be deduced from Theorem 1.1. This reinforces that Theorem 1.1 is in fact a generalisation of Matomäki's theorem.

Notation. Throughout the paper, the letter η denotes a sufficiently small, fixed positive number. The letter ε denotes a sufficiently small positive real number. Any statement in which ε occurs holds for each fixed $\varepsilon > 0$. The letter p, with or without subscript, denotes a prime number. Constants, both explicit and implicit, in Vinogradov symbols may depend on λ_1, λ_2. We write $e(x) = \exp(2\pi i x)$.

2. Outline of the method

We follow the modification of the Hardy–Littlewood method which first stated by Davenport and Heilbronn. Now let $0 < \tau < 1$(indeed, we take $\tau = X^{-\delta}$), X be some (large) positive quantity. We define

$$K(\alpha) = \left(\frac{\sin \pi\tau\alpha}{\pi\alpha}\right)^2, \quad A(x) = \int_{\mathbb{R}} K(\alpha) e(\alpha x) d\alpha. \tag{2.1}$$

Then, by [11], It is easy to show that

$$K(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \quad A(x) = \max(0, \tau - |x|). \tag{2.2}$$

We use the vector sieve. We need lower and upper bounds $\rho^-(n)$ and $\rho^+(n)$ for the characteristic function $\rho(n)$ of primes. Assuming $\rho^-(n) \leq \rho(n) \leq \rho^+(n)$, we have a simple inequality

$$\rho(m)\rho(n) \geq \rho^+(m)\rho^-(n) + \rho^-(m)\rho^+(n) - \rho^+(m)\rho^+(n) \tag{2.3}$$

given in Brüdern and Fouvry [2] (also see Lemma 10.1 in Harman [6]).

We write for $j = 1, 2$

$$S_j^{\pm}(\alpha) = \sum_{\eta X < n \le X} \rho^{\pm}(n) e(n\lambda_j \alpha); \quad I(\alpha) = \int_{\eta X}^{X} \frac{e(\alpha x)}{\log x} dx; \tag{2.4}$$

$$U(\alpha) = \sum_{\eta X < n \le X} e(n\alpha), \tag{2.5}$$

where η is a sufficiently small, fixed positive number.

We define further

$$F(\alpha) := S_1^+(\alpha) S_2^-(\alpha) + S_1^-(\alpha) S_2^+(\alpha) - S_1^+(\alpha) S_2^+(\alpha). \tag{2.6}$$

For any measurable subset $\mathfrak{X}$ of $\mathbb{R}$, we define

$$J_v(\mathfrak{X}) := \int_{\mathfrak{X}} F(\alpha) K(\alpha) e(-\alpha v) d\alpha. \tag{2.7}$$

Then by (2.1), (2.3),

$$\begin{aligned} &J_v(\mathbb{R}) \\ &= \sum_{n_1, n_2 \le X} (\rho^+(n_1)\rho^-(n_2) + \rho^-(n_1)\rho^+(n_2) - \rho^+(n_1)\rho^+(n_2)) A(\lambda_1 n_1 + \lambda_2 n_2 - v) \\ &\le \sum_{p_1, p_2 \le X} A(\lambda_1 p_1 + \lambda_2 p_2 - v) \end{aligned}$$

and

$$J_v(\mathbb{R}) \le \tau \psi(v), \tag{2.8}$$

where $\psi(v)$ counts the number of the solutions to the inequality (1.1).

We shall restrict our attention to those v satisfying $X/2 \le v \le X$. In general, one can consider $X2^{-j} \le v \le X2^{1-j}$, $j = 1, 2, \ldots$, and obtain a satisfactory bound for the exceptional set.

To estimate the integral in (2.7), we divide the real line into three parts: the major arc $\mathfrak{M}$, the minor arc $\mathfrak{m}$ and the trivial arc $\mathfrak{t}$ which are defined by

$$\mathfrak{M} = \{\alpha : |\alpha| \le \phi\}, \quad \mathfrak{m} = \{\alpha : \phi < |\alpha| \le \xi\}, \quad \mathfrak{t} = \{\alpha : |\alpha| > \xi\},$$

where $\phi = X^{-\frac{5}{9} - 3\varepsilon}, \xi = \tau^{-2} X^{1+\varepsilon}$.

For the contribution from the trivial arc, by (15) of [1], we know that

$$|J_v(\mathfrak{t})| \ll \tau^2 X^{1-\varepsilon}. \tag{2.9}$$

3. The vector sieve

We use the lower and upper bounds $\rho^-(n)$ and $\rho^+(n)$ given by [8], they can be written as sums of coefficients a_n that are either of the form

$$a_n = \sum_{\substack{mk=n\\ m\sim M}} b_m \qquad \text{(Type I sums)}$$

with $M \ll X^{7/9}$ or such that, for any $Q \in [X^{1/3}, X^{4/9}]$, there exists $M \in [Q, QX^{1/9}]$ such that

$$a_n = \sum_{\substack{lm=n\\ m\sim M}} b_m c_l. \qquad \text{(Type II sums)}$$

Here $m \sim M$ means that $M \leq m < 2M$. a_n, b_m and c_l are divisor–bounded. This means, for example, that $a_n \ll \tau(n)^C$ for some constant C.

Lemma 3.1. *([8], Lemma 7) Suppose that α is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a, q) = 1, \quad |q\alpha - a| < q^{-1}.$$

Then for any complex sequence $b_m, c_l \ll 1$, we have

$$\sum_{\substack{lm\sim X\\ m\sim M}} b_m c_l e(ml\alpha) \ll \left(Xq^{-1/2} + (Xq)^{1/2} + XM^{-1/2} + (XM)^{1/2}\right)(\log X)^2$$

and

$$\sum_{\substack{lm\sim X\\ m\sim M}} b_m e(ml\alpha) \ll (M + Xq^{-1} + q)(\log(2qX)).$$

Lemma 3.2. *([8], Lemma 10) Suppose that α is a real number, and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a, q) = 1, \quad |q\alpha - a| < q^{-1}.$$

Let A and Q be positive integers with $AQ \ll q^C$ and let $\mathcal{Q}$ be a set of distinct integers q_1 with $q_1 \sim Q$. Then for every $\varepsilon > 0$ and $\theta < 1/2$ the number of solutions to

$$\|q_1 n\alpha\| < \theta \quad \text{with } q_1 \in \mathcal{Q},\ 1 \leq n \leq A \tag{3.1}$$

is

$$\ll |\mathcal{Q}|A\theta + q^{\varepsilon}(Q + AQq^{-1} + q\theta),$$

where the implied constant depends only on α, C and ε.

4. The major arc

In this paper, we need a larger major arc than that in [8]. This brings additional difficulties.

Lemma 4.1. *There are positive real numbers u^- and u^+ with $2u^- > u^+$ such that for any $\vartheta \in [\frac{1}{6\phi X}, \frac{6}{\phi X}]$ and $A \geq 0$, we have*

$$\int_{\eta X}^{X} \left(\sum_{y \leq n < y + y\vartheta} \left(\rho^{\pm}(n) - \frac{u^{\pm}}{\log n} \right) \right)^2 dy \ll \frac{X}{\phi^2} (\log X)^{-A}. \tag{4.1}$$

Proof. Let $\vartheta' = \exp(-3(\log X)^{1/3})$. We write $\mathcal{A} = [y, y + y\vartheta)$ and $\mathcal{B} = [y, y + y\vartheta']$. We will first show that

$$\int_{\eta X}^{X} \left(\sum_{n \in \mathcal{A}} \rho^{\pm}(n) - \frac{\vartheta}{\vartheta'} \sum_{n \in \mathcal{B}} \rho^{\pm}(n) \right)^2 dy \ll \frac{X}{\phi^2} (\log X)^{-A}. \tag{4.2}$$

Clearly, it is enough to show that this holds when $\rho^{\pm}$ are replaced by our type I and type II sums.

Case 1: We have a type II sums $\sum_{\substack{ml \in \mathcal{A} \\ m \sim M}} b_m c_l$ with $M \in [X^{4/9}, X^{5/9}]$. We use the method of Heath–Brown [7] (also see Lemmas 7.2 and 9.3 of [6]). Let $T = \vartheta^{-1} X^{2\varepsilon}$ and $s = \frac{1}{2} + it$. Let

$$F(s) = \sum_{\substack{\eta X \leq ml < 2X \\ m \sim M}} b_m c_l (ml)^{-s}. \tag{4.3}$$

Obviously, $|F(s)| \ll X^{1/2} (\log X)^B$ for some positive number B, since b_m and c_l are divisor–bounded. Using Perron's formula (see page 1371 of [7]), for $\vartheta^* = \vartheta$ or ϑ', we have

$$\sum_{\substack{y \leq ml < y + y\vartheta^* \\ m \sim M}} b_m c_l = \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} F(s) \frac{(y + y\vartheta^*)^s - y^s}{s} ds + O\left(X^{\varepsilon} \left(1 + \frac{X}{T} \right) \right). \tag{4.4}$$

Let $T_0 = \exp((\log X)^{1/3})$. While for $s = \frac{1}{2} + it$, we have

$$\frac{(y + y\vartheta)^s - y^s}{s} = \begin{cases} y^s \vartheta + O(y^{1/2} |s| \vartheta^2), & \text{if } t \leq T_0; \\ O(y^{1/2} \vartheta), & \text{if } t > T_0. \end{cases} \tag{4.5}$$

Thus we have

$$\sum_{\substack{y\le ml<y+y\vartheta^* \\ m\sim M}} b_m c_l$$
$$= \frac{\vartheta^*}{2\pi i}\int_{\frac12-iT_0}^{\frac12+iT_0} F(s)y^s ds + E(\vartheta^*) + O\left(y(\log y)^B(\vartheta^*)^2T_0^2\right) + O\left(X^{1-\varepsilon}\vartheta\right), \tag{4.6}$$

where

$$E(\vartheta^*) = \frac{1}{2\pi i}\left(\int_{\frac12+iT_0}^{\frac12+iT} + \int_{\frac12-iT}^{\frac12-iT_0}\right) F(s)\frac{(1+\vartheta^*)^s-1}{s}y^s ds. \tag{4.7}$$

Then we have

$$\sum_{\substack{ml\in\mathcal{A} \\ m\sim M}} b_m c_l - \frac{\vartheta}{\vartheta'}\sum_{\substack{ml\in\mathcal{B} \\ m\sim M}} b_m c_l = E(\vartheta) + \frac{\vartheta}{\vartheta'}E(\vartheta') + O\left(X\vartheta\exp(-0.5(\log X)^{1/3})\right). \tag{4.8}$$

Hence we get

$$\int_{\eta X}^{X}\left(\sum_{\substack{ml\in\mathcal{A} \\ m\sim M}} b_m c_l - \frac{\vartheta}{\vartheta'}\sum_{\substack{ml\in\mathcal{B} \\ m\sim M}} b_m c_l\right)^2 dy$$
$$\ll \int_{\eta X}^{X}|E(\vartheta)|^2 dy + \frac{\vartheta^2}{\vartheta'^2}\int_{\eta X}^{X}|E(\vartheta')|^2 dy + \frac{X}{\phi^2}\exp(-(\log X)^{1/3}). \tag{4.9}$$

By Lemma 9.1 of [6], for $\vartheta^* = \vartheta$ or ϑ', we have

$$\int_{\eta X}^{X}|E(\vartheta^*)|^2 dy \ll X^2\log T\int_{\frac12+iT_0}^{\frac12+iT}\left|F(s)\frac{(1+\vartheta^*)^s-1}{s}\right|^2|ds|$$
$$\ll X^2(\vartheta^*)^2\log T\int_{\frac12+iT_0}^{\frac12+iT}|F(s)|^2|ds|. \tag{4.10}$$

Thus we have

$$\int_{\eta X}^{X}\left(\sum_{\substack{ml\in\mathcal{A}\\ m\sim M}} b_m c_l - \frac{\vartheta}{\vartheta'}\sum_{\substack{ml\in\mathcal{B}\\ m\sim M}} b_m c_l\right)^2 dy$$
$$\ll X^2\vartheta^2 \log T\int_{T_0}^{T}|F(1/2+it)|^2 dt + \frac{X}{\phi^2}\exp(-(\log X)^{1/3}). \qquad (4.11)$$

By Lemma 5.2 of [6], we have

$$\begin{aligned}\int_{T_0}^{T}|F(1/2+it)|^2 dt &\ll \int_{T_0}^{T}\left|\sum_{m\sim M} b_m m^{-1/2-it}\right|^2\left|\sum_{l\sim X/M} c_l l^{-1/2-it}\right|^2 dt\\ &\ll \max_{t\in[T_0,T]}\left|\sum_{m\sim M} b_m m^{-1/2-it}\right|^2 (X/M+T)\log^C X\\ &\ll X(\log X)^{-A}, \end{aligned} \qquad (4.12)$$

since $T=\vartheta^{-1}X^{2\varepsilon}\ll X^{4/9}\ll y/M$ and $|\sum_{m\sim M} b_m m^{-1/2-it}|\ll M^{1/2}(\log X)^{-A-C}$, which holds in interesting cases since the coefficients arise from the characteristic function of primes. For example, one can refer to the explanation of (7.2.3) in Harman [6]. Hence we have

$$\int_{\eta X}^{X}\left(\sum_{\substack{ml\in\mathcal{A}\\ m\sim M}} b_m c_l - \frac{\vartheta}{\vartheta'}\sum_{\substack{ml\in\mathcal{B}\\ m\sim M}} b_m c_l\right)^2 dy \ll \frac{X}{\phi^2}(\log X)^{-A}. \qquad (4.13)$$

Case 2: We have a type I sums with $M\le X^{7/9}$.
Case 2a: If $M\le X^{1-\varepsilon}\vartheta$, we have

$$\begin{aligned}\left|\sum_{\substack{mk\in\mathcal{A}\\ m\sim M}} b_m - \frac{\vartheta}{\vartheta'}\sum_{\substack{mk\in\mathcal{B}\\ m\sim M}} b_m\right| &= \left|\sum_{m\sim M} b_m\left(\left[\frac{y\vartheta}{m}\right]-\frac{\vartheta}{\vartheta'}\left[\frac{y\vartheta'}{m}\right]\right)\right|\\ &\le \sum_{m\sim M}|b_m| \ll MX^{\varepsilon/2}\ll X^{1-\frac{\varepsilon}{2}}\vartheta.\end{aligned}$$

Then

$$\int_{\eta X}^{X}\left(\sum_{\substack{mk\in\mathcal{A}\\ m\sim M}} b_m-\frac{\vartheta}{\vartheta'}\sum_{\substack{mk\in\mathcal{B}\\ m\sim M}} b_m\right)^2 dy \ll X^{3-\varepsilon}\vartheta^2 \ll \frac{X^{1-\varepsilon}}{\phi^2}.$$

Case 2b: If $X^{1-\varepsilon}\vartheta < M \le X^{7/9}$, we let

$$G(s)=\sum_{\substack{\eta X\le ml<2X\\ m\sim M}} b_m(ml)^{-s}. \tag{4.14}$$

Then similar to the case 1, we have

$$\begin{aligned}&\int_{\eta X}^{X}\left(\sum_{\substack{mk\in\mathcal{A}\\ m\sim M}} b_m-\frac{\vartheta}{\vartheta'}\sum_{\substack{mk\in\mathcal{B}\\ m\sim M}} b_m\right)^2 dy\\ &\ll X^2\vartheta^2\log T\int_{T_0}^{T}|G(1/2+it)|^2dt+\frac{X}{\phi^2}\exp(-(\log X)^{1/3}).\end{aligned} \tag{4.15}$$

Here we have

$$\int_{T_0}^{T}|G(1/2+it)|^2dt \ll \max_{\eta X\le Y\le X}\int_{T_0}^{T}\left|\sum_{m\sim M} b_m m^{-1/2-it}\sum_{k\sim K} k^{-1/2-it}\right|^2 dt, \tag{4.16}$$

where $K=Y/M$. Using the approximate functional equation for the Riemann zeta-function (see (4.12.4) of [9]), which with Re $s=1/2$ gives

$$\zeta(s)=\sum_{k\le K}k^{-s}+\chi(s)\sum_{l\le L}l^{s-1}+O(K^{-1/2}+L^{-1/2}),$$

where $2\pi KL=t$, $s=1/2+it$, $|\chi(s)|=1$. Thus when $t>K$, we have

$$\begin{aligned}\sum_{k\sim K}k^{-1/2-it}&=\chi(s)\sum_{\frac{t}{4\pi K}\le l\le\frac{t}{2\pi K}} l^{-1/2+it}+O\left(K^{-1/2}+\left(\frac{t}{K}\right)^{-1/2}\right)\\ &=O((t/K)^{1/2}).\end{aligned} \tag{4.17}$$

On the other hand, for $T_0 \le t \le K$, by Theorem 4.11 of [9], we have

$$\sum_{k\sim K} k^{-1/2-it} = \frac{(2K)^{1/2-it}-(K)^{1/2-it}}{1/2-it} + O(K^{-1/2}) = O(K^{1/2}/t). \tag{4.18}$$

Combining (4.15)–(4.18), by Lemma 5.2 of [6], we have

$$\begin{aligned}
&\int_{\eta X}^{X}\Bigg(\sum_{\substack{mk\in\mathcal{A}\\ m\sim M}} b_m - \frac{\vartheta}{\vartheta'}\sum_{\substack{mk\in\mathcal{B}\\ m\sim M}} b_m\Bigg)^2 dy\\
&\ll X^2\vartheta^2(\log X)\Big(\frac{MT}{X}+\frac{X}{MT_0^2}\Big)\int_{T_0}^{T}\Bigg|\sum_{m\sim M} b_m m^{-1/2-it}\Bigg|^2 dt\\
&\quad + \frac{X}{\phi^2}\exp(-(\log X)^{1/3})\\
&\ll X^2\vartheta^2(\log X)\Big(\frac{MT}{X}+\frac{X}{MT_0^2}\Big)(M+T)(\log X)^C + \frac{X}{\phi^2}\exp(-(\log X)^{1/3})\\
&\ll \frac{X}{\phi^2}(\log X)^{-A},
\end{aligned} \tag{4.19}$$

since we assume $X^{1-\varepsilon}\vartheta < M \le X^{7/9}$ in this case. This completes the proof of (4.2).

By the section 7 of [8], we have

$$\sum_{n\in\mathcal{B}} \rho^{\pm}(n) = \frac{u^{\pm}\vartheta'}{\vartheta}\sum_{n\in\mathcal{A}}\frac{1}{\log n} + O(X\exp(-3(\log X)^{1/3})), \tag{4.20}$$

where $u^- > 0.60$ and $u^+ < 1.19$. Thus $2u^- - u^+ > 0$.

Then (4.1) follows from (4.2) and (4.20). □

Lemma 4.2. *For $j = 1, 2$, we have*

$$\int_{-\phi}^{\phi} |S_j^{\pm}(\alpha) - u^{\pm}I(\lambda_j\alpha)|^2 d\alpha \ll X(\log X)^{-A}. \tag{4.21}$$

Proof. Obviously, we have

$$\begin{aligned}
&\int_{-\phi}^{\phi} |S_j^{\pm}(\alpha) - u^{\pm}I(\lambda_j\alpha)|^2 d\alpha\\
&\le \int_{-\phi}^{\phi} |S_j^{\pm}(\alpha) - u^{\pm}U(\lambda_j\alpha)|^2 d\alpha + \int_{-\phi}^{\phi} |u^{\pm}U(\lambda_j\alpha) - u^{\pm}I(\lambda_j\alpha)|^2 d\alpha.
\end{aligned} \tag{4.22}$$

First, by the Euler–Maclaurin summation formula, we have

$$|U(\lambda_j\alpha) - I(\lambda_j\alpha)| \ll 1 + |\alpha|X. \tag{4.23}$$

Thus

$$\begin{aligned}\int_{-\phi}^{\phi} |u^{\pm}U(\lambda_j\alpha) - u^{\pm}I(\lambda_j\alpha)|^2 d\alpha &\ll \int_{|\alpha|\le X^{-1}} d\alpha + \int_{X^{-1}<|\alpha|\le\phi} X^2\alpha^2 d\alpha \\ &\ll X^{-1} + X^2\phi^3 \ll X(\log X)^{-A}. \end{aligned} \tag{4.24}$$

Next, by Gallagher's lemma (Lemma 1 of [5]) and Lemma 4.1, we have

$$\begin{aligned}&\int_{-\phi}^{\phi} |S_j^{\pm}(\alpha) - u^{\pm}U(\lambda_j\alpha)|^2 d\alpha \\ &= \int_{-\phi}^{\phi} \left| \sum_{\eta X<n\le X} \left(\rho^{\pm}(n) - \frac{u^{\pm}}{\log n}\right) e(n\alpha)\right|^2 d\alpha \\ &\ll \phi^2 \left(\int_{\eta X}^{X} + \int_{\eta X - \frac{1}{2\phi}}^{\eta X}\right) \left| \sum_{n=y}^{y+\frac{1}{2\phi}} (\rho^{\pm}(n) - \frac{u^{\pm}}{\log n}) \right|^2 dy \\ &\ll X(\log X)^{-A} + \phi^{-1}X^{2\varepsilon} \ll X(\log X)^{-A}. \end{aligned} \tag{4.25}$$

Here, we have used the trivial bound

$$\left| \sum_{n=y}^{y+\frac{1}{2\phi}} (\rho^{\pm}(n) - \frac{u^{\pm}}{\log n}) \right| \ll \phi^{-1}X^{\varepsilon}.$$

Thus (4.21) follows from (4.22), (4.24) and (4.25). □

Lemma 4.3. *We have*

$$J_v(\mathfrak{M}) := \int_{\mathfrak{M}} F(\alpha)K(\alpha)e(-\alpha v)d\alpha \gg \tau^2 \frac{X}{(\log X)^2}. \tag{4.26}$$

Proof. This follows from Lemma 4.2 (the proof follows as Lemma 5 of [1]). □

5. The minor arc

By the definition of $F(\alpha)$, we have

$$\int_{\mathfrak{m}} |F(\alpha)|^2 K(\alpha) d\alpha \ll \int_{\mathfrak{m}} |S_1^+(\alpha) S_2^-(\alpha)|^2 K(\alpha) d\alpha + \int_{\mathfrak{m}} |S_1^-(\alpha) S_2^+(\alpha)|^2 K(\alpha) d\alpha + \int_{\mathfrak{m}} |S_1^+(\alpha) S_2^+(\alpha)|^2 K(\alpha) d\alpha. \tag{5.1}$$

We write for $i = 1, 2$,

$$S_i(\alpha) = \sum_{\eta X < n \le X} a_n e(n\lambda_i \alpha), \tag{5.2}$$

where a_n is of one of the types I and II, which are defined in section 3.

Without loss of generality we need only to prove that

$$\int_{\mathfrak{m}} |S_1(\alpha) S_2(\alpha)|^2 K(\alpha) d\alpha \ll \tau X^{1+2g(\delta)+\varepsilon}, \tag{5.3}$$

where

$$g(\delta) = \begin{cases} 7/9, & 1/6 \le \delta < 2/9; \\ 5/6 - \delta/3, & 0 < \delta < 1/6. \end{cases} \tag{5.4}$$

Let $\mathfrak{m}' = \mathfrak{m}_1 \cup \mathfrak{m}_2$, $\hat{\mathfrak{m}} = \mathfrak{m} \setminus \mathfrak{m}'$, where

$$\mathfrak{m}_1 = \{\alpha \in \mathfrak{m} : |S_1(\alpha)| \le X^{g(\delta)+2\varepsilon}\}, \quad \mathfrak{m}_2 = \{\alpha \in \mathfrak{m} : |S_2(\alpha)| \le X^{g(\delta)+2\varepsilon}\}. \tag{5.5}$$

Then it is easy to see

$$\int_{\mathfrak{m}'} |S_1(\alpha) S_2(\alpha)|^2 K(\alpha) d\alpha \ll \tau X^{1+2g(\delta)+5\varepsilon}. \tag{5.6}$$

It remains to discuss the set $\hat{\mathfrak{m}}$. We use the method first by Brüdern, Cook and Perelli [1] and followed by Matomäki [8], Wang [13]

Lemma 5.1. *We have*

$$\int_{\hat{\mathfrak{m}}} |S_1(\alpha) S_2(\alpha)|^2 K(\alpha) d\alpha \ll \tau X^{1+2g(\delta)+\varepsilon}. \tag{5.7}$$

Proof. We consider first the case that both $S_1(\alpha)$ and $S_2(\alpha)$ are type II sums.

Let $\mathcal{A}(Z_1, Z_2)$ be the subset of $\hat{\mathfrak{m}}$ satisfying $S_j(\alpha) \sim Z_j$ for $j = 1, 2$. Without loss the generality, we can assume that

$$Z_1 \geq Z_2 \geq X^{g(\delta)+2\varepsilon}. \tag{5.8}$$

Then by Lemma 3.1 and Dirichlet's theorem, for each $\alpha \in \mathcal{A}(Z_1, Z_2)$, there exist integers a_1, q_1, a_2, q_2 such that

$$\left|q_j\lambda_j\alpha - a_j\right| \ll \frac{X^{1+\varepsilon}}{Z_j^2}, \quad (a_j, q_j) = 1, \quad a_j \neq 0 \tag{5.9}$$

and

$$q_j \ll \frac{X^{2+\varepsilon}}{Z_j^2}. \tag{5.10}$$

Then for any $\alpha \in \mathcal{A}(Z_1, Z_2)$, we have

$$\left|\frac{a_j}{\alpha}\right| \ll q_j + \frac{X^{1+\varepsilon}}{Z_j^2}|\alpha|^{-1} \ll q_j. \tag{5.11}$$

Let $\mathcal{A}' = \mathcal{A}(Z_1, Z_2, Q_1, Q_2, k)$ be the subset of $\mathcal{A}(Z_1, Z_2)$ for which $q_j \sim Q_j$ and $a_j \asymp kQ_j$. To prove Lemma 5.1, we need to show that for every combination of Z_1, Z_2, Q_1, Q_1 and k,

$$Z_1^2 Z_2^2 \mu(\mathcal{A}') \min(\tau^2, k^{-2}) \ll \tau X^{1+2g(\delta)+\varepsilon}, \tag{5.12}$$

where $\mu(\mathcal{A}')$ is the Lebesgue measure of $\mathcal{A}'$. First, we notice that for each $\alpha \in \mathcal{A}'$ we have

$$\begin{aligned}\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| &= \left|\frac{a_2(q_1\lambda_1\alpha - a_1) + a_1(a_2 - q_2\lambda_2\alpha)}{\lambda_2\alpha}\right| \\ &\ll X^{1+\varepsilon} \max\left(\frac{Q_1}{Z_2^2}, \frac{Q_2}{Z_1^2}\right) := \theta. \end{aligned} \tag{5.13}$$

Case 1: $Z_1 Z_2 \gg X^{5/2-g(\delta)+2\varepsilon}$. In this case, by (5.8), (5.11) and (5.11)

$$\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| \ll \frac{X^{3+2\varepsilon}}{Z_1^2 Z_2^2} \ll \frac{1}{X^{2-2g(\delta)+2\varepsilon}}. \tag{5.14}$$

Since λ_1/λ_2 is irrational and algebraic, there exist a convergent a/q to λ_1/λ_2 with

$$X^{2-2g(\delta)} \ll q \ll X^{2-2g(\delta)+\varepsilon}. \tag{5.15}$$

Thus we have

$$\left\| a_2 q_1 \frac{\lambda_1}{\lambda_2} \right\| \le \frac{1}{4q}, \quad q_1 \sim Q_1, \ a_2 \asymp kQ_2, \tag{5.16}$$

since X is sufficiently large. Then by the pigeon–hole principle and the Legendre's law of best approximation for continued fractions, the above inequality (5.16) have $\ll \frac{kQ_1Q_2}{q}$ solutions of $|a_2q_1|$. Clearly, each value of $|a_2q_1|$ corresponds to $\ll X^\varepsilon$ values of a_1, a_2, q_1, q_2 by the well–known bound on the divisor function. Hence, we conclude that

$$\mu(\mathcal{A}') \ll X^\varepsilon \frac{kQ_1Q_2}{q} \min\left(\frac{X^{1+\varepsilon}}{Z_1^2 Q_1}, \frac{X^{1+\varepsilon}}{Z_2^2 Q_2} \right) \ll \frac{kX^{1+2\varepsilon} Q_1^{1/2} Q_2^{1/2}}{qZ_1Z_2}. \tag{5.17}$$

Then by (5.10), (5.15) and (5.17), the left–hand side of (5.12) is

$$\ll Z_1^2 Z_2^2 \tau k^{-1} \frac{kX^{1+2\varepsilon} Q_1^{1/2} Q_2^{1/2}}{qZ_1Z_2} \ll \tau \frac{X^{3+3\varepsilon}}{q} \ll \tau X^{1+2g(\delta)+3\varepsilon}. \tag{5.18}$$

Case 2: $Z_1 Z_2 \ll X^{5/2-g(\delta)+2\varepsilon}$. Let $\mathcal{Q}_1$ be the set of q_1 such that $|S_1(\alpha)| \sim Z_1$. By Lemma 3.2, the inequality

$$\left\| a_2 q_1 \frac{\lambda_1}{\lambda_2} \right\| \le \theta, \quad q_1 \in \mathcal{Q}_1, \ a_2 \asymp kQ_2$$

has

$$H \ll |\mathcal{Q}_1| kQ_2 \theta + (Q_1 + kQ_1Q_2 q^{-1} + q\theta) q^\varepsilon \tag{5.19}$$

solutions, where q is defined by (5.15). Then $\mathcal{A}'$ consists of $\ll HX^\varepsilon$ intervals of at most length

$$\min\left(\frac{X^{1+\varepsilon}}{Z_1^2 Q_1}, \frac{X^{1+\varepsilon}}{Z_2^2 Q_2} \right) := \gamma.$$

Note that

$$\theta\gamma = \frac{X^{2+2\varepsilon}}{Z_1^2 Z_2^2}. \tag{5.20}$$

We split into cases according to which term dominates in (5.19).

Case 2a: $H \ll (kQ_1Q_2q^{-1} + q\theta)q^{\varepsilon}$. In this case, the left–hand side of (5.12) is

$$\begin{aligned}
&\ll Z_1^2 Z_2^2 \min(\tau^2, k^{-2})(kQ_1Q_2q^{-1} + q\theta)q^{\varepsilon}X^{\varepsilon}\gamma \\
&\ll \tau\frac{X^{2\varepsilon}Z_1^2Z_2^2Q_1Q_2\gamma}{q} + \tau^2 Z_1^2 Z_2^2 X^{2\varepsilon} q\theta\gamma \\
&\ll \tau\frac{X^{1+3\varepsilon}Z_1Z_2Q_1^{1/2}Q_2^{1/2}}{q} + \tau^2 X^{2+4\varepsilon} q \\
&\ll \tau X^{3+4\varepsilon}q^{-1} + \tau^2 X^{2+4\varepsilon} q \ll \tau X^{1+2g(\delta)+4\varepsilon}.
\end{aligned}$$

Case 2b: $H \ll Q_1 q^{\varepsilon}$. The left–hand side of (5.12) is

$$\begin{aligned}
&\ll \tau^2 Z_1^2 Z_2^2 X^{1+3\varepsilon} Q_1 \min\left(\frac{1}{Z_1^2 Q_1}, \frac{1}{Z_2^2 Q_2}\right) \\
&\ll \tau^2 X^{1+3\varepsilon} Z_2^2 \ll \tau^2 X^{1+3\varepsilon} Z_1 Z_2 \\
&\ll \tau X^{1+5/2-g(\delta)-\delta+5\varepsilon} \ll \tau X^{1+2g(\delta)+5\varepsilon}
\end{aligned}$$

by our assumption $Z_1 \geq Z_2$ and $Z_1Z_2 \ll X^{5/2-g(\delta)+2\varepsilon}$.

Case 2c: $H \ll |\mathcal{Q}_1| kQ_2\theta$. We follow the method of Matomäki in [8] (one can see page 101 of [8]).

If $Z_1 > X^{8/9}$. Then by (5.10), $Q_1 \ll X^{2/9+\varepsilon}$. If $Q_1Q_2 \leq X^{2g(\delta)-1}$, then by the trivial bound $|\mathcal{Q}_1| \leq Q_1$, we have that the left–hand side of (5.12) is

$$\ll Z_1^2 Z_2^2 \min(\tau^2, k^{-2}) H X^{\varepsilon}\gamma \ll \tau X^{2+3\varepsilon}|\mathcal{Q}_1|Q_2 \ll \tau X^{1+2g(\delta)+3\varepsilon}.$$

Thus we can assume that $Q_1 \gg X^{2g(\delta)-1}/Q_2$. By Lemma 15 of [8], we have

$$|\mathcal{Q}_1| \ll \frac{X^{2+\varepsilon}}{Q_1 Z_1^2} + \frac{X^{13/9+\varepsilon}Q_1}{Z_1^2}. \tag{5.21}$$

Thus, the left–hand side of (5.12) is

$$\begin{aligned}
&\ll \tau X^{2+3\varepsilon} Q_2 \left(\frac{X^{2+\varepsilon}}{Q_1 Z_1^2} + \frac{X^{13/9+\varepsilon}Q_1}{Z_1^2}\right) \\
&\ll \tau X^{29/9-2g(\delta)+4\varepsilon} Q_2^2 + \tau X^{5/3+4\varepsilon} Q_1 Q_2 \qquad (5.22) \\
&\ll \tau X^{65/9-6g(\delta)+\varepsilon} + \tau X^{17/3-4g(\delta)+\varepsilon} \ll \tau X^{1+2g(\delta)+\varepsilon}
\end{aligned}$$

by (5.8) and (5.10).

If $\max(Z_1, Z_2) \le X^{8/9}$. We can argue as in the beginning of the proof with roles of q_1 and q_2 swapped to conclude that we can assume that

$$H \ll k\theta \min(|\mathcal{Q}_1|Q_2, |\mathcal{Q}_2|Q_1).$$

We renumber such that $Q_1 \ge Q_2$ (we do not anymore assume that $Z_1 \ge Z_2$). For $Q_2 \le Q_1 \le X^{1/3}$, by (5.21), then the left–hand side of (5.12) is

$$\begin{aligned}
&\ll \tau X^{2+3\varepsilon} Q_2 \left(\frac{X^{2+\varepsilon}}{Q_1 Z_1^2} + \frac{X^{13/9+\varepsilon} Q_1}{Z_1^2} \right) \\
&\ll \tau \frac{X^{4+4\varepsilon}}{Z_1^2} + \tau \frac{X^{31/9+4\varepsilon} Q_1^2}{Z_1^2} \\
&\ll \tau X^{4-2g(\delta)+\varepsilon} + \tau X^{37/9-2g(\delta)+\varepsilon} \ll \tau X^{1+2g(\delta)+\varepsilon}.
\end{aligned}$$

For $Q_1 \ge X^{1/3}$, by Lemma 14 of [8], we have

$$|\mathcal{Q}_1| \ll \frac{X^{4+\varepsilon}}{Z_1^4 Q_1^2}.$$

Thus the left–hand side of (5.12) is

$$\ll \tau X^{2+3\varepsilon} Q_2 \frac{X^{4+\varepsilon}}{Z_1^4 Q_1^2} \ll \tau \frac{X^{6+4\varepsilon}}{Z_1^4 Q_1} \ll \tau X^{17/3-4g(\delta)+\varepsilon} \ll \tau X^{1+2g(\delta)+\varepsilon}.$$

Thus we have proved Lemma for type II sums.

If one or both of $S_1(\alpha)$ and $S_2(\alpha)$, say $S_2(\alpha)$, is a type I sum, We use Lemma 3.1 and Dirichlet's theorem in Diophantine approximation. There exist integers a_2, q_2 depending on α such that

$$|q_2\lambda_2\alpha - a_2| \ll X^{\varepsilon}/Z_2, \quad (a_2, q_2) = 1, \quad a_2 \ne 0$$

and

$$q_2 \ll X^{1+\varepsilon}/Z_2$$

Then we have adopting the notion above that

$$\theta = \max\left(\frac{X^{\varepsilon} Q_1}{Z_2}, \frac{X^{1+\varepsilon} Q_2}{Z_1^2} \right), \quad \gamma = \min\left(\frac{X^{\varepsilon}}{Z_2 Q_2}, \frac{X^{1+\varepsilon}}{Z_1^2 Q_1} \right).$$

Thus we have

$$\gamma\theta = \frac{X^{1+2\varepsilon}}{Z_1^2 Z_2}.$$

By the discussion in the beginning of proof, we only need to consider the case $H \ll Q_1 Q_2 k\theta$. In this case, the left–hand side of (5.12) is

$$\ll Z_1^2 Z_2^2 \min(\tau^2, k^{-2}) H X^{\varepsilon}\gamma \ll \tau X^{\varepsilon} Z_1^2 Z_2^2 Q_1 Q_2 \theta\gamma \ll \tau \frac{X^{4+5\varepsilon}}{Z_1^2} \ll \tau X^{1+2g(\delta)+\varepsilon}.$$

This completes the proof of Lemma 5.1. □

Combining (5.1)–(5.7), we get the following lemma.

Lemma 5.2.

$$\int_{\mathfrak{m}} |F(\alpha)|^2 K(\alpha) d\alpha \ll \tau X^{1+2g(\delta)+\varepsilon}. \tag{5.23}$$

6. The proof of Theorem 1.1

We take $\tau = X^{-\delta}$. Let $\mathcal{V}$ be a well-spaced set. Then by the definition of $\psi(v)$ in section 2, we have $\psi(v) = 0$ for every $v \in \mathbb{E}(\mathcal{V}, X, \delta)$, where $\mathbb{E}(\mathcal{V}, X, \delta)$ is the set of $v \in \mathcal{V}$ with $v \le X$ such that the inequality (1.1) has no solution in primes p_1, p_2. Thus by (2.8), we have

$$\sum_{v\in\mathbb{E}(\mathcal{V},X,\delta)} (J_v(\mathfrak{M}) + J_v(\mathfrak{m}) + J_v(\mathfrak{t})) \le 0 \tag{6.1}$$

By (2.9) and Lemma 4.3, we get

$$\sum_{v\in\mathbb{E}(\mathcal{V},X,\delta)} (J_v(\mathfrak{M}) + J_v(\mathfrak{t})) \gg E(\mathcal{V}, X, \delta)\tau^2 X/(\log X)^2. \tag{6.2}$$

Thus we have

$$\left| \sum_{v\in\mathbb{E}(\mathcal{V},X,\delta)} (J_v(\mathfrak{m})) \right| \gg E(\mathcal{V}, X, \delta)\tau^2 X/(\log X)^3. \tag{6.3}$$

By Cauchy's inequality, we have

$$\begin{aligned}
&\left|\sum_{v\in\mathbb{E}(\mathcal{V},X,\delta)}(J_v(\mathfrak{m}))\right| \\
&\ll\left(\int_{-\infty}^{+\infty}\left|\sum_{v\in\mathbb{E}(\mathcal{V},X,\delta)}e(-v\alpha)\right|^2K(\alpha)d\alpha\right)^{1/2}\left(\int_{\mathfrak{m}}|F(\alpha)|^2K(\alpha)d\alpha\right)^{1/2} \\
&=\left(\sum_{v_1,v_2\in\mathbb{E}(\mathcal{V},X,\delta)}\int_{-\infty}^{+\infty}e((v_1-v_2)\alpha)K(\alpha)d\alpha\right)^{1/2}\left(\int_{\mathfrak{m}}|F(\alpha)|^2K(\alpha)d\alpha\right)^{1/2} \\
&\ll\tau^{1/2}(E(\mathcal{V},X,\delta))^{1/2}\left(\int_{\mathfrak{m}}|F(\alpha)|^2K(\alpha)d\alpha\right)^{1/2}. \qquad (6.4)
\end{aligned}$$

Here we have used that for every $v_1,v_2\in\mathbb{E}(\mathcal{V},X,\delta)\subset\mathcal{V}$, if $v_1\neq v_2$, then there is a constant c such that $|v_1-v_2|>c$, since $\mathcal{V}$ is a well-spaced set. Combining (5.23), (6.3) and (6.4), we have

$$E(\mathcal{V},X,\delta)\ll\tau^{-3}X^{-2}(\log X)^6\int_{\mathfrak{m}}|F(\alpha)|^2K(\alpha)d\alpha\ll X^{2g(\delta)-1+2\delta+\varepsilon}\ll X^{f(\delta)+\varepsilon}.$$

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