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Arithmetic properties for the minus space of weakly holomorphic modular forms

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Let $M_k^!(p)$ be the space of weakly holomorphic modular forms of weight k on $\Gamma_0(p)$, and let $M_k^{!-}(p)$ be the minus space which is the subspace of $M_k^!(p)$ consisting of all eigenforms of the Fricke involution W_p with eigenvalue -1. We are interested in finding a canonical basis for the minus space $M_k^{!-}(p)$ for certain levels. Using this result, along with previous works of Choi and Kim [CK13], we find a canonical basis for the space $M_k^!(p)$, and investigate its arithmetic properties. We also give another generalization of [CK13] to the cases of square-free integer levels.

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1. Introduction

Let $\Gamma_0^+(p)$ be the group generated by the Hecke group $\Gamma_0(p)$ and the Fricke involution $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$. Throughout the paper we assume that p is a prime number for which the genus of $\Gamma_0^+(p)$ is zero, that is, p belongs to the set

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

Let $M_k(p)$ (resp. $S_k(p)$) be the vector space of holomorphic modular forms (resp. cusp forms) of weight k for $\Gamma_0(p)$, and let $M_k^+(p)$ (resp. $S_k^+(p)$) be the space of weight k modular forms (resp. cusp forms) on $\Gamma_0^+(p)$. More precisely, the space $M_k^+(p)$ (resp. $S_k^+(p)$) is a subspace of $M_k(p)$ (resp. $S_k(p)$) consisting of all modular forms (resp. cusp forms) f which are invariant under W_p, i.e.,

$$M_k^+(p) := \{f \in M_k(p) : f\,|_k\, W_p = f\} \quad \text{and} \quad S_k^+(p) := \{f \in S_k(p) : f\,|_k\, W_p = f\}.$$

Similarly we define the other subspaces of $M_k(p)$ (resp. $S_k(p)$) as:

$$M_k^-(p) := \{f \in M_k(p) : f\,|_k\, W_p = -f\}, \quad S_k^-(p) := \{f \in S_k(p) : f\,|_k\, W_p = -f\}.$$

We call these the minus spaces of holomorphic modular forms, and cusp forms respectively.

Let $M_k^!(p)$ be the space of weakly holomorphic modular forms (that is, meromorphic with poles only at the cusps) of weight k for $\Gamma_0(p)$, and let $M_k^{!+}(p)$ be the space of weakly holomorphic modular forms of weight k for $\Gamma_0^+(p)$. The minus space $M_k^{!-}(p)$ is defined to be the subspace of $M_k^!(p)$ consisting of all eigenforms of W_p with eigenvalue -1.

For $f \in M_k^!(p)$, it can be easily seen that $f = \frac{f+f|_kW_p}{2} + \frac{f-f|_kW_p}{2}$, with $\frac{f+f|_kW_p}{2} \in M_k^{!+}(p)$, $\frac{f-f|_kW_p}{2} \in M_k^{!-}(p)$. Hence we have the following proposition.

Proposition 1.1. *Let $M_k^!(p)$, $M_k^{!+}(p)$ and $M_k^{!-}(p)$ be the spaces defined above. The space $M_k^!(p)$ is decomposed into the direct sum of the subspaces $M_k^{!+}(p)$ and $M_k^{!-}(p)$, that is,*

$$M_k^!(p) = M_k^{!+}(p) \oplus M_k^{!-}(p).$$

Choi and Kim [CK13] found a canonical basis for $M_k^{!+}(p)$ for any even integer k. Accordingly, Proposition 1.1 tells us that if we find a basis for the minus space $M_k^{!-}(p)$, we can construct a basis for the space $M_k^!(p)$. In this paper we address the question of finding a canonical basis for the space $M_k^{!-}(p)$, and investigate its arithmetic properties. In fact, the canonical basis we construct in this paper consists of the form $f_{k,m}^-$ whose Fourier expansion is given by

$$f_{k,m}^- = q^{-m} + \sum_{n > m_k^-} a_k^-(m,n) q^n \quad (q = e^{2\pi i z})$$

for every integer $m \geq -m_k^-$, where m_k^- is the maximal vanishing order at the cusp ∞ for a nonzero $f \in M_k^{!-}(p)$. The basis of the minus space $M_k^{!-}(p)$ has many properties similar to those of the space $M_k^{!+}(p)$. For example, the coefficients $a_k^-(m,n)$ of basis element $f_{k,m}^-$ for $M_k^{!-}(p)$ are also integral and satisfy the duality relation $a_k^-(n,m) = -a_{2-k}^-(m,n)$ as in the case of the space $M_k^{!+}(p)$.

In the theory of modular forms the classical j-invariant is of particular interest. The coefficients of the j-function have special arithmetic properties: for example, they appear as dimensions of a special graded representation of the Monster group. Let $c(n)$ be the n-th Fourier coefficient of j such that

$$j(z) = \frac{1}{q} + \sum_{n \geq 0} c(n)q^n.$$

In 1949 Lehner showed [Leh43,Leh49a,Leh49b] that for any positive integers a, b, c, n and a nonnegative integer d,

$$c(2^a 3^b 5^c 7^d 11n) \equiv 0 \pmod{2^{3a+8} 3^{2b+3} 5^{c+1} 7^d 11}.$$

Similar results to above congruences have recently been proven in higher level cases. Let $M_k^\sharp(p)$ be the subspace of $M_k^!(p)$ with poles allowed only at the cusp at ∞ (see [AJ13,JT15]). Andersen and Jenkins [AJ13] extended Lehner's theorem for all elements of a canonical basis for $M_0^\sharp(p)$ for $p \in \{2,3,5,7\}$.

Theorem. *[AJ13, Theorem 2] Let $p \in \{2,3,5,7\}$, and let*

$$f_{0,m}^{(p),\sharp}(z) = q^{-m} + \sum_{n=0}^{\infty} a_0^{(p),\sharp}(m,n)q^n$$

be an element of the canonical basis of $M_0^\sharp(p)$, with $m = p^\alpha m'$, $n = p^\beta n'$, $(m',p) = 1$, and $(n',p) = 1$. Then for $\beta > \alpha$,

$$\begin{aligned}
a_0^{(2),\sharp}(2^\alpha m', 2^\beta n') &\equiv 0 \pmod{2^{3(\beta-\alpha)+8}},\\
a_0^{(3),\sharp}(3^\alpha m', 3^\beta n') &\equiv 0 \pmod{3^{2(\beta-\alpha)+3}},\\
a_0^{(5),\sharp}(5^\alpha m', 5^\beta n') &\equiv 0 \pmod{5^{(\beta-\alpha)+1}},\\
a_0^{(7),\sharp}(7^\alpha m', 7^\beta n') &\equiv 0 \pmod{7^{(\beta-\alpha)}}.
\end{aligned}$$

For the other coefficients of $f_{0,m}^{(p),\sharp}$, Jenkins and D.J. Thornton [JT15] showed that similar congruences hold.

Theorem. *[JT15, Theorem 1] Let $p \in \{2,3,5,7,13\}$ and let $f_{0,m}^{(p),\sharp} = q^{-m} + \sum_{n\geq 1} a_0^{(p),\sharp} \times (m,n)q^n$ be a weakly holomorphic modular form in $M_0^\sharp(p)$. Let $m = p^\alpha m'$ and $n = p^\beta n'$*

with m', n' not divisible by p. Then for $\alpha > \beta$, we have

$$a_0^{(2),\sharp}(2^\alpha m', 2^\beta n') \equiv 0 \pmod{2^{4(\alpha-\beta)+8}},$$
$$a_0^{(3),\sharp}(3^\alpha m', 3^\beta n') \equiv 0 \pmod{3^{3(\alpha-\beta)+3}},$$
$$a_0^{(5),\sharp}(5^\alpha m', 5^\beta n') \equiv 0 \pmod{5^{2(\alpha-\beta)+1}},$$
$$a_0^{(7),\sharp}(7^\alpha m', 7^\beta n') \equiv 0 \pmod{7^{2(\alpha-\beta)}},$$
$$a_0^{(13),\sharp}(13^\alpha m', 13^\beta n') \equiv 0 \pmod{13^{(\alpha-\beta)}}.$$

Note that a basis for the space $M_k^!(p)$ is the union of the basis for $M_k^{!-}(p)$ which is built in this paper and the basis for $M_k^{!+}(p)$ found in [CK13]. Hence using these canonical bases for $M_k^{!+}(p)$ and $M_k^{!-}(p)$, we can extend the results of Andersen, Jenkins and Thornton [AJ13,JT15] to forms on $M_0^!(p)$. (See Theorem 5.1, Theorem 5.3, and Remark 5.4.)

Additionally, let $\Gamma_0^*(N)$ be the group generated by $\Gamma_0(N)$ and all Atkin–Lehner involutions $W_e = \left(\begin{smallmatrix} a\sqrt{e} & b/\sqrt{e} \\ cN/\sqrt{e} & d\sqrt{e}\end{smallmatrix}\right)$ of N where $a, b, c, d \in \mathbb{Z}$, $e||N$, and $\det W_e = 1$. We extend [CK13] to weakly holomorphic modular forms for $\Gamma_0^*(N)$ in the case of square-free integer level N for which the genus of $\Gamma_0^*(N)$ is zero. Indeed the following results of [CK13] will be extended in Section 6: the construction of a canonical basis, duality, and integrality of Fourier coefficients of basis elements.

This paper is organized as follows. A canonical basis for the space $M_k^{!-}(p)$ is constructed in Section 2 (see Theorem 2.5) and integrality is proved by giving the explicit recipe of construction for the basis elements in Section 3. We derive the duality relation in Section 4 and the divisibility properties in Section 5. Finally we generalize the results of [CK13] to the cases of square-free integer levels in Section 6.

2. Basis for the space $M_k^{!-}(p)$

In this section we construct a basis for the space $M_k^{!-}(p)$.

Lemma 2.1.

(1) *Let $k > 2$ be an even integer. Then we have*

$$\dim S_k^-(2) = \begin{cases} \left\lfloor \frac{k}{4} \right\rfloor - \left\lfloor \frac{k}{8} \right\rfloor, & k \equiv 2 \pmod 8, \\ \left\lfloor \frac{k}{4} \right\rfloor - \left\lfloor \frac{k}{8} \right\rfloor - 1, & k \not\equiv 2 \pmod 8, \end{cases}$$

$$\dim S_k^-(3) = \begin{cases} \left\lfloor \frac{k}{3} \right\rfloor - \left\lfloor \frac{k}{6} \right\rfloor, & k \equiv 2, 6 \pmod{12}, \\ \left\lfloor \frac{k}{3} \right\rfloor - \left\lfloor \frac{k}{6} \right\rfloor - 1, & k \not\equiv 2, 6 \pmod{12}, \end{cases}$$

and for $p > 3$

$$\dim S_k^-(p) = \begin{cases} (k-1)\left(\frac{p-13}{12}\right) - \left(\frac{p-7}{6}\right)\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \frac{k}{2} - 1, & p \equiv 1 \pmod{12}, \\ (k-1)\left(\frac{p-5}{12}\right) - \left(\frac{p+1}{6}\right)\left\lfloor \frac{k}{4} \right\rfloor + \frac{k}{2} - 1, & p \equiv 5 \pmod{12}, \\ (k-1)\left(\frac{p-7}{12}\right) - \left(\frac{p+5}{6}\right)\left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor + \frac{k}{2} - 1, & p \equiv 7 \pmod{12}, \\ (k-1)\left(\frac{p+1}{12}\right) - \left(\frac{p+13}{6}\right)\left\lfloor \frac{k}{4} \right\rfloor + \frac{k}{2} - 1, & p \equiv 11 \pmod{12}. \end{cases}$$

(2) *When* $k = 2$,

$$\dim S_2^-(p) = \begin{cases} \frac{p-13}{12}, & p \equiv 1 \pmod{12}, \\ \frac{p-5}{12}, & p \equiv 5 \pmod{12}, \\ \frac{p-7}{12}, & p \equiv 7 \pmod{12}, \\ \frac{p+1}{12}, & p \equiv 11 \pmod{12}. \end{cases}$$

Proof.

(1) By Proposition 1.1, $\dim S_k^-(p)$ can be found directly from $\dim S_k(p)$ and $\dim S_k^+(p)$. The dimension formula for $S_k^+(p)$ is presented in [CK13, Lemma 2.2]. Let $\nu_m = \nu_m(\Gamma_0(p))$ be the number of $\Gamma_0(p)$-inequivalent elliptic points of order m, and let $g = g(\Gamma_0(p))$ be the genus of $\Gamma_0(p)$. The dimension formula [DS05, Theorem 3.5.1] gives

$$\dim S_k(p) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor \nu_2 + \left\lfloor \frac{k}{3} \right\rfloor \nu_3 + (k-2).$$

In the cases $p \in \{2, 3\}$ it is not difficult to verify the assertion. For the case $p > 3$, using the formula [CK13, Lemma 2.1] for the genus g for each p, and data for the dimension formulas [DS05, Figure 3.3], we have the dimension of $S_k(p)$ as follows:

$$\dim S_k(p) = \begin{cases} (k-1)\left(\frac{p-13}{12}\right) + 2\left\lfloor \frac{k}{4} \right\rfloor + 2\left\lfloor \frac{k}{3} \right\rfloor - 1, & p \equiv 1 \pmod{12}, \\ (k-1)\left(\frac{p-5}{12}\right) + 2\left\lfloor \frac{k}{4} \right\rfloor - 1, & p \equiv 5 \pmod{12}, \\ (k-1)\left(\frac{p-7}{12}\right) + 2\left\lfloor \frac{k}{3} \right\rfloor - 1, & p \equiv 7 \pmod{12}, \\ (k-1)\left(\frac{p+1}{12}\right) - 1, & p \equiv 11 \pmod{12}. \end{cases}$$

Now we get the assertion.

(2) By [DS05, Theorem 3.5.1], $\dim S_2(\Gamma)$ is equal to the genus of Γ. Thus we easily get the assertion using [CK13, Lemma 2.1]. □

Proposition 2.2 *(Miller basis). Let k be a positive even integer. Suppose $d = \dim S_k^-(p) \geq 1$. Then there are $f_1, \ldots, f_d \in S_k^-(p)$ such that $\{f_1, \ldots, f_d\}$ is a basis for $S_k^-(p)$ with $f_i = q^i + Q(q^{d+1})$ for $i = 1, \ldots, d$.*

Proof. Let $t := \max\{\mathrm{ord}_\infty f \mid 0 \neq f \in S_k^-(p)\}$ and denote by f_t the unique cusp form having Fourier expansion of the form $q^t + O(q^{t+1})$. Multiplying f_t by the Hauptmodul

j_p^+ for $\Gamma_0^+(p)$, we obtain the set $\mathcal{B} = \{f_t, f_t j_p^+, \dots, f_t (j_p^+)^{t-1}\}$. It is clear that the set $\mathcal{B}$ is a basis for $S_k^-(p)$, and we are forced to have $t = d$. By an appropriate linear combination of elements in $\mathcal{B}$, we have a basis consisting of elements f_i with the Fourier expansion of the form $f_i = q^i + O(q^{d+1})$ for each $i = 1, \dots, d$. $\square$

Remark 2.3. Suppose that $d = \dim S_k^-(p) \geq 1$. By Lemma 2.2 there exists a unique cusp form f_d with a q-expansion of the form

$$f_d = q^d + O(q^{d+1}).$$

We denote $\Delta_{p,k}^-(z)$ by the unique cusp form f_d of $S_k^-(p)$. Let

$$E_k(z) = 1 - 2kB_k^{-1} \sum_{n \geq 1} \sigma_{k-1}(n) q^n, \quad E_{p,k}^-(z) = \frac{1}{1 - p^{k/2}} (E_k(z) - p^{k/2} E_k(pz))$$

where B_k is the k-th Bernoulli number and σ_{k-1} stands for the usual divisor sum. When $d = 0$, we set $\Delta_{p,k}^-(z) = E_{p,k}^-(z) = 1 + O(q)$.

For further discussion, we need to review the previous results [CK13] related to a canonical basis for the space $M_k^{!+}(p)$:

Lemma 2.4. *[CK13]*

(1) *The space $S_k^+(p)$ has a Miller basis which contains $\Delta_{p,k}^+$, where $\Delta_{p,k}^+$ is the cusp form of maximal vanishing order at infinity in the space $S_k^+(p) - \{0\}$. Note that* $\max\{ord_\infty f \mid f \neq 0 \in S_k^+(p)\} = \dim S_k^+(p)$.
(2) *Put*

$$\delta = \begin{cases} 8, & \text{if } p = 2, \\ 12, & \text{if } p = 3, \\ 12, & \text{if } p \equiv 1, 7 \pmod{12}, \\ 4, & \text{if } p \equiv 5, 11 \pmod{12}. \end{cases}$$

Then δ is the smallest positive weight k such that there exists a cusp form $f \in S_k^+(p)$ with

$$\dim S_k^+(p) = ord_\infty f = \frac{p+1}{24} k.$$

Furthermore if we let $\Delta_p^+(z) = (\eta(z)\eta(pz))^\delta$, then Δ_p^+ is the unique normalized cusp form in $S_\delta^+(p)$ such that $ord_\infty \Delta_p^+ = \frac{p+1}{24}\delta$.

(3) *Let* $m^+_{p,k} = \max\{ord_\infty f \mid 0 \neq f \in M^{!+}_k(p)\}$. *For integer* m *such that* $-m \leq m^+_{p,k}$, *there is a unique weakly holomorphic modular form* $f^+_{k,m}$ *with* q*-expansion of the form*

$$f^+_{k,m} = \frac{1}{q^m} + O(q^{m^+_{p,k}+1}),$$

and the set of these $f^+_{k,m}$ *forms a basis for the space* $M^{!+}_k(p)$. *In particular, if* $k = 0$, *then* $m^+_{p,k} = 0$ *and* $f^+_{0,m}$ *can be expressed as*

$$f^+_{0,m} = \frac{1}{q^m} + O(q) = F_m(j^+_p),$$

where $F_m(x)$ *is a monic polynomial of degree* m *in* x.

Unless otherwise noted, δ, $m^+_{p,k}$, Δ^+_p, and $\Delta^+_{p,k}$ are the same as given in Lemma 2.4. Now we are ready to find a canonical basis for the minus space $M^{!-}_k(p)$ of weakly holomorphic modular forms.

Theorem 2.5. *Let* $k \in 2\mathbb{Z}$. *We write* $k = \delta l_k + r_k$ *where* $r_k \in \{2, 4, 6, \ldots, \delta\}$. *Then*

(1) *For any non-zero* $f \in M^{!-}_k(p)$,

$$ord_\infty f \leq \frac{p+1}{24}\delta l_k + \dim S^-_{r_k}(p).$$

(2) *We put* $m^-_k = m^-_{p,k} = \frac{p+1}{24}\delta l_k + \dim S^-_{r_k}(p)$. *For each* $m \in \mathbb{Z}$, *such that* $-m \leq m^-_k$, *there exists a unique weakly holomorphic modular form* $f^-_{k,m} \in M^{!-}_k(p)$ *with a* q*-expansion of the form*

$$f^-_{k,m} = q^{-m} + O(q^{m^-_k+1}).$$

Proof.

(1) Suppose that $\mathrm{ord}_\infty f > \frac{p+1}{24}\delta l_k + \dim S^-_{r_k}(p)$. We set $g = f/(\Delta^+_p)^{l_k}$. Observing

$$\begin{aligned}\mathrm{ord}_\infty g &= \mathrm{ord}_\infty f - (\mathrm{ord}_\infty \Delta^+_p) l_k \\ &= \mathrm{ord}_\infty f - \frac{p+1}{24}\delta l_k > \dim S^-_{r_k}(p) \geq 0,\end{aligned}$$

we see that $g \in S^-_{r_k}(p)$. This contradicts Proposition 2.2.

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(2) We observe that

$$
\begin{aligned}
(\Delta_p^+)^{l_k}\Delta_{p,r_k}^-(j_p^+)^{m+m_k^-} &= q^{-m}+\cdots,\\
(\Delta_p^+)^{l_k}\Delta_{p,r_k}^-(j_p^+)^{m+m_k^- -1} &= q^{-m+1}+\cdots,\\
&\vdots\\
(\Delta_p^+)^{l_k}\Delta_{p,r_k}^- &= q^{m_k^-}+\cdots.
\end{aligned}
$$

Now $f_{k,m}^-$ is constructed by taking a suitable linear combination of the above forms. □

Corollary 2.6. *Let $k \in 2\mathbb{N}$. Then we have*

$$\dim S_k^-(p) = \frac{p+1}{24}\delta l_k + \dim S_{r_k}^-(p).$$

3. Explicit construction of $f_{k,m}^-$ and their integrality

In this section, we investigate the recipe of the explicit construction of the basis element $f_{k,m}^-$, and through it we prove that every $f_{k,m}$ has integral Fourier coefficients. Note that the set of these $f_{k,m}^-$ given in Theorem 2.5 forms a basis for the space $M_k^{!-}(p)$. We observe that $f_{k,-m_k^-}^- = (\Delta_p^+)^{l_k}\Delta_{p,r_k}^-$. Now for each positive integer n, we obtain $f_{k,n-m_k^-}^-$ by multiplying $f_{k,n-1-m_k^-}^-$ by j_p^+ and then subtracting off multiples of $f_{k,d-m_k^-}^-$ to successively kill the coefficients of $q^{-d+m_k^-}$ for $0 \le d \le n-1$. This construction shows that

$$f_{k,m}^- = (\Delta_p^+)^{l_k}\Delta_{p,r_k}^- F_{k,m+m_k^-}(j_p^+),$$

where $F_{k,D}(x)$ is a monic polynomial of degree D in x.

It is well known [CY96, p. 265] that the Hauptmodul j_p^+ has integral Fourier coefficients. Being an eta-product, $\Delta_p^+(z)$ also has integral Fourier coefficients. Thus for the explicit construction and the integrality question of the basis element $f_{k,m}$, we have only to consider $\Delta_{p,r_k}^-(z)$. Moreover it then follows from the integrality of the Fourier coefficients of such forms that the polynomial $F_{k,D}(x)$ has integral coefficients.

Recall from Remark 2.3 that Δ_{p,r_k}^- is the unique cusp form in $S_{r_k}^-(p)$ whose vanishing order at infinity is the same as the dimension of the space $S_{r_k}^-(p)$. We present dimensions of $S_{r_k}^-(p)$ obtained from Lemma 2.1 for each p, δ, and r_k in Table 1. Note that $2 \le r_k \le \delta$, and hence we don't need to consider the case of $r_k > \delta$. Accordingly, in the Table 1 below, when $r_k > \delta$, we leave the corresponding place blank.

Now we divide (p, r_k) into five cases:

Case (1) $\dim S_{r_k}^-(p) = 0$**:** In this case, according to Table 1, we only consider (p, r_k) in a set

Table 1
Dimensions of $S^-_{r_k}(p)$ for each case.

p	δ	$\dim S^-_{r_k}(p)$					
		$r_k = 2$	$r_k = 4$	$r_k = 6$	$r_k = 8$	$r_k = 10$	$r_k = 12$
2	8	0	0	0	0		
3	12	0	0	1	0	1	1
5	4	0	0				
7	12	0	0	2	1	3	3
11	4	1	0				
13	12	0	1	3	3	5	6
17	4	1	1				
19	12	1	1	5	4	8	8
23	4	2	1				
29	4	2	2				
31	12	2	2	8	7	13	13
41	4	3	3				
47	4	4	3				
59	4	5	4				
71	4	6	5				

$$S=\{(2,2),(2,4),(2,6),(2,8),(3,2),(3,4),(3,8),(5,2),(5,4),(7,2),(7,4),(11,4),(13,2)\}.$$

Note that we already set

$$\Delta^-_{p,r_k} = E^-_{p,r_k}(z) = \frac{1}{1-p^{r_k/2}}(E_{r_k}(z) - p^{r_k/2}E_{r_k}(pz))$$

in Remark 2.3 when $\dim S^-_{r_k}(p) = 0$. Recall that the Fourier expansion of E_k is given by $E_k(z) = 1 - 2kB_k^{-1}\sum_{n\geq 1}\sigma_{k-1}(n)q^n$. Since $B_2^{-1} = 6$, $B_4^{-1} = -30$, $B_6^{-1} = 42$, and $B_8^{-1} = -30$, we have

$$1 - p^{r_k/2} \mid 2r_k B^{-1}_{r_k},$$

for all $(p, r_k) \in S$. Thus $E^-_{p,r_k}(z) \in \mathbb{Z}[[q]]$.

Case (2) $\dim S^-_{r_k}(p) > 0$ **and** $p = 3$ **or** $p \equiv 7 \pmod{12}$**:** For $p \in \{3, 7, 19, 31\}$, we consider the function $f = \eta(z)^6\eta(pz)^6$. Then by [Ono04, Theorem 1.64], $f \in M_6(\Gamma_0(p))$. Moreover since the Dedekind eta function satisfies ([Ono04, Theorem 1.61])

$$\eta\left(-\frac{1}{z}\right) = (-iz)^{1/2}\eta(z),$$

we have

$$\begin{aligned}\eta(z)^6\eta(pz)^6 \mid_6 W_p &= (\sqrt{p}z)^{-6}\eta\left(-\frac{1}{pz}\right)^6\eta\left(-\frac{1}{z}\right)^6 \\ &= -\eta(z)^6\eta(pz)^6.\end{aligned}$$

This means $f \in S^-_6(p)$ with $\operatorname{ord}_\infty f = \frac{p+1}{4} = \dim S^-_6(p)$. Therefore $f = \eta(z)^6\eta(pz)^6$ is the cusp form $\Delta^-_{p,6}$ we desired, and has integral Fourier coefficients. For the cases of the other weights, it follows from Table 1 and definition of $\Delta^+_{p,k}$ in [CK13] that

$$\dim S_6^-(p) - 1 = \dim S_8^-(p), \tag{1}$$

and

$$\Delta_{p,14}^+ = \Delta_{p,2}^- \times \Delta_{p,12}^- = \Delta_{p,4}^- \times \Delta_{p,10}^- = \Delta_{p,6}^- \times \Delta_{p,8}^-. \tag{2}$$

The observation (1) yields that $\Delta_{p,8}^- = \Delta_{p,6}^- \times (-D(j_p^+))$, where D is the differential operator defined by $D = q\frac{d}{dq}$, and it implies that $\Delta_{p,8}^-$ also has integral Fourier coefficients.

And the observation (2) implies that it suffices to consider Δ_{p,r_k} for the case $r_k \in \{2, 4\}$ or $r_k \in \{10, 12\}$. Since the case when $(p, r_k) \in \{(3, 2), (3, 4), (7, 2), (7, 4)\}$ is included in the Case (1) above, now the only remaining case is when p is 19 or 31. We also observe that

$$\Delta_{p,10}^- = \Delta_{p,4}^+ \times \Delta_{p,6}^-, \quad \Delta_{p,12}^- = \Delta_{p,6}^+ \times \Delta_{p,6}^-,$$

for $p \in \{19, 31\}$. Using these identities, now we obtain the Δ_{p,r_k}^- for all cases, and clearly $\Delta_{p,10}^-$ and $\Delta_{p,12}^-$ have integral coefficients. Accordingly, $\Delta_{p,2}^-$ and $\Delta_{p,4}^-$ also have integral Fourier coefficients.

Case (3) $\dim S_{r_k}^-(p) > 0$ **and** $p \equiv 1 \pmod{12}$**:** In this case, the only p is 13. First we consider the eta-quotient $\eta(z)^2\eta(13z)^{-2}$. Then by [Ono04, Theorem 1.64] $\eta(z)^2\eta(13z)^{-2}$ is $\Gamma_0(13)$-invariant, and satisfies

$$\left.\frac{\eta(z)^2}{\eta(13z)^2}\right|_0 W_{13} = 13\frac{\eta(13z)^2}{\eta(z)^2}.$$

Let $\eta_{13}(z)$ be the form

$$\eta_{13}(z) := \frac{\eta(z)^2}{\eta(13z)^2} - 13\frac{\eta(13z)^2}{\eta(z)^2}.$$

Then $\eta_{13}(z)$ is not only a modular function on $\Gamma_0(13)$ but also an eigenform of the Fricke involution W_{13} with eigenvalue -1. Moreover its order of vanishing at ∞ is -1. Thus the function

$$\Delta_{13}^+(z)\eta_{13}(z) = \eta(z)^{14}\eta(13z)^{10} - 13\eta(z)^{10}\eta(13z)^{14}$$

lies in the space $S_{12}^-(13)$, and has vanishing order $\mathrm{ord}_\infty f_{13} = 6$. By the uniqueness of the cusp form with maximal order, we conclude that

$$\Delta_{13,12}^- = \Delta_{13}^+\eta_{13},$$

and clearly $\Delta_{13,12}^-$ has integral Fourier coefficients. In a similar way, it is not difficult to find

$$\Delta_{13,8}^- = \Delta_{13,8}^+\eta_{13}.$$

From Table 1 and definition of $\Delta^+_{p,k}$ in [CK13], we observe that

$$\Delta^-_{13,10} = \Delta^-_{13,6} \times \Delta^+_{13,4}, \quad \Delta^-_{13,8} = \Delta^-_{13,4} \times \Delta^+_{13,4}$$

and

$$\Delta^+_{13,14} = \Delta^-_{13,2} \times \Delta^-_{13,12} = \Delta^-_{13,4} \times \Delta^-_{13,10} = \Delta^-_{13,6} \times \Delta^-_{13,8}.$$

Since we have $\Delta^-_{13,12}$ and $\Delta^-_{13,8}$, we can get $\Delta^-_{13,k}$ for each k, using these identities. Note that every Δ_{13,r_k} obtained above also has integer coefficients.

Case (4) $\dim S^-_{r_k}(p) > 0$ **and** $p \equiv 11 \pmod{12}$**:** This case occurs when p belongs to $\{11, 23, 47, 59, 71\}$. From Table 1, we observe that

$$\Delta^-_{p,4} = \Delta^-_{p,2} \times (-D(j^+_p)).$$

That means we need only to construct $\Delta^-_{p,2}$ for this case. It is not difficult to find

$$\Delta^-_{p,2} = \eta(z)^2\eta(pz)^2,$$

and $\Delta^-_{p,2}$ and $\Delta^-_{p,4}$ have integral Fourier coefficients.

Case (5) $\dim S^-_{r_k}(p) > 0$ **and** $p \equiv 5 \pmod{12}$**:** In this case, the only p are those in the set $\{5, 17, 29, 41\}$. For $p = 5$, we have $\dim S^-_2(5) = \dim S^-_4(5) = 0$, and hence this case is included in Case (1) described above. In addition, from Table 1, we have that

$$\Delta^-_{p,4} \times \Delta^-_{p,2} = \Delta^+_{p,6}.$$

Hence we only need to consider either $\Delta^-_{p,2}$ or $\Delta^-_{p,4}$, for each p. Thus we concentrate the cases when (p, r_k) belongs to the set $\{(17, 2), (29, 2), (41, 2)\}$.

Note that $M^-_2(p) = \mathbb{C}(E^-_{p,2}(z)) \oplus S^{-,new}_2(p)$, where $S^{-,new}_2(p)$ is the subspace of $S^-_2(p)$ consisting of newforms. (See [Kri95, Theorem 1].) It means $S^-_2(p) = S^{-,new}_2(p)$. It follows from the well-known theory [Zag89, p. 263] that $S^-_2(p) = S^{-,new}_2(p)$ splits as sum of subspaces of some dimensions $d_1, \ldots, d_r \geq 1$, each of which is spanned by some normalized Hecke eigenform with integral Fourier coefficients in a totally real number field K_i of degree d_i over $\mathbb{Q}$, and the algebraic conjugates of this form. It means that we can obtain the forms by considering the various real embeddings of K_i. More precisely, let $f = \sum a_n q^n$ be a normalized Hecke eigenform in $S^{-,new}_2(p)$, and let K_f be a number field which is extended by coefficients of f. Let $[K_f : \mathbb{Q}] = d$. Then $S^{-,new}_2(p)$ has a d-dimensional splitting factor S_f which is spanned by $f^{\sigma_1}, f^{\sigma_2}, \ldots, f^{\sigma_d}$, where $\sigma_1, \sigma_2, \ldots, \sigma_d$ are embeddings of K_f. Indeed, let $\alpha_1, \alpha_2, \ldots, \alpha_d$ form an integral basis of K_f. Then the coefficients $a_n = \sum_{j=1}^{d} a_{n,j}\alpha_j$ where $a_{n,j} \in \mathbb{Z}$, and hence

$$f = \sum a_n q^n = \sum_{j=1}^{d} f_j \alpha_j,$$

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where f_j is a q-expansion with rational integer coefficients $a_{n,j}$. For some conjugate of f, if we write

$$f^{\sigma_i} = \alpha_1^{\sigma_i} f_1^{\sigma_i} + \cdots + \alpha_d^{\sigma_i} f_d^{\sigma_i} = \alpha_1^{\sigma_i} f_1 + \cdots + \alpha_d^{\sigma_i} f_d,$$

we have a system of linear equations of the matrix form

$$\begin{pmatrix} f^{\sigma_1} \\ \vdots \\ f^{\sigma_d} \end{pmatrix} = \begin{pmatrix} \alpha_1^{\sigma_1} & \alpha_2^{\sigma_1} & \cdots & \alpha_d^{\sigma_1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{\sigma_2} & \alpha_2^{\sigma_d} & \cdots & \alpha_d^{\sigma_d} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}.$$

Furthermore we also have $f^{\sigma_i} = (-f \mid_2 W_p)^{\sigma_i}$, since $f \in S_2^{-,new}(p)$. Accordingly for any conjugate f^{σ_i}, we obtain

$$\begin{aligned} f^{\sigma_i} &= (-f \mid_2 W_p)^{\sigma_i} \\ &= \left(-\sum_{j=1}^{d} (f_j \mid W_p)\alpha_j \right)^{\sigma_i} \\ &= -\sum_{j=1}^{d} (f_j \mid W_p)^{\sigma_i} \alpha_j^{\sigma_i} = -\sum_{j=1}^{d} (f_j \mid W_p)\alpha_j^{\sigma_i}, \end{aligned}$$

which can be expressed as another system of linear equations of the matrix form

$$\begin{pmatrix} f^{\sigma_1} \\ \vdots \\ f^{\sigma_d} \end{pmatrix} = \begin{pmatrix} \alpha_1^{\sigma_1} & \alpha_2^{\sigma_1} & \cdots & \alpha_d^{\sigma_1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{\sigma_2} & \alpha_2^{\sigma_d} & \cdots & \alpha_d^{\sigma_d} \end{pmatrix} \begin{pmatrix} -f_1 \mid W_p \\ \vdots \\ -f_d \mid W_p \end{pmatrix}.$$

Since $\det(\alpha_j^{\sigma_i})^2 \neq 0$, which is the discriminant of the number field K_f, the linear system above is invertible. This means that each f_i belongs to $S_2^-(p)$, and $f_1, f_2, \ldots, f_d$ span the splitting factor S_f. Taking union of such f_j's in each subspace S_f, we can get a basis for $S_2^-(p)$ consisting of the form with rational integer coefficients. Denote these basis elements by $g_1, g_2, \ldots, g_t$ where $\dim S_2^-(p) = t$. Then we can obtain a Miller basis which contains $\Delta_{p,2}^-$ by proper $\mathbb{Q}$-linear combination of the cusp forms

$$\Delta_{p,2}^- = \frac{n_1}{d_1} g_1 + \cdots + \frac{n_t}{d_t} g_t,$$

where $n_i, d_i \in \mathbb{Z}$ for each i. Letting $D = \operatorname{lcm}(d_1, d_2, \ldots, d_t)$, we see that $D\Delta_{p,2}^-$ has integral Fourier coefficients. However, we cannot be sure that $\Delta_{p,2}^-$ also has integral Fourier coefficients. To investigate the integrality of $\Delta_{p,2}^-$, the following lemma is required.

Lemma 3.1 *(Sturm's bound for the minus space). Let K be a fixed number field, $\mathcal{O}_K$ be the ring of integers in K, and l be a prime ideal of $\mathcal{O}_K$. For $f = \sum_{n=1}^{\infty} a_n q^n \in M_k^-(p)$*

with $a_n \in \mathcal{O}_K$*, define* $ord_l f := \inf\{n \in \mathbb{Z}_{\geq 0} \mid a_n \equiv 0 \pmod{l}\}$*, with the convention* $ord_l f = \infty$ *if* $l \mid a_n$ *for all* n*. If*

$$ord_l f > \frac{p+1}{24}k,$$

then $ord_l f = \infty$*.*

Proof. Let $f \in M_k^-(p)$, with $\mathrm{ord}_l f > \frac{p+1}{24}k$. Then we have $\mathrm{ord}_l f^\delta > \frac{p+1}{24}k\delta$. We know that Δ_p^+ has a Fourier expansion

$$\Delta_p^+(z) = q^{\frac{p+1}{24}\delta} + O(q^{\frac{p+1}{24}\delta+1}),$$

and hence we have

$$f(z)^\delta \left(\Delta_p^+(z)\right)^{-k} = \sum_{n=-\frac{p+1}{24}k\delta}^{\infty} c(n)q^n,$$

where the coefficients $c(n)$ are in $\mathcal{O}_K$. On the other hand, since δ is even, $f^\delta \left(\Delta_p^+\right)^{-k}$ is a weakly holomorphic modular form of weight 0 on $\Gamma_0^+(p)$. As we have seen in Lemma 2.4, the space $M_0^{!+}(p)$ has a canonical basis consisting of

$$f_{0,m} = \frac{1}{q^m} + O(q).$$

Therefore $f^\delta \left(\Delta_p^+\right)^{-k}$ can be expressed as

$$f^\delta(\Delta_p^+)^{-k} = \sum_{m=0}^{(p+1)k\delta/24} c(-m)f_{0,m}.$$

Furthermore since $f_{0,m}$ is a monic polynomial in j_p^+, we have

$$f^\delta(\Delta_p^+)^{-k} = \sum_{m=0}^{(p+1)k\delta/24} c(-m)f_{0,m} \in \mathcal{O}_K[j_p^+].$$

Since $\mathrm{ord}_l f^\delta > \frac{p+1}{24}\delta k$, we have $c(t) \equiv 0 \pmod{l}$ for $-\frac{p+1}{24}\delta k \leq t \leq 0$. That is, $f^\delta(\Delta_p^+)^{-k} \in l\mathcal{O}_K[j_p^+]$, and hence $f^\delta \in l \cdot \mathcal{O}_K[j_p^+](\Delta_p^+)^k$ which implies $\mathrm{ord}_l f^\delta = \infty$. Consequently we see that $\mathrm{ord}_l f = \infty$. □

It follows from Lemma 2.1 that $\dim S_2^-(p) = \frac{p-5}{12}$. Thus by definition of $\Delta_{p,2}^-$, $D\Delta_{p,2}^-$ has q-expansion of the form

$$Dq^{\frac{p-5}{12}} + O(q^{\frac{p+7}{12}}).$$

Let l be any prime such that $l \mid D$. Then we have

$$\mathrm{ord}_l(D\Delta^-_{p,2}) \geq \frac{p+7}{12} > \frac{p+1}{12}.$$

It follows from Lemma 3.1 that $\mathrm{ord}_l(D\Delta^-_{p,2}) = \infty$, that is $D\Delta^-_{p,2} \equiv 0 \pmod{l}$. Hence $\frac{D}{l}\Delta^-_{p,2}$ also has rational integer coefficients. Repeating this argument, we see that $\Delta^-_{p,2}$ has integral Fourier coefficients.

Example 3.1. Using the data for list of newforms [The13a], we can compute the Fourier expansion of Δ^-_{p,r_k} explicitly. For instance, when $p = 29$, the space $S^-_2(29)$ is two-dimensional. From [The13a], we find a Hecke eigenform f in $S_2(29)$ whose Fourier expansion of the form

$$f = q + \alpha q^2 - \alpha q^3 + (-2\alpha - 1)q^4 - q^5 + (2\alpha - 1)q^6 + \cdots$$

where α is a root of the polynomial $x^2 + 2x - 1$. Let $\alpha = -1 + \sqrt{2}$. Clearly its coefficient field is $\mathbb{Q}(\sqrt{2})$ which is a number field of degree 2 over $\mathbb{Q}$. That is, the space $S_2(29) = S^-_2(29) = S^{-,new}_2(29)$ is spanned by f and its conjugate f^σ. Letting $\alpha_1 = 1$, $\alpha_2 = \sqrt{2}$ form an integral basis for $\mathbb{Q}(\sqrt{2})$, we have the following linear equation given in a matrix form

$$\begin{pmatrix} f \\ f^\sigma \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where f_1, f_2 are q-series with integral Fourier coefficients of the form

$$\begin{aligned} f_1 &= q - q^2 + q^3 + q^4 - q^5 - 3q^6 + \cdots, \\ f_2 &= q^2 - q^3 - 2q^4 + 2q^6 + \cdots. \end{aligned}$$

Since $\det \begin{pmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{pmatrix} \neq 0$, we know that f_1, f_2 also span the space $S^-_2(29)$, and $\Delta^-_{29,2}$ is none other than f_2 by uniqueness of cusp form with a maximal order.

4. Duality

In this section we show that the basis elements $f^-_{k,n}$ have a generating function and as its application we obtain the beautiful duality of Fourier coefficients. Let $f^-_{k,m}$ be the weakly holomorphic modular form defined in Theorem 2.5, and let $F_k(z)$ be the function $f^-_{k,-m^-_k}(z)$, that is,

$$F^-_k(z) := f^-_{k,-m^-_k}(z) = (\Delta^+_p(z))^{l_k}\Delta^-_{p,r_k}(z) = q^{m^-_k} + \sum_{l=m^-_k+1}^{\infty} a_{F^-_k}(l)q^l.$$

We write

$$\frac{1}{F_k^-(z)} = \sum_{l=-m_k^-}^{\infty} a_{1/F_k^-}(l)q^l.$$

Lemma 4.1. *For each even integer k, we have that*

$$f_{k,n}^- = F_k^- \sum_{r+s=n} a_{1/F_k^-}(r) f_{0,s}^+.$$

Proof. We have

$$\begin{aligned}
&F_k^-(z) \sum_{r+s=n} a_{1/F_k^-}(r) f_{0,s}^+(z) \\
&= \left(\sum_{l=m_k^-}^{\infty} a_{F_k^-}(l)q^l\right) \Big(a_{1/F_k^-}(-m_k^-) f_{0,m_k^-+n}^+(z) + a_{1/F_k^-}(-m_k^-+1) f_{0,m_k^-+n-1}^+(z) + \cdots \\
&\quad + a_{1/F_k^-}(n-1) f_{0,1}^+(z) + a_{1/F_k^-}(n) f_{0,0}^+(z)\Big) \\
&= \sum_{l=m_k^-}^{2m_k^-+n} a_{F_k^-}(l) a_{1/F_k^-}(-m_k^-) q^{l-m_k^- -n} \\
&\quad + \sum_{l=m_k^-}^{2m_k^-+n-1} a_{F_k^-}(l) a_{1/F_k^-}(-m_k^-+1) q^{l-m_k^- -n+1} + \cdots \\
&\quad + \sum_{l=m_k^-}^{m_k^-+1} a_{F_k^-}(l) a_{1/F_k^-}(n-1) q^{l-1} + a_{F_k^-}(m_k^-) a_{1/F_k^-}(n) q^{m_k^-} + O(q^{m_k^-+1}) \\
&= \sum_{r=0}^{m_k^-+n} \sum_{s+t=r} a_{F_k^-}(m_k^-+s) a_{1/F_k^-}(-m_k^-+t) q^{-n+r} + O(q^{m_k^-+1}).
\end{aligned}$$

On the other hand, since $F_k^-(z)(1/F_k^-(z)) = 1$, we have

$$\sum_{s+t=r} a_{F_k^-}(m_k^-+s) a_{1/F_k^-}(-m_k^-+t) = 0,$$

for each positive r, and for $r=0$,

$$\sum_{s+t=0} a_{F_k^-}(m_k^-+s) a_{1/F_k^-}(-m_k^-+t) = a_{F_k^-}(m_k^-) a_{1/F_k^-}(-m_k^-) = 1.$$

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It means that

$$F_k^- \sum_{r+s=n} a_{1/F_k^-}(r) f_{0,s}^- = q^{-n} + O(q^{m_k^-+1}).$$

By the uniqueness of $f_{k,n}^-$, we obtain the assertion. □

Let $f_{k,m}^+$ be the unique weakly holomorphic modular form defined in Lemma 2.4. Then for $\tau \in \mathfrak{H}$, the function

$$\Psi_p(z,\tau) = 1 + \sum_{n=1}^{\infty} e_\tau f_{0.n}(\tau) q^n$$

is a meromorphic modular form of weight 2 for $\Gamma_0^+(p)$, where $1/e_\tau$ is the cardinality of $\Gamma_0^+(p)_\tau/\{\pm 1\}$. See [CK13, Theorem 3.1], and [Cho06, Theorem 3.2]. Then we have the following theorem:

Theorem 4.2.

$$\frac{f_{2,1}^+(z) F_k^-(\tau)}{(f_{0,1}^+(z) - f_{0,1}^+(\tau)) F_k^-(z)} = \sum_{n=m_k^-}^{\infty} f_{k,n}^-(\tau) q^n.$$

Proof. By definition of Ψ_p, we have

$$\begin{aligned}
\frac{F_k^-(\tau)}{F_k^-(z)} \Psi_p(z,\tau) &= \frac{F_k^-(\tau)}{F_k^-(z)} \left(1 - e_\tau + \sum_{n=0}^{\infty} e_\tau f_{0,n}(\tau) q^n\right) \\
&= \frac{F_k^-(\tau)}{F_k^-(z)}(1 - e_\tau) + F_k^-(\tau) \left(\sum_{r=-m_k^-}^{\infty} a_{1/F_k^-}(r) q^r\right) \left(\sum_{s=0}^{\infty} e_\tau f_{0,s}(\tau) q^s\right) \\
&= \frac{F_k^-(\tau)}{F_k^-(z)}(1 - e_\tau) + e_\tau \sum_{n=-m_k^-}^{\infty} \left(F_k^-(\tau) \sum_{r+s=n} a_{1/F_k^-}(r) f_{0,s}(\tau)\right) q^n.
\end{aligned}$$

And by Lemma 4.1, it is equal to

$$\frac{F_k^-(\tau)}{F_k^-(z)}(1 - e_\tau) + e_\tau \sum_{n=-m_k^-}^{\infty} f_{k,n}^- q^n.$$

On the other hand, using the fact [CK13, Theorem 3.2]

$$\Psi_p(z,\tau) = \frac{e_\tau f_{2,1}^+(z)}{f_{0,1}^+(z) - f_{0,1}^+(\tau)} - e_\tau + 1,$$

we have that

$$\frac{F_k^-(\tau)}{F_k^-(z)}\Psi_p(z,\tau) = \frac{F_k^-(\tau)}{F_k^-(z)}(1-e_\tau) + \frac{e_\tau f_{2,1}^+(z)F_k^-(\tau)}{(f_{0,1}^+(z)-f_{0,1}^+(\tau))F_k^-(z)}.$$

Now the assertion is immediate. □

Consider the function $F_{2-k}^-(z)$ which is equal to $(\Delta_p^+)^{l_{2-k}}\Delta_{p,r_{2-k}}^-$. Since we have $2-k = 2-\delta l_k - r_k = \delta(-l_k-1)+(\delta-r_k+2)$, we have $l_{2-k} = -l_k-1$ and $r_{2-k} = \delta - r_k + 2$. Moreover by uniqueness of cusp form with maximal order, we obtain that $\Delta_{p,r_k}^-\Delta_{p,r_{2-k}}^- = \Delta_{p,\delta+2}^+$. From these facts, we have

$$\begin{aligned} F_{2-k}^- &= (\Delta_p^+)^{l_{2-k}}\Delta_{p,r_{2-k}}^- \\ &= (\Delta_p^+)^{-l_k-1}\frac{\Delta_{p,\delta+2}^+}{\Delta_{p,r_k}^-} \\ &= \frac{\Delta_{p,\delta+2}^+}{(\Delta_p^+)^{l_k}\cdot\Delta_p^+\cdot\Delta_{p,r_k}^-} = \frac{1}{F_k^-}\cdot\frac{\Delta_{p,\delta+2}^+}{\Delta_p^+}. \end{aligned}$$

Noting that $f_{2,1}^+ = \Delta_{p,\delta+2}^+/\Delta_p^+$, we have the following relation

$$F_{2-k}^- = \frac{f_{2,1}^+}{F_k^-}.$$

From this relation, we get the following theorem:

Theorem 4.3. *For each even integer k, we have that*

$$\sum_{n=-m_k^-}^{\infty} f_{k,n}^-(\tau)q^n = \sum_{m=-m_{2-k}^-}^{\infty} -f_{2-k,m}^-(z)e^{2\pi i m\tau}.$$

Proof. As we already have shown,

$$F_{2-k}^- = \frac{f_{2,1}^+}{F_k^-}.$$

It follows from Theorem 4.2 that

$$\begin{aligned} \sum_{n=-m_k^-}^{\infty} f_{k,n}^-(\tau)q^n &= \frac{f_{2,1}^+(z)F_k^-(\tau)}{(f_{0,1}^-(z)-f_{0,1}^-(\tau))F_k^-(z)} = \frac{f_{2,1}^+(\tau)F_{2-k}^-(z)}{(f_{0,1}^-(z)-f_{0,1}^-(\tau))F_{2-k}^-(\tau)} \\ &= -\sum_{m=-m_{2-k}^-}^{\infty} f_{2-k,m}^-(z)e^{2\pi i m\tau}. \qquad \square \end{aligned}$$

Let $a_k^-(n,m)$ be the m-th coefficient of $f_{k,n}^-$, i.e.,

$$f_{k,n}^-(z) = q^{-n} + \sum_{m > m_k^-} a_k^-(n,m) q^m.$$

Then as a corollary the following duality of Fourier coefficients holds.

Corollary 4.4. *For any even integer k and any integers m, n the equality*

$$a_k^-(n,m) = -a_{2-k}^-(m,n)$$

holds for the Fourier coefficients of the weakly holomorphic modular forms $f_{k,n}^-$ and $f_{2-k,m}^-$.

Example 4.1. Recall that $m_{5,0}^- = \frac{5+1}{24} \cdot 4 \cdot (-1) + \dim S_4^-(5) = -1$ and $m_{5,2}^- = \frac{5+1}{24} \cdot 4 \cdot 0 + \dim S_2^-(5) = 0$. Hence for each integer $m \geq 1$, we get the form $(\Delta_5^+)^{-1} \Delta_{5,4}^-(j_5^+)^{m-1} = q^{-m} + \cdots \in M_0^{!-}(5)$, and for each $m \geq 0$, we get the form $\Delta_{5,2}^-(j_5^+)^m = q^{-m} + \cdots \in M_2^{!-}(5)$. Thus, as we have seen in Theorem 2.5, by taking a suitable linear combination of the forms $(\Delta_5^+)^{-1} \Delta_{5,4}^-(j_5^+)^{t-1}$ with $1 \leq t \leq m$, the canonical basis $f_{0,m}^- = q^{-m} + O(1)$ are constructed. Similarly by taking a suitable linear combination of the forms $\Delta_{5,2}^-(j_5^+)^t$ with $0 \leq t \leq m$, $f_{2,m}^- = q^{-m} + O(q)$ are constructed.

The first four basis elements for $M_0^{!-}(5)$ and the first five basis elements for $M_2^{!-}(5)$ are given below.

$$f_{0,1}^- = \frac{1}{q} - 6 - 116q - 740q^2 - 3405q^3 - 12244q^4 + \cdots,$$

$$f_{0,2}^- = \frac{1}{q^2} - 18 - 1480q - 24604q^2 - 227808q^3 - 1553740q^4 + \cdots,$$

$$f_{0,3}^- = \frac{1}{q^3} - 24 - 10215q - 341712q^2 - 5601356q^3 - 61459920q^4 + \cdots,$$

$$f_{0,4}^- = \frac{1}{q^4} - 42 - 48976q - 3107480q^2 - 81946560q^3 - 1345808364q^4 + \cdots,$$

$$f_{2,0}^- = 1 + 6q + 18q^2 + 24q^3 + 42q^4 + \cdots,$$

$$f_{2,1}^- = \frac{1}{q} + 116q + 1480q^2 + 10215q^3 + 48976q^4 + \cdots,$$

$$f_{2,2}^- = \frac{1}{q^2} + 740q + 24604q^2 + 341712q^3 + 3107480q^4 + \cdots,$$

$$f_{2,3}^- = \frac{1}{q^3} + 3405q + 227808q^2 + 5601356q^3 + 81946560q^4 + \cdots,$$

$$f_{2,4}^- = \frac{1}{q^4} + 12244q + 1553740q^2 + 61459920q^3 + 1345808364q^4 + \cdots.$$

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By comparing rows of coefficients in weight 0 to columns of coefficients in weight 2, the duality relation $a_0^-(n,m) = -a_2^-(m,n)$ is clear.

5. Divisibility properties

In this section we show that the basis elements for $M_k^!(p)$ have divisibility properties when $p \in \{2,3,5,7,11\}$. To emphasize the level of the space and to describe both the case of the spaces $M_k^{!+}(p)$ and the space $M_k^{!-}(p)$, we need to rearrange the notation. Note that the space $M_k^{!+}(p)$ and $M_k^{!-}(p)$ can be expressed using character. Let χ be a character on $\Gamma_0^+(p)$ satisfying

$$\chi\,|_{\Gamma_0(p)} \equiv 1, \quad \chi(W_p) = \varepsilon \in \{-1,1\}.$$

Then $M_k^!(\Gamma_0^+(p),\chi)$ stands for the space

$$M_k^!(\Gamma_0^+(p),\chi) = \begin{cases} M_k^{!+}(p) & \text{if } \varepsilon = 1, \\ M_k^{!-}(p) & \text{if } \varepsilon = -1. \end{cases}$$

Throughout this section, we denote by $f_{k,m}^{(p),+}$ a basis element of $M_k^{!+}(p)$ defined in Lemma 2.4, and denote by $f_{k,m}^{(p),-}$ a basis element of $M_k^{!-}(p)$. Let $a_k^{(p),+}(m,n)$ and $a_k^{(p),-}(m,n)$ be the n-th Fourier coefficient of $f_{k,m}^{(p),+}$ and $f_{k,m}^{(p),-}$ respectively, which means

$$f_{k,m}^{(p),+}(z) = q^{-m} + \sum_{n > m_{p,k}^+} a_k^{(p),+}(m,n) q^n,$$

$$f_{k,m}^{(p),-}(z) = q^{-m} + \sum_{n > m_{p,k}^-} a_k^{(p),-}(m,n) q^n.$$

Let us use the character notation to express the cases of $M_k^{!+}(p)$ and $M_k^{!-}(p)$ at once. In other words if $\varepsilon = 1$, $f_{k,m}^{(p),\varepsilon}$ is an element of the basis $f_{k,m}^{(p),+}$, otherwise $f_{k,m}^{(p),\varepsilon}$ stands for $f_{k,m}^{(p),-}$. Similarly $a_k^{(p),\varepsilon}(m,n)$ means the n-th coefficient of $f_{k,m}^{(p),\varepsilon}$, and $m_{p,k}^{\varepsilon}$ denotes the maximal vanishing order at ∞ for a nonzero $f \in M_k^{!,\varepsilon}(p)$. Here we note that the coefficients $a_k^{(p),\varepsilon}(m,n)$ are integral.

For $p \in \{2,3,5,7,11\}$, these basis elements have divisibility properties which bear a remarkable resemblance to the divisibility properties of $j(z)$ as follows.

Theorem 5.1. *Let $a_0^{(p),\varepsilon}(m,n)$ be the n-th coefficient of $f_{0,m}^{(p),\varepsilon}$ with $m = p^\alpha m'$, $n = p^\beta n'$, $(m',p) = (n',p) = 1$. Then for all nonnegative integers α and β with $\beta > \alpha$ we have that*

$$a_0^{(2),\varepsilon}(2^\alpha m', 2^\beta n') \equiv 0 \pmod{2^{3(\beta-\alpha)+8}},$$

$$a_0^{(3),\varepsilon}(3^\alpha m', 3^\beta n') \equiv 0 \pmod{3^{2(\beta-\alpha)+3}},$$

$$a_0^{(5),\varepsilon}(5^\alpha m', 5^\beta n') \equiv 0 \pmod{5^{(\beta-\alpha)+1}},$$
$$a_0^{(7),\varepsilon}(7^\alpha m', 7^\beta n') \equiv 0 \pmod{7^{(\beta-\alpha)}},$$

and for all positive integer β with $\alpha = 0$ or for all positive integers α, β with $\beta > 1$ we have that

$$a_0^{(11),\varepsilon}(11^\alpha m', 11^\beta n') \equiv 0 \pmod{11}.$$

Example 5.1. Let $p = 5$. Then the Hauptmodul $t = f_{0,1}^{(5),+}$ for $\Gamma_0^+(5)$ is given by

$$\left(\frac{\eta(z)}{\eta(5z)}\right)^6 + 6 + 5^3\left(\frac{\eta(5z)}{\eta(z)}\right)^6 = q^{-1} + \sum_{n>0} a_0^{(5),+}(1,n)q^n,$$

from which we find that

$$a_0^{(5),+}(1,5) = 39350 = 2 \cdot 5^2 \cdot 787,$$
$$a_0^{(5),+}(1,10) = 4298600 = 2^3 \cdot 5^2 \cdot 21493,$$
$$a_0^{(5),+}(1,15) = 172859325 = 3 \cdot 5^2 \cdot 2304791,$$
$$a_0^{(5),+}(1,20) = 4049168800 = 2^5 \cdot 5^2 \cdot 17 \cdot 173 \cdot 1721,$$
$$a_0^{(5),+}(1,25) = 66640520250 = 2 \cdot 3^2 \cdot 5^3 \cdot 29618009,$$

as desired from Theorem 5.1. Moreover, we observe that

$$f_{0,2}^{(5),+} = t^2 - 268,$$
$$f_{0,3}^{(5),+} = t^3 - 402t - 2280,$$
$$f_{0,4}^{(5),+} = t^4 - 536t^2 - 3040t + 22532,$$
$$f_{0,5}^{(5),+} = t^5 - 670t^3 - 3800t^2 + 73055t + 447920,$$

which enable us to compute

$$a_0^{(5),+}(5,25) = 1218832843304227769954718 50 = 2 \cdot 5^2 \cdot 719239 \cdot 3389229013733203483,$$

as expected from Theorem 5.1.

Remark 5.2.

(1) We emphasize that our result covers the case $p = 11$. As far as we know, the known literatures, for example, [AJ13,DJ10] do not cover the case $p = 11$, except for the Lehner's classical result.

(2) By the duality $a_0^{(p),\varepsilon}(n,m) = -a_2^{(p),\varepsilon}(m,n)$ (see [CK13] and Corollary 4.4), Theorem 5.1 also gives the corresponding results for $a_2^{(p),\varepsilon}(m,n)$.

From uniqueness of the form $f_{k,m}^{(p),\varepsilon}$, it is not difficult to check that

$$D(f_{0,m}^{(p),\varepsilon}) = -m f_{2,m}^{(p),\varepsilon},$$

which implies

$$n a_0^{(p),\varepsilon}(m,n) = -m a_2^{(p),\varepsilon}(m,n).$$

It follows from Remark 5.2(2) that

$$a_0^{(p),\varepsilon}(m,n) = -a_2^{(p),\varepsilon}(n,m) = \frac{m}{n} a_0^{(p),\varepsilon}(n,m). \tag{3}$$

Let $m = p^\alpha m'$, $n = p^\beta n'$ with $p \nmid m'$, $p \nmid n'$. Assume that $\alpha > \beta$. Then by (3), we find the relation

$$a_0^{(p),\varepsilon}(m,n) = p^{\alpha-\beta}\frac{m'}{n'} a_0^{(p),\varepsilon}(n,m).$$

Applying this relation to Theorem 5.1, we have

$$\begin{cases} 2^{3(\alpha-\beta)+8} \mid 2^{(\beta-\alpha)} \cdot \frac{n'}{m'} \cdot a_0^{(2),\varepsilon}(m,n) \\ 3^{2(\alpha-\beta)+3} \mid 3^{(\beta-\alpha)} \cdot \frac{n'}{m'} \cdot a_0^{(3),\varepsilon}(m,n) \\ 5^{(\alpha-\beta)+1} \mid 5^{(\beta-\alpha)} \cdot \frac{n'}{m'} \cdot a_0^{(5),\varepsilon}(m,n) \\ 7^{(\alpha-\beta)} \mid 7^{(\beta-\alpha)} \cdot \frac{n'}{m'} \cdot a_0^{(7),\varepsilon}(m,n) \\ 11 \mid 11^{(\beta-\alpha)} \cdot \frac{n'}{m'} \cdot a_0^{(11),\varepsilon}(m,n). \end{cases}$$

In addition, it is clear that

$$1 \mid 13^{\beta-\alpha} \cdot \frac{n'}{m'} \cdot a_0^{(13),\varepsilon}(m,n).$$

Now we obtain the following theorem:

Theorem 5.3. *Let $a_0^{(p),\varepsilon}$ be the n-th Fourier coefficient of $f_{0,m}^{(p),\varepsilon}$ with $m = p^\alpha m'$, $n = p^\beta n'$, $(m',p) = (n',p) = 1$, Then for any $\alpha > \beta$, we have*

$$a_0^{(2),\varepsilon}(2^\alpha m', 2^\beta n') \equiv 0 \pmod{2^{4(\alpha-\beta)+8}},$$

$$a_0^{(3),\varepsilon}(3^\alpha m', 3^\beta n') \equiv 0 \pmod{3^{3(\alpha-\beta)+3}},$$

$$a_0^{(5),\varepsilon}(5^\alpha m', 5^\beta n') \equiv 0 \pmod{5^{2(\alpha-\beta)+1}},$$

$$a_0^{(7),\varepsilon}(7^\alpha m', 7^\beta n') \equiv 0 \pmod{7^{2(\alpha-\beta)}},$$

$$a_0^{(11),\varepsilon}(11^\alpha m', 11^\beta n') \equiv 0 \pmod{11^{(\alpha-\beta)+1}},$$
$$a_0^{(13),\varepsilon}(13^\alpha m', 13^\beta n') \equiv 0 \pmod{13^{(\alpha-\beta)}}.$$

Remark 5.4. For $p \in \{2,3,5,7,13\}$, recall that $m_{p,0}^+ = 0$ and $m_{p,0}^- = -1$. Hence for any $m \geq 0$, there is a unique form $f_{0,m}^{(p),+} = \frac{1}{q^m} + O(q) \in M_0^{!+}(p)$, and for any $m \geq 1$, there exists a unique form $f_{0,m}^{(p),-} = q^{-m} + O(1) \in M_0^{!-}(p)$. Note that $m = 0$ implies that $f_{0,m}^{(p),\sharp} = f_{0,m}^{(p),+} = 1$. For $m \geq 1$, observing that

$$\left.\frac{f_{0,m}^{(p),+} + f_{0,m}^{(p),-}}{2}\right| W_p = \frac{f_{0,m}^{(p),+} - f_{0,m}^{(p),-}}{2} = O(1),$$

and

$$\frac{f_{0,m}^{(p),+} + f_{0,m}^{(p),-}}{2} = q^{-m} + O(1),$$

we have

$$f_{0,m}^{(p),\sharp} = q^{-m} + O(q) = \frac{1}{2}(f_{0,m}^{(p),+} + f_{0,m}^{(p),-} - a_0^{(p),-}(m,0))$$

from uniqueness of $f_{0,m}^{(p),\sharp}$. Therefore our results for $M_0^{!+}(p)$ and $M_0^{!-}(p)$ can be applied to prove the results of Andersen, Jenkins and Thornton [AJ13,JT15] for $p \in \{3,5,7,13\}$.

Now all that remains is to prove Theorem 5.1. To prove it, we follow the main idea in [Cho12] that combines the idea of Doud and Jenkins [DJ10] with that of Lehner [Leh43, Leh49a,Leh49b]. To get a relation among the Fourier coefficients of weakly holomorphic modular forms which plays a crucial role in finding p-divisible properties of Fourier coefficients Doud and Jenkins [DJ10, Corollary 3.2] used Hecke operators T_p. In this paper we find an analogy (see Lemma 5.6) of [DJ10, Corollary 3.2] by making use of U_p-operator instead of T_p and the fact that $f(z) + f(-1/(pz))$ is a weakly holomorphic modular function for $\Gamma_0^+(p)$ if f is a weakly holomorphic modular function for $\Gamma_0(p)$.

The concluding remarks of Lehner's last paper [Leh49b] say that the coefficients of certain level p modular functions having a pole of order less than p at the cusp ∞ have the same p-divisible properties as the coefficients $c(n)$ of $j(z)$ (for a precise statement, see [AJ13, Theorem 1]). A necessary condition in the statement of Lehner's theorem is that the order of the pole at the cusp ∞ must be less than p. In this paper by using Lemma 5.6 we remove this restriction on the order of the pole to show that all functions $f_{0,m}^{(p),\varepsilon}$ in our basis have p-divisible properties as stated in Theorem 5.1.

For $f \in M_0^{!,\varepsilon}(p)$, we introduce the linear operator

$$U_p f(z) = \frac{1}{p}\sum_{\lambda=0}^{p-1} f\left(\frac{z+\lambda}{p}\right).$$

It is well known from [Apo90, Theorem 4.5] and [Ono04, Proposition 2.22] that U_pf is a weakly holomorphic modular function for $\Gamma_0(p)$ and if $f(z) = \sum_{n\geq s} a_nq^n$, then

$$f_p := U_pf = \sum_{n\geq s/p} a_{pn}q^n.$$

For each positive integer a we denote $U_p(U_p^a f)$ by $U_p^{a+1}f$, where $U_p^1 f = U_pf$.

Lemma 5.5. *Let f be a weakly holomorphic modular function in the space $M_0^{!,\varepsilon}(p)$. Then*

$$pf_p(-1/(pz)) = -f(z) + pf_p(pz) + \varepsilon f(pz).$$

Further, $pf_p(-1/(pz))$ is a weakly holomorphic modular function for $\Gamma_0(p)$.

Proof. The proof of this lemma is in fact identical to the proof of [DJ10, Lemma 4.1] and [Apo90, Theorem 4.6]. However, here we will prove again considering the change due to the difference of an eigenvalue for Fricke involution W_p of f.

By the definition of f_p, it is easily seen that

$$pf_p(-1/z) = f\left(\frac{-1}{pz}\right) + \sum_{\lambda=1}^{p-1} f\left(\frac{\lambda z-1}{pz}\right). \tag{4}$$

Since $f \in M_0^{!,\varepsilon}(p)$, the transformation law $f(-1/pz) = \varepsilon f(z)$ holds. Hence the right hand side of equation (4) is equal to

$$\varepsilon f(z) + \sum_{\lambda=1}^{p-1} f\left(\left(\begin{smallmatrix} \lambda & -1 \\ p & 0 \end{smallmatrix}\right) z\right).$$

For an integer λ with $1 \leq \lambda \leq p-1$, let λ' be the unique integer with $-(p-1) \leq \lambda' \leq -1$ such that $\lambda\lambda' \equiv 1 \pmod p$, and let $b_\lambda = (\lambda\lambda' - 1)/p$. Then we have

$$\begin{pmatrix} \lambda & -1 \\ p & 0 \end{pmatrix} = \begin{pmatrix} \lambda & b_\lambda \\ p & \lambda' \end{pmatrix} \begin{pmatrix} 1 & -\lambda' \\ 0 & p \end{pmatrix},$$

and hence the right hand side of equation (4) can be written as

$$\sum_{\lambda=1}^{p-1} f\left(\left(\begin{smallmatrix} \lambda & b_\lambda \\ p & \lambda' \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\lambda' \\ 0 & p \end{smallmatrix}\right) z\right) + \varepsilon f(z).$$

Noticing that $\left(\begin{smallmatrix} \lambda & b_\lambda \\ p & \lambda' \end{smallmatrix}\right) \in \Gamma_0(p)$, we get

$$f\left(\left(\begin{smallmatrix} \lambda & b_\lambda \\ p & \lambda' \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\lambda' \\ 0 & p \end{smallmatrix}\right) z\right) = f\left(\frac{z-\lambda'}{p}\right).$$

Therefore we obtain

$$\begin{aligned} pf_p(-1/z) &= \sum_{\lambda=1}^{p-1} f\left(\frac{z-\lambda'}{p}\right) + \varepsilon f(z) = \sum_{\lambda=1}^{p-1} f\left(\frac{z+\lambda}{p}\right) + \varepsilon f(z) \\ &= -f(z/p) + \sum_{\lambda=0}^{p-1} f\left(\frac{z+\lambda}{p}\right) + \varepsilon f(z) \\ &= -f(z/p) + pf_p(z) + \varepsilon f(z). \end{aligned}$$

By replacing z by pz, we easily get the assertion. □

Lemma 5.6. *Let m' and n' be any positive integers with $(m',p) = (n',p) = 1$. Then we have that*

(1) $a_0^{(p),\varepsilon}(pm', p^\beta n') = a_0^{(p),\varepsilon}(m', p^{\beta-1}n') + \varepsilon p a_0^{(p),\varepsilon}(m', p^\beta n') + p a_0^{(p),\varepsilon}(m', p^{\beta+1}n')$ *for all positive integer β.*
(2) $a_0^{(p),\varepsilon}(p^{\alpha+1}m', p^\beta n') = a_0^{(p),\varepsilon}(p^\alpha m', p^{\beta-1}n') + p a_0^{(p),\varepsilon}(p^\alpha m', p^{\beta+1}n') - p a_0^{(p),\varepsilon}(p^{\alpha-1}m', p^\beta n')$ *for all positive integers α, β.*

Proof. Let $f(z)$ be a basis element $f_{0,m}^{(p),\varepsilon}$ of $M_0^{!,\varepsilon}(p)$, and let $f_p := f \mid U_p$. Then we know $pf_p(z)$ is a weakly holomorphic modular function for $\Gamma_0(p)$ and hence $pf_p(-1/(pz)) + \varepsilon pf_p(z)$ is a weakly holomorphic modular function in $M_k^{!,\varepsilon}(p)$. Noticing that a weakly holomorphic modular function $pf_p(-1/(pz)) + \varepsilon pf_p(z) = -f(z) + pf_p(pz) + \varepsilon f(pz) + \varepsilon pf_p(z)$ has a Fourier expansion of the form

$$\begin{aligned} &-f(z) + pf_p(pz) + \varepsilon f(pz) + \varepsilon pf_p(z) \\ &= \begin{cases} \varepsilon q^{-pm} - q^{-m} + O(q^{m_{p,0}^{\varepsilon}+1}), & \text{if } p \nmid m, \\ \varepsilon q^{-pm} + (p-1)q^{-m} + \varepsilon p q^{-m/p} + O(q^{m_{p,0}^{\varepsilon}+1}), & \text{if } p \mid m, \end{cases} \end{aligned}$$

we have that

$$\begin{aligned} &-f_{0,m}^{(p),\varepsilon} + pf_p(pz) + \varepsilon f_{0,m}^{(p),\varepsilon}(pz) + \varepsilon pf_p(z) \\ &= \begin{cases} -f_{0,m}^{(p),\varepsilon} + \varepsilon f_{0,pm}^{(p),\varepsilon}, & \text{if } p \nmid m, \\ \varepsilon p f_{0,m/p}^{(p),\varepsilon} + (p-1) f_{0,m}^{(p),\varepsilon} + \varepsilon f_{0,pm}^{(p),\varepsilon}, & \text{if } p \mid m. \end{cases} \end{aligned} \tag{5}$$

We now obtain the assertion by comparing the Fourier coefficients of weakly holomorphic modular functions in both sides of (5). □

Note that for each $p \in \{2, 3, 5, 7, 13\}$, the genus of $\Gamma_0(p)$ is zero. Hence we may take a univalent function $\Phi(z)$ [Leh49a,Leh49b] as follows:

$$\Phi(z) = \Phi_p(z) = (\frac{\eta(pz)}{\eta(z)})^r = q + \cdots ,$$

with

$$r(p-1) = 24.$$

Let $j_p(z) = 1/\Phi_p(z)$. We then have that j_p is holomorphic on the upper half plane $\mathfrak{H}$, has a simple pole at the cusp ∞ and

$$j_p(-1/(pz)) = p^{r/2}\Phi_p(z). \tag{6}$$

For (6), see [Leh43, (8.83)]. In fact, by using the transformation law of η we can show (6). We know from the definitions that j_p and Φ have integral Fourier coefficients.

In what follows, for each positive integer m with $m = p^\alpha m'$ and $(m', p) = 1$, we write

$$f(z) = f_{0,m}^{(p),\varepsilon}(z) = \frac{1}{q^m} + O(q^{m^{\varepsilon}_{p,0}+1}).$$

If $\alpha = 0$, that is, $m = m'$, then f_p is holomorphic on $\mathfrak{H}$ and at the cusp ∞. Moreover it follows from Lemma 5.5 that

$$pf_p(-1/(pz)) = -f(z) + pf_p(pz) + \varepsilon f(pz)$$

is a weakly holomorphic modular function for $\Gamma_0(p)$, which is holomorphic at the cusp 0 and meromorphic at the cusp ∞ and has integral Fourier coefficients in the q-expansion at ∞. Hence for each $p \in \{2, 3, 5, 7, 13\}$, we have

$$pf_p(-1/(pz)) = \sum_{t\geq 0} A_{t,p}j_p(z)^t$$

for some integers $A_{t,p}$. Under the same notation as above, replacing z by $-1/(pz)$, we have the following theorem.

Theorem 5.7. *Assume that* $\alpha = 0$. *Then for each* $p \in \{2, 3, 5, 7, 13\}$, *we obtain*

$$f_p(z) = D_{0,p} + \sum_{t\geq 1} D_{t,p}p^{rt/2-1}\Phi(z)^t$$

for some integers $D_{t,p}$.

Following a main idea in [Cho12] we now prove Theorem 5.1. We use similar notations to [Cho12]. We will use induction on α. Assume that $\alpha = 0$. We can rewrite f_p in Theorem 5.7 as

$$f_p = \begin{cases} B_0 + 2^{11} \sum_{t\geq 1} B_t 2^{8(t-1)} \Phi^t = B_0 + 2^{11} R, & \text{if } p = 2, \\ C_0 + 3^5 \sum_{t\geq 1} C_t 3^{4(t-1)} \Phi^t = C_0 + 3^5 T, & \text{if } p = 3, \\ D_0 + \sum_{t\geq 1} D_t 5^{3t-1} \Phi^t = D_0 + 5^2 Q_5, & \text{if } p = 5, \\ E_0 + \sum_{t\geq 1} E_t 7^{2t-1} \Phi^t = E_0 + Q_7, & \text{if } p = 7, \end{cases} \tag{7}$$

for some integers B_t, C_t, D_t, E_t. Here R is a polynomial of the form $R = \sum_{t\geq 1} b_t 2^{8(t-1)} \Phi^t$, T is a polynomial of the form $T = \sum_{t\geq 1} c_t 3^{4(t-1)} \Phi^t$, Q_5 is a polynomial of the form $Q_5 = d_1 \Phi + \sum_{t\geq 2} d_t 5^t \Phi^t$, and Q_7 is a polynomial of the form $Q_7 = e_1 \Phi + \sum_{t\geq 2} e_t 7^t \Phi^t$ for some integers b_t, c_t, d_t, e_t. Also R, T, Q_5 and Q_7 will denote polynomials of these types, not necessarily the same one at each appearance.

Proposition 5.8. *For each positive integer h, we have that*

$$\begin{aligned} &2^{8(h-1)} U_2 \Phi^h = 2^3 R, \\ &3^{4(h-1)} U_3 \Phi^h = 3^2 T, \\ &U_5 \Phi = 5 Q_5, \quad 5^{h+1} U_5 \Phi^{h+1} = 5 Q_5, \\ &U_7 \Phi = 7 Q_7, \quad 7^{h+1} U_7 \Phi^{h+1} = 7 Q_7. \end{aligned}$$

Proof. See [Leh49b, (3.4), (3.24)] and [Leh49a, (5.13), (5.14), Section 6]. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. For each positive integer β, applying the operator U_p^β to both sides in (7) we obtain from Proposition 5.8 that

$$U_p^\beta f = \begin{cases} B_0' + 2^{11} 2^{3(\beta-1)} R \equiv B_0' \pmod{2^{3\beta+8}}, & \text{if } p = 2, \\ C_0' + 3^{2\beta+3} T \equiv C_0' \pmod{3^{2\beta+3}}, & \text{if } p = 3, \\ D_0' + 5^{\beta+1} Q \equiv D_0' \pmod{5^{\beta+1}}, & \text{if } p = 5, \\ E_0' + 7^\beta Q \equiv E_0' \pmod{7^\beta}, & \text{if } p = 7. \end{cases} \tag{8}$$

Proposition 5.8 gives

$$a_0^{(2),\varepsilon}(m', 2^\beta n') \equiv 0 \pmod{2^{3\beta+8}}, \tag{9}$$

$$a_0^{(3),\varepsilon}(m', 3^\beta n') \equiv 0 \pmod{3^{2\beta+3}}, \tag{10}$$

$$a_0^{(5),\varepsilon}(m', 5^\beta n') \equiv 0 \pmod{5^{\beta+1}}, \tag{11}$$

$$a_0^{(7),\varepsilon}(m', 7^\beta n') \equiv 0 \pmod{7^\beta}. \tag{12}$$

Thus the assertion holds for all $\beta > 0$ when $\alpha = 0$. Now consider $p = 2$. We then obtain from (9) and Lemma 5.6(1) that

$$a_0^{(2),\varepsilon}(2m', 2^\beta n') \equiv 0 \pmod{2^{3(\beta-1)+8}}$$

for all $\beta > 1$. Thus the assertion holds when $\alpha = 1$. Let α be some positive integer and assume that $a_0^{(2),\varepsilon}(2^i m', 2^\beta n') \equiv 0 \pmod{2^{3(\beta-i)+8}}$ for all positive integer i with $0 < i \leq \alpha$ and for each $\beta > i$. Then Lemma 5.6(2) implies that the assertion holds when $m = 2^{\alpha+1}m'$ and for each positive integer β with $\beta > \alpha + 1$. Consequently by induction we obtain the assertion when $p = 2$. By the same argument as the case of $p = 2$ we obtain the assertion for other primes $p = 3, 5, 7$.

In the case $p = 11$, we notice that the genus of $\Gamma_0(11)$ is not zero, so we need a new approach. In fact, by adopting an argument similar to [Cho12] we can obtain the assertion. For the convenience of readers we provide a proof. Following the notation in [Leh43] we have modular functions for $\Gamma_0(11)$ which are holomorphic on $\mathfrak{H}$ and have integral Fourier coefficients [Leh43, (4.51), (6.44), (6.46) and Lemma 3] as follows:

$$A(z) = A(\frac{-1}{11z}) = \frac{1}{q} + 6 + 17q + 46q^2 + \cdots ,$$
$$C(z) = q + 5q^2 + \cdots ,$$
$$11^2 C(\frac{-1}{11z}) = \frac{1}{q^2} + \frac{2}{q} + \cdots .$$

Letting

$$\alpha(z) = 11^2 C(\frac{-1}{11z}) = \frac{1}{q^2} + \cdots ,$$
$$\beta(z) = 11^2 C(\frac{-1}{11z}) A(z) = \frac{1}{q^3} + \cdots ,$$

we obtain

$$11 f_{11}(\frac{-1}{11z}) = \sum_{a\geq 0, b\geq 0} D_{ab}\alpha(z)^a \beta(z)^b$$

for some integers D_{ab} because the genus of $\Gamma_0(11)$ is not zero. Now replacing z by $-1/11z$ we obtain that

$$11 f_{11}(z) = \sum_{a\geq 0, b\geq 0} D_{ab}\alpha(\frac{-1}{11z})^a \beta(\frac{-1}{11z})^b = \sum_{a\geq 0, b\geq 0} D_{ab} 11^{2(a+b)} C(z)^{a+b} A(z)^b,$$

which implies that $f_{11}(z) \equiv A_0 \pmod{11}$ for some integer A_0 and hence $a_0^{(11)}(m', 11^\beta n) \equiv 0 \pmod{11}$ for all positive integers β. Thus Lemma 5.6 implies the assertion when $p = 11$. $\square$

6. Square-free level cases

Until now we only considered weakly holomorphic modular forms of prime levels. As another extension of [CK13], we will consider square-free level cases. We recall that

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$\Gamma_0^*(N)$ is the group generated by $\Gamma_0(N)$ and all Atkin–Lehner involutions of N. Note that if N is prime, then $\Gamma_0^*(N)$ is the same as $\Gamma_0^+(N)$ which we have discussed so far. Throughout this section, we assume that N is a square-free composite integer for which the genus of $\Gamma_0^*(N)$ is zero, that is, N belongs to the set

$$\mathfrak{S}_0 = \{6, 10, 14, 15, 21, 22, 26, 30, 33, 34, 35, 38, 39, 42, 46, 51, 55, 62, 66, 69, 70, 78, 87, 94, 95, 105, 110, 119\}.$$

Let $M_k^*(N)$ (resp. $S_k^*(N)$) be the space of holomorphic modular forms (resp. cusp forms) of weight k for $\Gamma_0^*(N)$, and let $M_k^{!*}(N)$ be the space of weakly holomorphic modular forms for $\Gamma_0^*(N)$. In this section we will generalize the results of [CK13] to the space $M_k^{!*}(N)$ of weakly holomorphic modular forms in the cases of square-free levels. For a square-free integer N, it is well known from [JST16,JST17] that $\Gamma_0^*(N)$ has only one inequivalent cusp, and hence we can generalize the results of [CK13] without difficulty.

Let $k > 2$ be an even integer. Then $\dim S_k^*(N)$ is finite, and it follows from [Miy06, Theorem 2.5.2] that

$$\dim S_k^*(N) = \nu_2 \left\lfloor \frac{k}{4} \right\rfloor + \nu_3 \left\lfloor \frac{k}{3} \right\rfloor + \nu_4 \left\lfloor \frac{3k}{8} \right\rfloor + \nu_6 \left\lfloor \frac{5k}{12} \right\rfloor - \frac{k}{2},$$

where ν_i denotes the number of inequivalent elliptic points of order i of $\Gamma_0^*(N)$. Using [CL04, Table 4], one can compute $\dim S_k^*(N)$ for each $N \in \mathfrak{S}_0$.

Remark 6.1. By finite-dimensionality and existence of the Hauptmodul j_N^* of $\Gamma_0^*(N)$, one can show that the space $S_k^*(N)$ also has a Miller basis by adopting the same arguments as in Lemma 2.2. Furthermore, for $d = \dim S_k^*(N) \geq 1$, there exists a unique cusp form $\Delta_{N,k}^*$ with q-expansion of the form

$$\Delta_{N,k}^*(z) = q^d + O(q^{d+1}),$$

and for $d = 0$, we define $\Delta_{N,k}^*(z) = E_{N,k}^* := \frac{1}{\sigma_{k/2}(N)} \sum_{d|N} d^{k/2} E_k(dz)$ where we set $E_{N,0}^* = 1$.

Next step to find the canonical basis of $M_k^{!*}(N)$ is, as in prime level cases, to define δ_N and Δ_N^* for each N. In fact, it was done by [JST16,JST17].

Lemma 6.2. *(See [JST16, Theorem 16] and [JST17, Proposition 4 and Corollary 5].) Let N be a square-free integer with r distinct prime factors.*

(1) *Put*

$$\delta = \delta_N = lcm\left(4, 2^{r-1} \frac{24}{\gcd(24, \sigma(N))}\right),$$

where $\sigma(N)$ is a divisor sum. Then δ is the smallest weight k such that there exists a cusp form $f \in S_k^*(N)$ vanishing only at the cusps.

(2) *There exists a unique normalized cusp form $\Delta_N^* \in S_\delta^*(N)$ such that $ord_\infty \Delta_N^* = \frac{\sigma(N)}{24\cdot 2^{r-1}}k$. More explicitly,*

$$\Delta_N^*(z) = \left(\prod_{d|N} \eta(dz)\right)^{\ell_N}$$

where $\ell_N = 2^{1-r}\delta_N$.

Theorem 6.3. *Let $k \in 2\mathbb{Z}$ and δ be the integer given in Lemma 6.2. We have unique l_k and r_k such that*

$$k = \delta l_k + r_k \quad \textit{where } r_k = \begin{cases} \delta + 2 & \textit{if } k \equiv 2 \pmod{\delta}, \\ k - \left\lfloor \frac{k}{\delta} \right\rfloor \delta & \textit{otherwise.} \end{cases}$$

(1) *For $f \in M_k^{!*}(N)$,*

$$ord_\infty f \le \frac{\sigma(N)}{24\cdot 2^{r-1}}\delta l_k + \dim S_{r_k}^*(N).$$

(2) *We put $m_{N,k}^* = \frac{\sigma(N)}{24\cdot 2^{r-1}}\delta l_k + \dim S_{r_k}^*(N)$. For each $m \in \mathbb{Z}$, such that $-m \le m_{N,k}^*$, there exists a unique weakly holomorphic modular form $f_{k,m}^* \in M_k^{!*}(N)$ with*

$$f_{k,m}^* = q^{-m} + O(q^{m_{N,k}^*+1}).$$

Proof. Same as the proof of Theorem 2.5.

Note that for each integer $m \ge -m_{N,k}^*$, the canonical basis $f_{k,m}^*$ for $M_k^{!*}(N)$ is given by

$$f_{k,m}^* = (\Delta_N^*)^{l_k}\Delta_{N,r_k}^* F_{k,m+m_{N,k}^*}(j_N^*),$$

where $F_{k,D}(x)$ is a monic polynomial in x of degree D and j_N^* is the Hauptmodul for $\Gamma_0^*(N)$. Since Δ_N^* is an eta product, Δ_N^* has integer Fourier coefficients. Integrality of the coefficients in the q-expansion of j_N^* is proved in [JST16, Section 3]. Thus for integrality of the coefficients of the canonical basis we have only to consider Δ_{N,r_k}^*, which is the unique cusp form in $S_{r_k}^*(N)$ whose vanishing order at infinity is the same as the dimension of the space $S_{r_k}^*(N)$. In Table 2 we list the dimensions of $S_{r_k}^*(N)$ for each case.

Remark 6.4. From the Table 2, we observe that $\dim S_{r_k}^*(N) + \dim S_{\delta+2-r_k}^*(N) = \dim S_{\delta+2}^*(N)$, and $\dim S_\delta^*(N) = \dim S_{\delta+2}^*(N) + 1$. In other words, the relation

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Table 2
Dimensions of $S^*_{r_k}(N)$.

N	δ	$\dim S^*_{r_k}(N)$					
		$r_k = 4$	$r_k = 6$	$r_k = 8$	$r_k = 10$	$r_k = 12$	$r_k = 14$
6	4	1	0				
10	8	1	1	3	2		
14	4	2	1				
15	4	2	1				
21	12	2	2	5	5	8	7
22	4	3	2				
26	8	3	3	7	6		
30	4	3	2				
33	4	4	3				
34	8	4	4	9	8		
35	4	4	3				
38	4	5	4				
39	12	4	4	9	9	14	13
42	4	4	3				
46	4	6	5				
51	4	6	5				
55	4	6	5				
62	4	8	7				
66	4	6	5				
69	4	8	7				
70	4	6	5				
78	4	7	6				
87	4	10	9				
94	4	12	11				
95	4	10	9				
105	4	8	7				
110	4	9	8				
119	4	12	11				

$$\dim S^*_{r_k}(N) + \dim S^*_{\delta+2-r_k}(N) = \frac{\sigma(N)\ell_N}{24} - 1$$

holds for every $N \in \mathfrak{S}_0$, having similarities with [CK13, Lemma 3.7].

Corollary 6.5. *The duality relation*

$$a^*_k(n,m) = -a^*_{2-k}(m,n)$$

also holds for the weakly holomorphic modular forms $f^*_{k,n} = q^{-n} + \sum a^*_k(n,m)q^m$ and $f^*_{2-k,m} = q^{-m} + \sum a^*_{2-k}(m,n)q^n$.

Proof. The assertion immediately follows by combining the arguments in [CK13, Remark 3.8] with Remark 6.4. □

It follows from Remark 6.4 that $\Delta^*_{N,\delta} = \Delta^*_N$, and $\Delta^*_{N,\delta+2} = \Delta^*_N \times (-D(j^*_N))$. Accordingly, when $\delta = 4$, we can construct Δ^*_{N,r_k} explicitly for every pair (N, r_k). The remaining cases of N are $N \in \{10, 21, 26, 34, 39\}$. Before we look at each case, recall that $W_eW_f \equiv W_fW_e \equiv W_g \pmod{\Gamma_0(N)}$ where $g = ef/\gcd(e,f)^2$.

Case (1) $N = 10$**:** Since we have $\Delta^*_{10,4} \times \Delta^*_{10,6} = \Delta^*_{10,10}$, we have only to consider $\Delta^*_{10,4}$. First we consider the function $f = \Delta^+_5 + \Delta^+_5 \mid W_2$. Then it is not difficult to check that f is a modular form of weight 4 for $\Gamma_0(10)$ and invariant under all Atkin–Lehner involutions of 10. Further, since $\Delta_5^+(z) \mid W_2 = 4 \cdot \Delta_5^+(2z)$, we get the q-expansion of f as follows:

$$\begin{aligned}\Delta_5^+ + \Delta_5^+ \mid W_2 &= q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + \cdots + 4(q^2 - 4q^4 + 2q^6 + \cdots) \\ &= q + 2q^3 - 8q^4 - 5q^5 + \cdots .\end{aligned}$$

Consequently, $f = \Delta_5^+ + \Delta_5^+ \mid W_2$ is the unique cusp form $\Delta_{10,4}^*$, and has rational integer Fourier coefficients.

Case (2) $N = 21$**:** In this case, we need to find $\Delta_{21,4}^*$ and $\Delta_{21,6}^*$. Using [The13b], we can find a newform f_1 of weight 4 on $\Gamma_0(21)$ with integral Fourier coefficients, which is invariant under all Atkin–Lehner involutions of 21. On the other hand, we have another holomorphic cusp form $f_2 = \Delta_{7,4}^+ + \Delta_{7,4}^+ \mid W_3$, and f_2 is also invariant under all Atkin–Lehner involutions of 21. In addition, f_1 and f_2 have q-expansions of the form

$$\begin{aligned}f_1 &= q + 4q^2 - 3q^3 + 8q^4 - 4q^5 - 12q^6 - 7q^7 + 9q^9 + \cdots , \\ f_2 &= q - q^2 + 7q^3 - 7q^4 + 16q^5 - 7q^6 - 7q^7 + 15q^8 - 41q^9 + \cdots .\end{aligned}$$

Then we have

$$f_1 - f_2 = 5q^2 - 10q^3 + 15q^4 - 20q^5 - 5q^6 - 15q^8 + 50q^9 + \cdots \equiv 0 \pmod 5,$$

using classical Sturm bound for modular forms in $M_4(21)$. Therefore, $\Delta_{21,4}^* = (f_1 - f_2)/5$.

We can also obtain $\Delta_{21,6}^*$ with similar arguments. From [The13f], the function $g_1 \in \mathbb{Z}[[q]]$ with q-expansion

$$g_1 = q + q^2 - 9q^3 - 31q^4 - 34q^5 - 9q^6 - 49q^7 - 63q^8 + 81q^9 + \cdots$$

is a newform of weight 6 on $\Gamma_0(21)$, and its eigenvalues of all Atkin–Lehner involutions are equal to 1. Thus g_1 lies in the space $S_6^*(21)$. There is another holomorphic cusp form in $S_6^*(21)$. Consider the holomorphic cusp form g_2 in $S_6(21)$ given by

$$g_2 = \Delta_{7,6}^+ + \Delta_{7,6}^+ \mid W_3 = \Delta_{7,6}^+ + 27\Delta_{7,6}(3z).$$

Then g_2 is also invariant under all Atkin–Lehner involutions of 21, that is, $g_2 \in S_6^*(21)$. The Fourier expansion of g_2 has the form

$$g_2 = q - 10q^2 + 13q^3 + 68q^4 - 56q^5 - 130q^6 - 49q^7 - 360q^8 - 425q^9 + \cdots .$$

Then we have

$$g_1 - g_2 = 11q^2 - 22q^3 - 99q^4 + 22q^5 + 121q^6 + 297q^8 + 506q^9 + \cdots \equiv 0 \pmod{11}$$

by classical Sturm bound for modular forms in $M_6(21)$. Therefore we have $\Delta^*_{21,6} = (g_1 - g_2)/11$.

Case (3) $N = 26$**:** For this case, it suffices to find $\Delta^*_{26,4}$. Let f_1 be the newform from [The13c], and let f_2, f_3 be defined by

$$\begin{aligned} f_2 &= \Delta^+_{13,4} + \Delta^+_{13,4} \mid W_2, \\ f_3 &= (\Delta^+_{13,4} \cdot j^+_{13}) + (\Delta^+_{13,4} \cdot j^+_{13}) \mid W_2. \end{aligned}$$

Then it is not difficult to check that f_1, f_2, and f_3 are holomorphic cusp forms in $M^*_4(26)$ with integral Fourier coefficients. Moreover the q-expansions of these functions are given by

$$\begin{aligned} f_1 &= q - 2q^2 + 3q^3 + 4q^4 + 11q^5 - 6q^6 + 19q^7 - 8q^8 - 18q^9 + \cdots, \\ f_2 &= q^2 - 3q^3 + 5q^4 + q^5 - 11q^6 + 11q^7 - 7q^8 - 15q^9 + \cdots, \\ f_3 &= q + q^2 + 13q^3 - 19q^4 - 5q^5 + 37q^6 - 43q^7 + 9q^8 + 70q^9 + \cdots. \end{aligned}$$

Since $\dim S^*_4(26) = 3$, the space $S^*_4(26)$ is spanned by f_1, f_2, and f_3. Hence $\Delta^*_{26,4} = (-f_1 - 3f_2 + f_3)/19$, and it has integral Fourier coefficients.

Case (4) $N = 34$**:** In this case, we only need to construct $\Delta^*_{34,4}$. Recall that the vanishing order of Δ^+_{17} at ∞ is 3. Thus we have three different holomorphic cusp forms on $\Gamma^*_0(34)$ as follows:

$$\begin{aligned} f_1 &= \Delta^+_{17} + \Delta^+_{17} \mid W_2, \\ f_2 &= (\Delta^+_{17} \cdot j^+_{17}) + (\Delta^+_{17} \cdot j^+_{17}) \mid W_2, \\ f_3 &= \left(\Delta^+_{17} \cdot (j^+_{17})^2\right) + \left(\Delta^+_{17} \cdot (j^+_{17})^2\right) \mid W_2. \end{aligned}$$

Additionally, let f_4 be the newform from [The13d] which is also a holomorphic cusp form on $\Gamma^*_0(34)$. Then the cusp forms

$$\begin{aligned} f_1 &= q^3 - 4q^4 + 2q^5 + 12q^6 - 5q^7 - 20q^8 - 10q^9 + \cdots, \\ f_2 &= q^2 - 4q^3 + 13q^4 - 6q^5 - 34q^6 + 14q^7 + 53q^8 + 22q^9 + \cdots, \\ f_3 &= q + 16q^3 - 36q^4 + 18q^5 + 96q^6 - 40q^7 - 156q^8 - 49q^9 + \cdots, \\ f_4 &= q - 2q^2 - 2q^3 + 4q^4 + 16q^5 + 4q^6 + 24q^7 - 8q^8 - 23q^9 + \cdots \end{aligned}$$

span the space $S^*_4(34)$. Therefore $\Delta^*_{34,4} = -(26f_1 + 2f_2 + f_4 - f_3)/38$ and it has integral Fourier coefficients.

Case (5) $N = 39$**:** For this case, we need to find $\Delta^*_{39,4}$ and $\Delta^*_{39,6}$. Applying similar arguments to Example 3.1 to the data [The13e], we get two holomorphic cusp forms f_1 and f_2 in $S^*_4(39)$ with integral Fourier coefficients, and f_1, f_2 have q-expansions of the form

$$f_1 = q - 3q^3 + 5q^4 + 14q^5 + 2q^7 + 26q^8 + \cdots,$$
$$f_2 = q^2 + 2q^4 - 2q^5 - 3q^6 - 2q^7 + q^8 + \cdots.$$

Let $f_3 = \Delta^+_{13,4} + \Delta^+_{13,4} \mid W_3$ and let $f_4 = (\Delta^+_{13,4} \cdot j^+_{13}) + (\Delta^+_{13,4} \cdot j^+_{13}) \mid W_3$. Then f_3 and f_4 have q-expansions of the form

$$f_3 = q^2 - 3q^3 + q^4 + q^5 + 10q^6 + 11q^7 - 11q^8 + \cdots,$$
$$f_4 = q - 3q^2 + 22q^3 - 7q^4 - 5q^5 - 42q^6 - 43q^7 + 37q^8 + \cdots.$$

Taking a suitable linear combination of f_1, f_2, f_3 and f_4, we obtain

$$\Delta^*_{39,4} = q^4 - 4q^6 - 4q^7 + 6q^8 + \cdots \in \mathbb{Z}[[q]].$$

Similarly, from the data [The13g], we get two holomorphic cusp forms in $S^*_6(39)$ with integral Fourier coefficients. Let g_1 and g_2 be two cusp forms coming from [The13g], and let $g_3 = \Delta^+_{13,6} + \Delta^+_{13,6} \mid W_3$ and $g_4 = (\Delta^+_{13,6} \cdot j^+_{13}) + (\Delta^+_{13,6} \cdot j^+_{13}) \mid W_3$. Then g_1, g_2, g_3 and g_4 have q-expansions of the form

$$g_1 = q - 9q^3 + 20q^4 - 34q^5 + 14q^7 + \cdots,$$
$$g_2 = q^2 - 4q^4 - 5q^5 - 9q^6 - 11q^7 + \cdots,$$
$$g_3 = q^2 - 6q^3 - 5q^4 + 40q^5 + 28q^6 - 70q^7 + \cdots,$$
$$g_4 = q - 6q^2 + 34q^3 - 4q^4 - 161q^5 - 156q^6 + 227q^7 + \cdots.$$

By a suitable linear combination, we have $\Delta^*_{39,6} = q^4 - 3q^5 - q^6 + 5q^7 + \cdots \in \mathbb{Z}[[q]]$.

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