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On a class of generalized Fermat equations of signature $(2, 2n, 3)$

Karolina Chałupka, Andrzej Dąbrowski and Gökhan Soydan

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Abstract. We consider the Diophantine equation $7x^2 + y^{2n} = 4z^3$. We determine all solutions to this equation for $n = 2, 3, 4$ and 5. We formulate a Kraus type criterion for showing that the Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no non-trivial proper integer solutions for specific primes $p > 7$. We computationally verify the criterion for all primes $7 < p < 10^9$, $p \neq 13$. We use the symplectic method and quadratic reciprocity to show that the Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no non-trivial proper solutions for a positive proportion of primes p. In the paper [10] we consider the Diophantine equation $x^2 + 7y^{2n} = 4z^3$, determining all families of solutions for $n = 2$ and 3, as well as giving a (mostly) conjectural description of the solutions for $n = 4$ and primes $n \geq 5$.

Key words: Diophantine equation, modular form, elliptic curve, Galois representation, Chabauty method

2010 Mathematics Subject Classification: 11D61, 11B39

1 Introduction

Fix nonzero integers A, B and C. For given positive integers p, q, r satisfying $1/p + 1/q + 1/r < 1$, the generalized Fermat equation

$$Ax^p + By^q = Cz^r \tag{1}$$

has only finitely many primitive integer solutions. Modern techniques coming from Galois representations and modular forms (methods of Frey–Hellegouarch curves and variants of Ribet's level-lowering theorem, and of course,

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the modularity of elliptic curves or abelian varieties over the rationals or totally real number fields) allow to give partial (sometimes complete) results concerning the set of solutions to (1) (usually, when a radical of ABC is small), at least when (p,q,r) is of the type (n,n,n), $(n,n,2)$, $(n,n,3)$, $(2n,2n,5)$, $(2,4,n)$, $(2,6,n)$, $(2,n,4)$, $(2,n,6)$, $(3,3,p)$, $(2,2n,3)$, $(2,2n,5)$. Recent papers by Bennett, Chen, Dahmen and Yazdani [1] and by Bennett, Mihăilescu and Siksek [2] survey approaches to solving the equation (1) when $ABC = 1$. When $1/p + 1/q + 1/r$ is close to one (for instance, for $(p,q,r) = (2,3,5)$, $(2,3,7)$ or $(2,3,8)$), then one needs new methods (Chabauty method or its refinements [6, 7, 8], or a combination of Chabauty type method with a modular approach [18, 23]).

The Diophantine equation

$$x^2 + y^{2n} = z^3 \tag{2}$$

was studied by Bruin [7], Chen [11], Dahmen [16], and Bennett, Chen, Dahmen and Yazdani [1]. It is known that the equation (2) has no solutions for a family of n's of natural density one. Moreover, a Kraus type criterion is known which allows to check non-existence of solutions for all exponents n up to 10^7 (or more). Let us also mention that nonexistence of solutions for $n = 7$ follows from a more general result of Poonen, Schaefer and Stoll [23].

One of our motivations was to extend the above results (and methods) of Bruin, Chen and Dahmen, by considering some Diophantine equations $Ax^2 + By^{2n} = Cz^3$ with (A,B,C)'s different from $(1,1,1)$ (assuming for simplicity that the class number of $\mathbb{Q}(\sqrt{-AB})$ is one).

Another motivation was to extend our previous results in the case $k = 3$. To be precise, given odd, coprime integers a, b ($a > 0$), we consider the Diophantine equation

$$ax^2 + b^{2n} = 4y^k, \quad x,y \in \mathbb{Z}, \ n,k \in \mathbb{N}, \ k \, odd \, prime, \gcd(x,y) = 1. \tag{3}$$

The equation (3) was completely solved in [14] for $a \in \{7, 11, 19, 43, 67, 163\}$, and b a power of an odd prime, under the conditions $2^{k-1}b^n \not\equiv \pm 1 \pmod{a}$ and $\gcd(k,b) = 1$. The paper [9] extends these results by removing the first of these assumptions, namely that b is an odd prime. In this paper we fix $k = 3$, but b is arbitrary.

Let us briefly explain why we were unable to handle the Diophantine equations $7x^2 + y^{2n+1} = 4z^3$ and $x^2 + 7y^{2n+1} = 4z^3$. In [23], Poonen, Schaefer and Stoll find the primitive integer solutions to $x^2 + y^7 = z^3$. Their method

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combine the modular method together with determination of rational points on certain genus-3 algebraic curves. This case (and possible generalizations to $Ax^2 + By^7 = Cz^3$) is very difficult (as explained by the authors in the Introduction to [23]). Freitas, Naskrecki, Stoll [18] considered a general Diophantine equation $x^2 + y^p = z^3$ (with p any prime > 7). They follow and refine the arguments of [23] by combining new ideas around the modular method with recent approaches to determination of the set of rational points on certain algebraic curves. As a result, they were able to find (under GRH) the complete set of solutions of the Diophantine equation $x^2 + y^p = z^3$ only for $p = 11$.

It is the aim of this paper (and the next one [10]) to consider the Diophantine equations

$$ax^2 + y^{2n} = 4z^3, \quad x, y, z \in \mathbb{Z}, \ \gcd(x, y) = 1, \ n \in \mathbb{N}_{\geq 2}, \tag{4}$$

and

$$x^2 + ay^{2n} = 4z^3, \quad x, y, z \in \mathbb{Z}, \ \gcd(x, y) = 1, \ n \in \mathbb{N}_{\geq 2}, \tag{5}$$

where the class number of $\mathbb{Q}(\sqrt{-a})$ with $a \in \{7, 11, 19, 43, 67, 163\}$ is 1. An easy observation (see Corollary 4) is that we may only have solutions for $a = 7$. Hence, below we will treat in some detail the equation (4) for $a = 7$ (the equation (5) for $a = 7$ will be treated in our next paper [10]).

We show that the Diophantine equation $7x^2 + y^{2n} = 4z^3$ has no nontrivial solutions for $n = 3, 4, 5$ (using Chabauty method for $n = 5$ [Theorem 12], and calculating the Mordell-Weil group of the corresponding elliptic curve for $n = 3$ [Theorem 6]). In Section 8 we will describe a Kraus type criterion (Theorem 14), and its refinement (Theorem 16) for the first equation. The computational verification of the criteria for all primes $7 < p < 10^9$ gives the following result (Theorem 13):

Theorem 1. *The Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no primitive solutions for all primes $11 \leq p < 10^9$, $p \neq 13$.*

Let us briefly explain why the cases $p = 7, 13$ are omitted in the statement. It seems that the only available method (at present) for treating the Diophantine equation $7x^2 + y^{14} = 4z^3$ is to follow the methods of [23] (the modular methods used in our article are not sufficient). On the other hand, it seems possible that further improvements of the Kraus type criterion (as

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in sections 8.1, 8.2 and 8.3) may allow to treat the remaining case $p = 13$, but we were unsuccessful.

We also use the symplectic method and quadratic reciprocity to show that the Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no non-trivial proper solutions for a positive proportion of primes p (Theorem 19):

Theorem 2. *The Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no primitive solutions for a family of primes p satisfying:*

$$p \equiv 3 \ or \ 55 \pmod{106} \quad or \quad p \equiv 47, 65, 113, 139, 143 \ or \ 167 \pmod{168}.$$

The article is structured as follows.

In the preliminary Section 2 we show that, if $a \in \{11, 19, 43, 67, 163\}$, then the Diophantine equations (4) and (5) have no solutions.

In Section 3 we will determine all families of solutions to the Diophantine equation $7x^2 + y^4 = 4z^3$ (variant of Zagier's result in the case $x^2 + y^4 = z^3$ [4, 7]). Detailed proof of this result is given in the Appendix A.

In Section 4 we completely solve the Diophantine equations $7x^2 + y^6 = 4z^3$.

In Section 5 we completely solve the Diophantine equation $7x^2 + y^8 = 4z^3$.

In Section 6 we use the modular approach to the Diophantine equation $7x^2 + y^{2p} = 4z^3$, with $p \geq 7$ a prime. The main results of this section are crucial for the proofs of Theorems 1 and 2. The Appendix B contains the proof of Lemma 11.

In Section 7 we completely solve the Diophantine equation $7x^2 + y^{10} = 4z^3$, and comment on the Diophantine equation $7x^2 + y^{14} = 4z^3$.

In Section 8 we formulate a Kraus type criterion (actually, two criteria) for showing that this equation has no non-trivial proper integer solutions for specific primes $p > 7$. We computationally verify the criterion for all primes $7 < p < 10^9$, $p \neq 13$. The Appendix C (resp. D) contains proof of Corollary 15 (resp. 17).

In Section 9 we use the symplectic method and quadratic reciprocity to show that the Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no non-trivial proper solutions for a positive proportion of primes p.

2 Preliminaries

As the class number of $\mathbb{Q}(\sqrt{-a})$ with $a \in \{7, 11, 19, 43, 67, 163\}$ is 1, we have the following factorization for the left hand side of (4)

$$\frac{y^n + x\sqrt{-a}}{2} \cdot \frac{y^n - x\sqrt{-a}}{2} = z^3.$$

Now we have

$$\frac{y^n + x\sqrt{-a}}{2} = \left(\frac{u + v\sqrt{-a}}{2}\right)^3,$$

where u, v are odd rational integers. Note that necessarily $\gcd(u, v) = 1$. Replacing x with y^n, we obtain a similar factorization for the left hand side of (5). Equating the real and imaginary parts, we obtain the following result.

Lemma 3. *(a) Suppose that (x, y, z) is a solution to (4). Then*

$$(x, y^n, z) = \left(\frac{v(3u^2 - av^2)}{4}, \frac{u(u^2 - 3av^2)}{4}, \frac{u^2 + av^2}{4}\right) \tag{6}$$

for some odd $u, v \in \mathbb{Z}$ with $\gcd(u, v) = 1$, $uv \neq 0$.

(b) Suppose that (x, y, z) is a solution to (5). Then

$$(x, y^n, z) = \left(\frac{u(u^2 - 3av^2)}{4}, \frac{v(3u^2 - av^2)}{4}, \frac{u^2 + av^2}{4}\right) \tag{7}$$

for some odd $u, v \in \mathbb{Z}$ with $\gcd(u, v) = 1$, $uv \neq 0$.

Corollary 4. *If $a \in \{11, 19, 43, 67, 163\}$, then the Diophantine equations (4) and (5) have no solutions.*

Proof. We are reduced to the equation $u(u^2 - 3av^2) = 4y^n$ or $v(3u^2 - av^2) = 4y^n$. Now, if $a \in \{11, 19, 43, 67, 163\}$, then $u(u^2 - 3av^2)$ is congruent to 0 modulo 8, while $4y^n$ is congruent to 4 modulo 8, a contradiction. Similarly in the second case. $\square$

In what follows, we will consider the equations (4) and (5) for $a = 7$. From Lemma 3 we obtain infinitely many solutions with $n = 1$. Below we will treat the cases $n \in \{2, 3, 4\}$ and $n \geq 5$ a prime, separately.

3 The Diophantine equation $7x^2 + y^4 = 4z^3$

In this section we will determine all families of solutions to the title equation (variants of Zagier's result in the case $x^2 + y^4 = z^3$ [4, 7]).

Theorem 5. *Let x, y, z be coprime integers such that $7x^2 + y^4 = 4z^3$. Then there are rational numbers s, t such that one of the following holds.*

$$
\begin{aligned}
x = {}&\pm(1911s^4 + 1260ts^3 + 378t^2s^2 + 12t^3s + 7t^4) \\
&(-5078115s^8 - 11928168ts^7 - 2556036t^2s^6 - 1802808t^3s^5 \\
&\quad - 929922t^4s^4 - 38808t^5s^3 + 46620t^6s^2 + 9912t^7s + 461t^8), \\
y = {}&\pm 3(21s^2 - 14ts - 3t^2)(2499s^4 + 1764ts^3 + 378t^2s^2 + 84t^3s - 5t^4), \\
z = {}&6828444s^8 + 7260624ts^7 + 6223392t^2s^6 + 1728720t^3s^5 \\
&\quad + 156408t^4s^4 + 49392t^5s^3 + 28224t^6s^2 + 3696t^7s + 268t^8,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
x = {}&\pm(343s^4 - 84ts^3 + 378t^2s^2 - 180t^3s + 39t^4) \\
&(1106861s^8 - 3399816ts^7 + 2284380t^2s^6 + 271656t^3s^5 \\
&\quad - 929922t^4s^4 + 257544t^5s^3 - 52164t^6s^2 + 34776t^7s - 2115t^8), \\
y = {}&\pm 3(21s^2 - 14ts - 3t^2)(-245s^4 - 588ts^3 + 378t^2s^2 - 252t^3s + 51t^4), \\
z = {}&643468s^8 - 1267728ts^7 + 1382976t^2s^6 - 345744t^3s^5 + 156408t^4s^4 \\
&\quad - 246960t^5s^3 + 127008t^6s^2 - 21168t^7s + 2844t^8.
\end{aligned}
\tag{9}
$$

For the proof of this result see the Appendix A.

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4   The Diophantine equation $7x^2 + y^6 = 4z^3$

Theorem 6. *Diophantine equation $7x^2 + y^6 = 4z^3$ has no non-trivial solutions.*

Proof. This follows from the fact that the Mordell-Weil group of the elliptic curve given by $Y^2 = X^3 - 2^4 7^3$ is trivial. □

5   The Diophantine equation $7x^2 + y^8 = 4z^3$

We will prove variant of Bruin's result from section 5 in [7].

Any primitive solution of the Diophantine equation $7x^2 + y^8 = 4z^3$ satisfies, of course, the equation $7x^2 + (y^2)^4 = 4z^3$. Hence, using Theorem 5, we obtain formulas describing x, y^2 and z. In particular we have the following formulas for y^2:

$$y^2 = \pm 3(21s^2 - 14ts - 3t^2)(2499s^4 + 1764ts^3 + 378t^2s^2 + 84t^3s - 5t^4),$$
$$y^2 = \pm 3(21s^2 - 14ts - 3t^2)(-245s^4 - 588ts^3 + 378t^2s^2 - 252t^3s + 51t^4).$$

Note that $t = 0$ implies $y = 0$. Therefore, nontrivial solutions correspond to affine rational points on one of the following genus two curves:

$$\mathcal{C}_1 : Y^2 = 3(21X^2 - 14X - 3)(2499X^4 + 1764X^3 + 378X^2 + 84X - 5),$$
$$\mathcal{C}_2 : Y^2 = -3(21X^2 - 14X - 3)(2499X^4 + 1764X^3 + 378X^2 + 84X - 5),$$
$$\mathcal{C}_3 : Y^2 = 3(21X^2 - 14X - 3)(-245X^4 - 588X^3 + 378X^2 - 252X + 51),$$
$$\mathcal{C}_4 : Y^2 = -3(21X^2 - 14X - 3)(-245X^4 - 588X^3 + 378X^2 - 252X + 51).$$

Theorem 7. *The Diophantine equation $7x^2 + y^8 = 4z^3$ has no non-trivial solutions.*

Proof. We check that the curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ have no affine rational points. Indeed, the Magma command `HasPointsEverywhereLocally`$(f, 2)$, gives $\mathcal{C}_1(\mathbb{Q}_2) = \mathcal{C}_2(\mathbb{Q}_2) = \mathcal{C}_3(\mathbb{Q}_2) = \mathcal{C}_4(\mathbb{Q}_2) = \emptyset$. □

6   A modular approach to the Diophantine equation $7x^2 + y^{2n} = 4z^3$

Now we will assume that $n = p$ is a prime ≥ 7. By Lemma 3(a) we have reduced the problem of solving the title equation to solving the equation $4y^p = u(u^2 - 21v^2)$ with odd u, v and y. Since $\gcd(u, v) = 1$, we have $d = \gcd(u, u^2 - 21v^2)|3$. We can solve the equation corresponding to $d = 1$ with $p \geq 11$ and $p \neq 19$ (Proposition 10). In the case $d = 3$ we obtain a partial result (Lemma 11). We will continue with the modular approach in sections 8 and 9.

It is possible to use modular approach for $p = 7$, i.e. we can associate a Frey type curve, use the modularity theorem and Ribet's level-lowering theorem. However, the methods of eliminating newforms that we use to prove our main results fail in this case (see the proof of Proposition 10 and Lemma 11). For more remarks see Section 7.2.

6.1   Reducing to the case of signature $(2p, p, 2)$

Below we will reduce the problem to solving the equation $4y^p = u(u^2 - 21v^2)$ with odd u, v and y, to the problem of solving equations of signature $(2p, p, 2)$.

(i) $\underline{d = 1}$. Writing $u = \alpha^p$, $u^2 - 21v^2 = 4\beta^p$, we arrive at

$$\alpha^{2p} - 4\beta^p = 21v^2. \tag{10}$$

(ii) $\underline{d = 3}$. Since $v_3(u^2 - 21v^2) = 1$ we have

$$\begin{cases} u = 3^{p-1}\alpha^p \\ u^2 - 21v^2 = 12\beta^p, \end{cases}$$

with odd α, β satisfying $\gcd(\alpha, \beta) = 1$. This leads to the equation

$$3^{2p-3}\alpha^{2p} - 4\beta^p = 7v^2. \tag{11}$$

6.2   The equation $\alpha^{2p} - 4\beta^p = 21v^2$

First, let us give short elementary proof that the title equation (the equation (10)) has no solution for infinitely many primes p. We will use the following result, which is a variant of Lemmas 14 and 15 from [16].

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Lemma 8. *(a) $\beta - \alpha^2$ is a square modulo* 7.
(b) $7 \nmid \alpha$ and $(\beta/\alpha^2)^p \equiv 2 \pmod{7}$.

Proof. (a) We have $\beta - \alpha^2 \mid 4(\beta^p - \alpha^{2p}) = -3(u^2 + 7v^2)$. Assume that q is an odd prime dividing $\beta - \alpha^2$. Then $-(3u)^2 \equiv 7(3v)^2 \pmod{q}$, hence $q = 3$ or $\left(\frac{-7}{q}\right) = 1$. Note that $v_3(\beta - \alpha^2) = 1$, hence $\left(\frac{3^{v_3(\beta-\alpha^2)}}{7}\right) = \left(\frac{3}{7}\right) = -1$. Moreover, $\left(\frac{-1}{7}\right) = -1$, $\left(\frac{2}{7}\right) = 1$, $\left(\frac{q}{7}\right) = \left(\frac{-7}{q}\right) = 1$, and $\beta - \alpha^2 < 0$. Hence $\left(\frac{\beta-\alpha^2}{7}\right) = 1$ as wanted.

(b) Note that $7 \mid \alpha$ implies $7 \mid \beta$, a contradiction. Now $(\beta/\alpha^2)^p = (1 - 21(v/u)^2)/4 \equiv 2 \pmod{7}$ as wanted. $\qquad\square$

Proposition 9. *Let p be a prime with $p \equiv 5 \pmod{6}$. Then (10) has no non-trivial solutions.*

Proof. From part (a) of Lemma 8 we get that $\beta/\alpha^2 - 1$ is a square modulo 7, while from part (b) we get $\beta/\alpha^2 - 1 \equiv 3 \pmod{7}$, a contradiction. $\qquad\square$

Now, let us use modular approach to prove a much stronger result.

Proposition 10. *The Diophantine equation (10) has no solutions in coprime odd integers for $p \geq 11$, $p \neq 19$.*

Proof. We will apply the Bennett-Skinner strategy [3] to a more general equation $X^p - 4Y^p = 21Z^2$, $p \geq 7$. We are in case (iii) of [3, p.26], hence from Lemma 3.2 it follows, that we need to consider the newforms of weight 2 and levels $N \in \{1764, 3528\}$.

a) There are 13 Galois conjugacy classes of forms of weight 2 and level 1764. We can compute systems of Hecke eigenvalues for conjugacy classes of newforms using Magma [5] or use Stein's Modular Forms Database [24]. We will use numbering as in Stein's tables.

We can eliminate f_{12}, when $p \geq 11$, as follows. We have $c_5(f_{12}) = \pm\sqrt{2}$ and so, by [3, Prop. 4.3], p must divide one of 2, 14, 34. On the other hand, $c_{13}(f_{12}) = \pm 3\sqrt{2}$ and so p must divide one of 2, 14, 18. Similarly, we can eliminate f_{13} when $p \geq 11$, considering $c_5(f_{13})$: in this case p must divide one of 4, 100, 196.

Similarly, we can use [3, Prop. 4.3] to eliminate f_1, f_3, f_4, f_6, f_8 and f_{10}. Elimination of f_1 and f_{10}: we have $c_5(f_1) = -3$ and $c_5(f_{10}) = 3$, hence p must divide one of 1, 3, 5, 7, 9. Elimination of f_3 and f_8: we have $c_{13}(f_3) = 3$

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and $c_{13}(f_8) = -3$, hence p must divide one of 1, 3, 5, 7, 9, 11, 17; we have $c_{19}(f_3) = 1$ and $c_{19}(f_8) = -1$, hence p must divide one of 1, 3, 5, 7, 9, 19, 21. Elimination of f_4 and f_6: we have $c_{13}(f_4) = -5$ and $c_{13}(f_6) = 5$, hence p must divide one of 1, 3, 5, 7, 9, 11, 19; we have $c_{19}(f_4) = 1$ and $c_{19}(f_6) = -1$, hence p must divide one of 1, 3, 5, 7, 9, 19, 21.

The newforms f_2, f_5, f_7, f_9 and f_{11} correspond to isogeny classes of elliptic curves A, G, H, I and K respectively (using notation from Cremona's online tables or from LMFDB) [13], with non-integral j-invariants $u/(3^v 7^w)$, where $\gcd(u, 21) = 1$ and $v^2 + w^2 > 0$. Hence we can eliminate all these forms using [3, Prop. 4.4].

b) There are 39 Galois conjugacy classes of forms of weight 2 and level 3528. Again, we will use numbering as in Stein's tables.

Let us consider 12 classes with irrational Fourier coefficients first. To eliminate f_{30}, f_{31}, f_{32}, f_{33}, f_{34}, f_{35}, f_{36}, and f_{36}, it is enough to consider c_5. To eliminate the remaining 4 newforms, we need to consider c_5 and c_{11}.

Now let us consider the newforms with rational Fourier coefficients. The newforms f_2, f_4, f_5, f_7, f_{13}, f_{14}, f_{20}, f_{21}, f_{22}, f_{23} and f_{24} correspond to isogeny classes of elliptic curves B, D, E, G, M, N, T, U, V, W and X respectively, with non-integral j-invariants $u/(3^v 7^w)$, where $\gcd(u, 21) = 1$ and $v^2 + w^2 > 0$. Hence we can eliminate all these forms using [3, Prop. 4.4].

To eliminate the remaining 16 newforms, we will use [3, Prop. 4.3]. To eliminate f_8, f_9, f_{10}, f_{11}, f_{12}, f_{15}, f_{16}, f_{17}, f_{18} and f_{19}, it is enough to consider c_5. To eliminate the remaining 6 newforms, we consider c_{13} (hence p must divide one of 1, 3, 5, 7, 9, 11, 17) and use Proposition 9 to exclude $p \in \{11, 17\}$. $\qquad\square$

6.3 The equation $3^{2p-3}\alpha^{2p} - 4\beta^p = 7v^2$

Let (a, b, c) be a solution in coprime odd integers of the title equation (i.e. the equation (11)). Following [3], we consider the following Frey type curve associated to (a, b, c)

$$E = E(a, b, c) : Y^2 = X^3 + 7cX^2 - 7b^p X. \tag{12}$$

We have $\Delta_E = 2^4 \cdot 3^{2p-3} \cdot 7^3 (ab)^{2p}$ and $N_E = 588 \cdot \prod_{l|ab} l$ (resp. $1176 \cdot \prod_{l|ab} l$) if $b \equiv 3 \pmod 4$ (resp. $b \equiv 1 \pmod 4$). Using [3, Corollary 3.1] we obtain, that the associated Galois representation

$$\overline{\rho}_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_p)$$

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is irreducible for all primes $p \geq 7$. From [3, Lemma 3.3] we know, that $\overline{\rho}_{E,p}$ arises from a cuspidal newform f of weight 2, level $N = 588$ (resp. 1176), and trivial Nebentypus character. Applying [3, Propositions 4.3 and 4.4] and [17, Proposition 13] yields the following result.

Lemma 11. *Let p be a prime. Suppose that (a, b, c) is a solution in coprime odd integers to the equation (11). Let $E = E(a, b, c)$ be the associated Frey type curve.*

1. *If $p \geq 13$, then $\overline{\rho}_{E,p} \cong \overline{\rho}_{F,p}$ for an elliptic curve F in one of the following isogeny classes: $588C, 588E, 1176G, 1176H$.*

2. *If $p = 11$, then $\overline{\rho}_{E,p} \cong \overline{\rho}_{F,p}$ for an elliptic curve F in one of the following isogeny classes: $588C, 588E, 1176A, 1176F, 1176G, 1176H$.*

We prove Lemma 11 in Appendix B. We use here Cremona labels (see [13]).

7 The Diophantine equation $7x^2 + y^{2n} = 4z^3$ for $n = 5, 7$

7.1 The Diophantine equation $7x^2 + y^{10} = 4z^3$

Theorem 12. *The Diophantine equation $7x^2 + y^{10} = 4z^3$ has no non-trivial solutions.*

Proof. We may consider the equations (10) and (11) for $p = 5$, and in this case they lead to the curves $\mathcal{C}_1 : Y^2 = 84X^5 + 21$ and $\mathcal{C}_2 : Y^2 = 28X^5 + 3^7 \times 7$, respectively. Now $Jac(\mathcal{C}_i)$ $(i = 1, 2)$ have $\mathbb{Q}$-rank 0, and using `Chabauty0`, we obtain $\mathcal{C}_i(\mathbb{Q}) = \{\infty\}$ $(i = 1, 2)$, and the assertion follows. $\square$

7.2 The Diophantine equation $7x^2 + y^{14} = 4z^3$

We expect that the title equation has no solution in coprime odd integers. However, as we wrote at the start of Section 6, we have been unable to do so. Here we discuss a few approaches to this equation and the obstacles to making them work here.

(i) *The modular method.* We may consider the equations (10) and (11) for $p = 7$: $X^{14} - 4Y^7 = 21Z^2$ and $3^{11}X^{14} - 4Y^7 = 7Z^2$, respectively. In

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both cases, we could not exclude the possibility that the Galois representation associated to the Frey type curve arises from newform with nonrational Fourier coefficients (see the proof of Proposition 2 and the proof of Lemma 11 in Appendix B).

(ii) *Chabauty type approach in genus* 3. The Diophantine equations from (i) lead to the genus 3 curves $\mathcal{D}_1 : y^2 = x^7 + 2^{12} \cdot 3^7 \cdot 7^7$ and $\mathcal{D}_2 : y^2 = x^7 + 2^{12} \cdot 3^{11} \cdot 7^7$, respectively. Magma calculations show that the only rational points on $\mathcal{D}_i(\mathbb{Q})$ (with bounds 10^9) are points at infinity, as expected. Magma also shows that ranks of $Jac(\mathcal{D}_i)(\mathbb{Q})$ ($i = 1, 2$) are bounded by 1. There are two technical problems to use Chabauty method: one needs explicit rational points of infinite order (not easy to find) and there is no readily available implementation of Chabauty's method for (odd degree) hyperelliptic genus 3 curves. Professor Stoll suggested to try the methods of his papers [25, 26], but we were not able to follow his advise yet.

(iii) *Combination of the modular and Chabauty methods*. One may consider a more general Diophantine equation $7x^2 + y^7 = 4z^3$, try to follow the paper by Poonen, Schaefer and Stoll [23], and then deduce the solutions for the original Diophantine equation. It seems a very difficult task, but maybe the only available way ...

8   Solving the Diophantine equation $7x^2 + y^{2p} = 4z^3$ for a fixed prime $p \geq 11$

In this and the next section we will continue the modular approach to the title equation that we started in Section 6. Here we will assume that $p \geq 11$ is a prime and apply variants of the method introduced by Kraus in [21]. Kraus stated a very interesting criterion [21, Théorème 3.1] that often allows to prove that the Diophantine equation $x^3 + y^3 = z^p$ (p an odd prime) has no primitive solutions for fixed p, and verified his criterion for all primes $17 \leq p < 10^4$. Such a criterion has been formulated (and refined) in other situations (see, for instance, [11, 12, 15, 16]). In Subsection 8.1 we will formulate such a criterion (Theorem 14) in the case of Diophantine equation (11). In Subsection 8.2 we will give a refined version of the criterion, and we will apply it to those values of p, for which Theorem 14 is not sufficient. In Subsection 8.3 we use Kraus method to the equation (10) with the exponent $p = 19$. Magma calculations based on the corresponding algorithms allow to

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state the following result.

Theorem 13. *The Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no primitive solutions for all primes $11 \leq p < 10^9$, $p \neq 13$.*

As was shown in section 6, it is enough to deal with the equation (11) with $p \geq 11$ and the equation (10) with $p = 19$ (see Proposition 10). The proof of Theorem 13 will take the whole section and follow from Corollaries 15 and 17 and Proposition 18.

8.1 Kraus type criterion

Let $q \geq 11$ be a prime number, and let $k \geq 1$ be an integer factor of $q - 1$. Let $\mu_k(\mathbb{F}_q)$ denote the group of k-th roots of unity in $\mathbb{F}_q^\times$. Set

$$A_{k,q} := \{\xi \in \mu_k(\mathbb{F}_q) : \frac{1 - 2^2 3^3 \xi}{3^3 \cdot 7} \text{ is a square in } \mathbb{F}_q\}.$$

For each $\xi \in A_{k,q}$, we denote by δ_ξ the least non-negative integer such that

$$\delta_\xi^2 \bmod q = \frac{1 - 2^2 3^3 \xi}{3^3 \cdot 7}.$$

We associate with each $\xi \in A_{k,q}$ the following equation

$$Y^2 = X^3 + 7\delta_\xi X^2 - 7\xi X.$$

Its discriminant equals $2^4 3^{-3} 7^3 \xi^2$, so it defines an elliptic curve E_ξ over $\mathbb{F}_q$. We put $a_q(\xi) := q + 1 - \#E_\xi(\mathbb{F}_q)$.

Theorem 14. *Let $p \geq 13$ be a prime (resp. $p = 11$). Suppose that for each elliptic curve*

$$F \in \{588C1, 1176G1\} \qquad (resp. \ F \in \{588C1, 1176A1, 1176G1\})$$

there exists a positive integer k such that the following three conditions hold

(1) $q := kp + 1$ is a prime,

(2) $a_q(F)^2 \not\equiv 4 \pmod{p}$,

(3) $a_q(F)^2 \not\equiv a_q(\xi)^2 \pmod{p}$ for all $\xi \in A_{k,q}$.

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Then the equation $3^{2p-3}x^{2p} - 4y^p = 7z^2$ has no solutions in coprime odd integers.

Proof. Let $p \geq 11$ be a prime. Suppose that (a, b, c) is a solution of the equation (11), where a, b, c are coprime odd integers. Let E denote the Frey type curve associated to (a, b, c). From Lemma 11 it follows that $\overline{\rho}_{E,p} \cong \overline{\rho}_{F,p}$, where F is one of the curves:

$$588C1, 588E1, 1176G1, 1176E1 \quad \text{if } p \geq 13,$$
$$588C1, 588E1, 1176A1, 1176F1, 1176G1, 1176H1 \quad \text{if } p = 11.$$

We have $a_l(E) \equiv a_l(F) \pmod{p}$ for all primes l such that $l \nmid N_E$ and $a_l(F) \equiv \pm(l+1) \pmod{p}$ for all primes l such that $l \mid N_E$ but $l \nmid N_F$.

Suppose k is an integer that satisfies conditions (1) – (3). If $q \mid N_E$, then $a_q(F) \equiv \pm(q+1) \equiv \pm 2 \pmod{p}$, a contradiction. So the curve E has good reduction at q and $a_q(F) \equiv a_q(E) \pmod{p}$. From (11) it follows that

$$\left(\frac{c}{(3a)^p}\right)^2 = \frac{1 - 2^2 3^3 (\frac{b}{9a^2})^p}{3^3 \cdot 7}.$$

Let $\xi = (\overline{b}/9\overline{a}^2)^p \in A_{k,q}$, where $\overline{a}$ and $\overline{b}$ are reductions of a and b modulo q. The reduction of E modulo q is a quadratic twist of E_ξ by $(3\overline{a})^p$ or $-(3\overline{a})^p$. Hence $a_q(E)^2 = a_q(\xi)^2$ and so $a_q(F)^2 \equiv a_q(\xi)^2 \pmod{p}$, a contradiction.

The elliptic curves $588C1$ and $588E1$ have the same j-invariant, which is different from 0 and 1728, so they are isomorphic over some quadratic extension of $\mathbb{Q}$ and $a_l(588C1)^2 = a_l(588E1)^2$ for every prime l. Hence, it suffices to consider only the first of the two curves. A similar remark applies to the other two pairs of curves: $1176A1$, $1176F1$ and $1176G1$, $1176H1$. □

Corollary 15. *Let $11 \leq p < 10^9$ and $p \neq 13, 17$ be a prime. Then there are no triples (x, y, z) of coprime odd integers satisfying $3^{2p-3}x^{2p} - 4y^p = 7z^2$.*

The proof of Corollary 15 is based on computations in Magma and is contained in the Appendix C. We applied Theorem 14 for all primes in range $11 \leq p < 10^9$. We could not find an integer k satisfying the conditions (1) – (3) in the following two cases $(F, p) = (588C1, 13)$ and $(F, p) = (1176G1, 17)$. We found such k in any other case. We will use this information later in Section 8.2.

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8.2 A refined version of Kraus type criterion

As we have seen in Subsection 8.1 Kraus criterion is not sufficient to prove that the equation (11) with $p = 13$ or 17 has no solution in coprime odd integers. Following [16] we will refine the method and apply it successfully in case $p = 17$.

Recall from subsection 6.1 case (ii) that if $3 \mid y$ we have

$$3u = (3\alpha)^p \quad \text{and} \quad u^2 - 21v^2 = 12\beta^p$$

for some coprime odd integers u, v and coprime odd integers α, β such that $y = 3\alpha\beta$. Let $R = \mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{21}}{2}$, be the ring of integers of the number field $\mathbb{Q}(\sqrt{21})$. Observe that R has class number one. If we factor in R the both sides of the second equation, then we obtain

$$u + \sqrt{21}v = (3 \pm \sqrt{21})x_1^p \varepsilon \quad \text{and} \quad u - \sqrt{21}v = (3 \mp \sqrt{21})x_2^p \varepsilon^{-1},$$

where $x_1, x_2 \in R$ and $\varepsilon \in R^*$.

Suppose that $q = kp + 1$ is a prime that splits in R. Let $\mathfrak{q}$ be a prime in R lying above q. We have $R/\mathfrak{q} \simeq \mathbb{F}_q$. Write $\overline{x}$ for the reduction of $x \in R$ modulo $\mathfrak{q}$ and write r_{21} for $\overline{\sqrt{21}}$. From the above equalities it follows that for some $\xi_0, \xi_1, \xi_2 \in \mu_k(\mathbb{F}_q)$

$$3\overline{u} = \xi_0, \quad \overline{u} + r_{21}\overline{v} = (3 \pm r_{21})\xi_1\overline{\varepsilon} \quad \text{and} \quad \overline{u} - r_{21}\overline{v} = (3 \mp r_{21})\xi_2\overline{\varepsilon}^{-1}.$$

If we divide the second and the third equality by $3\overline{u}$ we obtain

$$\frac{1}{3} + r_{21}\frac{\overline{v}}{3\overline{u}} = (3 \pm r_{21})\xi_1'\overline{\varepsilon} \quad \text{and} \quad \frac{1}{3} - r_{21}\frac{\overline{v}}{3\overline{u}} = (3 \mp r_{21})\xi_2'\overline{\varepsilon}^{-1},$$

where $\xi_1' = \xi_1/\xi_0$ and $\xi_2' = \xi_2/\xi_0$. Suppose further that $\overline{\varepsilon_f} \in \mu_k(\mathbb{F}_q)$ for a fundamental unit $\varepsilon_f \in R^*$. Then also $\xi_1'\overline{\varepsilon}, \xi_2'\overline{\varepsilon} \in \mu_k(\mathbb{F}_q)$. Hence $\frac{\overline{v}}{3\overline{u}}$ is an element of $S_{k,q} \cup S_{k,q}'$, where

$$S_{k,q} = \left\{ \delta \in \mathbb{F}_q : \frac{1}{3 + r_{21}}\left(\frac{1}{3} + r_{21}\delta\right), \ \frac{1}{3 - r_{21}}\left(\frac{1}{3} - r_{21}\delta\right) \in \mu_k(\mathbb{F}_q) \right\},$$

$$S_{k,q}' = \left\{ \delta \in \mathbb{F}_q : \frac{1}{3 - r_{21}}\left(\frac{1}{3} + r_{21}\delta\right), \ \frac{1}{3 + r_{21}}\left(\frac{1}{3} - r_{21}\delta\right) \in \mu_k(\mathbb{F}_q) \right\}.$$

For $\delta \in S_{k,q} \cup S_{k,q}'$ we define $\xi_\delta = \frac{1 - 3^3 \cdot 7\delta^2}{3^3 2^2}$, which is an element of $\mu_k(\mathbb{F}_q)$. The equation

$$Y^2 = X^3 + 7\delta X^2 - 7\xi_\delta X$$

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defines an elliptic curve E_δ over $\mathbb{F}_q$. We put $a_q(\delta) := q + 1 - \#E_\delta(\mathbb{F}_q)$.

The above consideration and the argumentation as the proof of Theorem 14 imply the following result.

Theorem 16. *Let $p > 11$ be a prime. Suppose that for each elliptic curve $F \in \{588C1, 1176G1\}$ there exists a positive integer k such that the following conditions hold*

(1) $q := kp + 1$ is a prime,

(2) q splits in $\mathbb{Z}[\frac{1+\sqrt{21}}{2}]$,

(3) $q \mid \mathrm{Norm}_{\mathbb{Q}(\sqrt{21})/\mathbb{Q}}((\frac{5+\sqrt{21}}{2})^k - 1)$,

(4) $a_q(F)^2 \not\equiv 4 \pmod{p}$,

(5) $a_q(F)^2 \not\equiv a_q(\delta)^2 \pmod{p}$ for all $\delta \in S_{k,q} \cup S'_{k,q}$.

Then the equation $3^{2p-3}x^{2p} - 4y^p = 7z^2$ has no solutions in coprime odd integers.

Compared to Theorem 14, we have two additional conditions that must be satisfied by the integer k. However, we get more information about the hypothetical solution and so there are less congruences to check in the latter condition. The set $S_{k,q} \cup S'_{k,q}$ has significantly fewer elements than the set $A_{k,q}$, which appears in Theorem 14. For example, if $(F, p) = (1176G1, 17)$ and $k = 374$ we have $\#(S_{k,q} \cup S'_{k,q}) = 18$ and $\#A_{k,q} = 176$. In this case, conditions $(1) - (5)$ are satisfied. On the other hand, we could not find an integer k satisfying conditions $(1) - (5)$ for $(F, p) = (588C1, 17)$. Nevertheless, combining arguments of Theorem 14 and Theorem 16 allow us to prove the following result.

Corollary 17. *The equation $3^{31}x^{34} - 4y^{34} = 7z^2$ has no solutions in coprime odd integers.*

See Appendix D for the proof.

Application of the refined Kraus method does not give any new information about the equation (11) with $p = 13$. If $F = 588C1$, we are unable to find an integer k satisfying the conditions $(1) - (5)$ of Theorem 16 (see Appendix D). Thus, the only conclusion we can make in case $p = 13$ is that the equation (11) has no solution in coprime odd integers a, b, c with $b \equiv 3 \pmod{4}$.

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8.3   No proper solution to the equation $7x^2 + y^{38} = 4z^3$

To complete the poof of Theorem 13, we need to deal with the equation (10) for $p = 19$. We will treat this case with the same method as we treated the equation (11) for $p \geq 11$ in Subsection 8.1.

Proposition 18. *The Diophantine equation $\alpha^{38} - 4\beta^{19} = 21v^2$ has no solution in coprime odd integers.*

Proof. Let (a, b, c) be a solution of the equation (10), where a, b, c are odd and coprime. Following [3], we associate to (a, b, c) the following Frey type curve

$$E = E(a, b, c) : Y^2 = X^3 + 21cX^2 - 21b^pX. \tag{13}$$

For a prime $p \geq 7$, the Galois representation $\overline{\rho}_{E,p}$ arises from a newform $f \in S_2(N)$ with $N \in \{1764, 3528\}$. If $p = 19$ the only two newforms which cannot be eliminated by the methods applied in the proof of Proposition 10 are the newforms f_4 and f_6 of level 1764. These newforms correspond to the isogeny classes of elliptic curves represented by

$$F : Y^2 = X^3 - 28 \qquad \text{and} \qquad G : Y^2 = X^3 - 259308$$

respectively. Observe that the two curves are isomorphic over $\mathbb{Q}(\sqrt{21})$, hence $a_q(F)^2 = a_q(G)^2$ for all primes q.

Let $q \geq 11$ be a prime number and let $k > 1$ be an integer such that $k \mid q - 1$. We define

$$B_{k,q} = \{\xi \in \mu_k(\mathbb{F}_q) : \frac{1 - 4\xi}{21} \text{ is a square in } \mathbb{F}_q\}.$$

For each $\xi \in B_{k,q}$, we denote by δ_ξ the least non-negative integer such that

$$\delta_\xi^2 \bmod q = \frac{1 - 4\xi}{21}.$$

We associate with each $\xi \in B_{k,q}$ the following equation

$$Y^2 = X^3 + 21\delta_\xi X^2 - 21\xi X,$$

which defines an elliptic curve E_ξ over $\mathbb{F}_q$. We put $a_q(\xi) := q + 1 - \#E_\xi(\mathbb{F}_q)$.

Proceeding as in the proof of Theorem 14, we obtain the following conclusion. If there exists an integer k such that $q = 19k + 1$ is a prime, $a_q(F)^2 \not\equiv 4 \pmod{19}$ and $a_q(F)^2 \not\equiv a_q(\xi)^2 \pmod{19}$ for all $\xi \in B_{k,q}$, then neither $\overline{\rho}_{E,19} \cong \overline{\rho}_{F,19}$ nor $\overline{\rho}_{E,19} \cong \overline{\rho}_{G,19}$, and the Proposition follows. It can be checked (e.g. in Magma) that the least such k is equal 34. $\qquad\square$

9 No solutions to the Diophantine equation $7x^2 + y^{2p} = 4z^3$ for infinitely many prime p's

In this section we will use ideas of the papers [1, 11, 12, 16, 17, 22] to prove the following result.

Theorem 19. *The Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no primitive solutions for a family of primes p satisfying:*

$$p \equiv 3 \ or \ 55 \pmod{106} \quad or \quad p \equiv 47, 65, 113, 139, 143 \ or \ 167 \pmod{168}.$$

If the title equation has a primitive solution, then one of the equations (10) or (11) is solvable in coprime odd integers (see 6.1). Proposition 10 together with Proposition 18 say that there is no such solution of equation (10) for a prime exponent ≥ 11. Now we would like to establish an infinite family of prime exponents for which the equation (11) has no solution in coprime odd integers.

9.1 Application of the symplectic method

Let $p \geq 3$ be a prime. Let E and E' be elliptic curves over $\mathbb{Q}$ and write $E[p]$ and $E'[p]$ for their p-torsion modules. Write $G_{\mathbb{Q}}$ for the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\phi : E[p] \to E'[p]$ be a $G_{\mathbb{Q}}$-modules isomorphism. There is an element $d(\phi) \in \mathbb{F}_p^{\times}$ such that, for all $P, Q \in E[p]$, the Weil pairings satisfy $e_{E',p}(\phi(P), \phi(Q)) = e_{E,p}(P, Q)^{d(\phi)}$. We say that ϕ is a *symplectic isomorphism* if $d(\phi)$ is a square modulo p and an *anti-symplectic* otherwise. If the Galois representation $\overline{\rho}_{E,p}$ is irreducible then all $G_{\mathbb{Q}}$-isomorphisms have the same symplectic type.

Write Δ and Δ' for minimal discriminants of E and E'. Suppose E and E' have potentially good reduction at a prime l. Set $\tilde{\Delta} = \Delta/l^{v_l(\Delta)}$ and $\tilde{\Delta}' = \Delta'/l^{v_l(\Delta')}$. Define a *semistability defect* e as the order of the group $\mathrm{Gal}(\mathbb{Q}_l^{un}(E[p])/\mathbb{Q}_l^{un})$. Define e' in the same way. Note that if $E[p] \cong E'[p]$ then $e = e'$ [17, Proposition 13]. If $l \geq 5$ then e is the denominator of $v_l(\Delta)/12$ [20].

We apply the following criterion [17, Theorem 5].

Lemma 20. *Let $p \geq 5$ and $l \equiv 3 \pmod 4$ be prime numbers. Let E and E' be elliptic curves over $\mathbb{Q}_l$ with potentially good reduction and $e = 4$. Set*

$$r = \begin{cases} 0 & \text{if } v_l(\Delta) \equiv v_l(\Delta') \pmod 4, \\ 1 & \text{otherwise,} \end{cases} \qquad t = \begin{cases} 1 & \text{if } \left(\frac{\tilde{\Delta}}{l}\right)\left(\frac{\tilde{\Delta}'}{l}\right) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $E[p]$ and $E'[p]$ are isomorphic $\mathrm{G}_{\mathbb{Q}_l}$-modules. Then

$$E[p] \text{ and } E'[p] \text{ are symplectically isomorphic} \iff \left(\frac{l}{p}\right)^r \left(\frac{2}{p}\right)^t = 1.$$

Proposition 21. *The Diophantine equation $3^{2p-3}X^{2p} - 4Y^p = 7Z^2$ has no solution in coprime odd integers for any prime $p \equiv 47, 65, 113, 139, 143$ or $167 \pmod{168}$.*

Proof. Let $p > 11$ be a prime and let E be the Frey curve given by the equation (12). There exists an elliptic curve F in the isogeny class with Cremona label $588C$, $588E$, $1176G$ or $1176H$, such that the p-torsion modules $E[p]$ and $F[p]$ are $\mathrm{G}_\mathbb{Q}$-isomorphic (see Lemma 11). Since $\overline{\rho}_{E,p}$ is irreducible, this isomorphism is either symplectic or anti-symplectic.

Both curves E and F have potentially good reduction at 7 and their semistability defects are equal 4, so we may apply Lemma 20. If F is isomorphic to $588C1$ or $1176G1$, we obtain that $E[p]$ and $F[p]$ are symplectically isomorphic. We assume that the same is true for other choices of F. This happens if and only if $\left(\frac{7}{p}\right) = \left(\frac{2}{p}\right) = 1$.

Now we use [22, Proposition 2] with $l = 3$, which says that $E[p]$ and $F[p]$ are symplectically isomorphic if and only if

$$\left(\frac{v_3(\Delta_E)v_3(\Delta_F)}{p}\right) = 1.$$

To obtain a contradiction we assume that the last equality doesn't hold for any curve F in the considered isogeny classes, i.e. -3 and -6 aren't squares modulo p. Summarizing, we have

$$\left(\frac{7}{p}\right) = \left(\frac{2}{p}\right) = 1 \quad \text{and} \quad \left(\frac{-3}{p}\right) = -1.$$

This is equivalent to the congruence condition stated in the Proposition. $\square$

9.2 Application of quadratic reciprocity

For a given field K, let $(,)_K : K^\times \times K^\times \to \{\pm 1\}$ be the Hilbert symbol defined by

$$(A, B)_K = \begin{cases} 1 & \text{if } z^2 = Ax^2 + By^2 \text{ has a nonzero solution in } K, \\ -1 & \text{otherwise.} \end{cases}$$

Note that the Hilbert symbol is symmetric and multiplicative. We will let $(,)_q$, $(,)$ and $(,)_\infty$ to denote $(,)_{\mathbb{Q}_q}$, $(,)_{\mathbb{Q}}$ and $(,)_{\mathbb{R}}$, respectively. Let $A = q^\alpha u$, $B = q^\beta v$, with u, v q-adic units. If q is an odd prime, then

$$(A, B)_q = (-1)^{\alpha\beta\frac{q-1}{2}} \left(\frac{u}{q}\right)^\beta \left(\frac{v}{q}\right)^\alpha,$$

and

$$(A, B)_2 = (-1)^{\frac{u-1}{2}\frac{v-1}{2} + \alpha\frac{v^2-1}{8} + \beta\frac{u^2-1}{8}}.$$

For all nonzero rationals a and b, we have

$$\prod_{q \leq \infty} (a, b)_q = 1 \tag{14}$$

(Quadratic Reciprocity in terms of the Hilbert symbol). We will use the following result [1, Proposition 15].

Lemma 22. *Let r and s be nonzero rational numbers. Assume that the Diophantine equation*

$$A^2 - rB^{2p} = s(C^p - B^{2p})$$

has a solution in coprime nonzero integers A, B and C, with BC odd. Then

$$(r, s(C - B^2))_2 \prod_{2 < q < \infty} (r, s(C - B^2))_q = 1,$$

where the product is over all odd primes q such that $v_q(r)$ or $v_q(s)$ is odd.

Combining modular method and Lemma 22 we prove the following result.

Proposition 23. *The Diophantine equation* $3^{2p-3}X^{2p} - 4Y^p = 7Z^2$ *has no solution in coprime odd integers for any prime* p *satisfying* $p \equiv 3$ *or* 55 (mod 106).

Proof. We rewrite the equation as follows

$$7A^2 - \frac{1}{27}B^{2p} = -4C^p. \tag{15}$$

By adding $4B^{2p}$ to both sides of (15) and dividing by 7 we obtain

$$A^2 + \frac{107}{189}B^{2p} = -\frac{4}{7}\left(C^p - B^{2p}\right).$$

From Lemma 22 and the definition of the Hilbert symbol we have

$$\prod_{q \in \{2,3,7,107\}} \left(-3 \cdot 7 \cdot 107, -7(C - B^2)\right)_q = 1.$$

The terms with $q \in \{2, 3, 7\}$ are equal 1 (for $q = 7$ observe that (15) with $p \equiv \pm 1$ (mod 6) imply that $\frac{C}{B^2} - 1$ is a square modulo 7). Hence either $107 \mid C - B^2$ or $(-7(C - B^2)/107) = 1$. Using modular method we will derive a contradiction for $p \equiv 3$ or 55 (mod 106).

The Frey curve E associated to (A, B, C) is given by the equation

$$y^2 = x^3 + 7Ax^2 - 7C^p x$$

(this is the equation (12) with $(a, b, c) = (B/3, C, A)$). We know that $\overline{\rho}_{E,p} \cong \overline{\rho}_{F,p}$ for some elliptic curve F in the isogeny class with Cremona label $588C$, $588E$, $1176G$ or $1176H$ (see Lemma 11). We have $a_{107}(F) = -14$ or 8. If $107 \mid BC$ (i.e. $107 \parallel N_E$) then $a_{107}(F) \equiv \pm 108$ (mod p). So in this case $p \in \{29, 47, 61\}$. But from Corollary 15 it follows that for such p the equation (15) has no solution in coprime odd integers. Hence $107 \nmid BC$ and $a_{107}(E) \equiv a_{107}(F)$ (mod p). Since $|a_{107}(E) - a_{107}(F)| < 2\sqrt{107} + 14 < 35$, then for $p > 35$ we have $a_{107}(E) = -14$ or 8.

If $107 \mid C - B^2$, then $107 \mid A$ and we obtain a contradiction since $a_{107}(E) = 0$. In case $107 \nmid C - B^2$ the assumption $a_{107}(E) = -14$ or 8 implies that

$$\frac{\overline{C^p}}{\overline{B^{2p}}} \in S \subset \mathbb{F}_{107}, \quad \text{where } S = \{11, 26, 34, 53, 70, 87, 90, 101\}.$$

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Here $\overline{C^p}$ and $\overline{B^{2p}}$ are reductions modulo 107 of C^p and B^{2p}. Next we check which exponents p modulo 106 have the property that p is coprime to 106 and for each $\zeta \in \mathbb{F}_{107}$ such that $\zeta^p \in S$ the element $-7(\zeta-1)$ isn't a square in $\mathbb{F}_{107}$ (this contradicts the condition $(-7(C-B^2)/107) = 1$). Such exponents p satisfy $p \equiv 3$ or $55 \pmod{106}$.

To complete the proof we use Theorem 6 (the case $p = 3$). $\qquad\square$

Remark. If we add $-4B^{2p}$ instead of $4B^{2p}$ to both sides of (15), then by replacing C with $-C$, we also may apply Lemma 22. But combining this result with modular method as in the proof above gives no additional information on p.

Appendix A. Proof of Theorem 5

Proof. By Lemma 3(a), we have reduced the problem to solving the equation $4y^2 = u(u^2 - 21v^2)$ with odd u, v and y. Since $\gcd(u,v) = 1$, we have $d = \gcd(u, u^2 - 21v^2)|3$. In this case, problem of solving the equation $7x^2 + y^4 = 4z^3$ is reduced to solving the following equations

$$dX^4 - (21/d)Y^2 = CZ^2, \tag{16}$$

where $d = 1$ or 3, $C = \pm 1$, and X, Y are odd, with $\gcd(X,Y) = 1$. Below we will analyse all these cases in some detail.

(*i*) If $(d,C) = (1,-1)$ or $(3,1)$, then (16) has no solutions. In these cases, we obtain a contradiction by reducing the equations modulo 3.

(*ii*) Now we are in the case $(d,C) = (1,1)$. Consider the Diophantine equation

$$Z^2 + 21Y^2 = X^4. \tag{17}$$

So, we have $\gcd(X^2 - Z, X^2 + Z) = 1$ or 2. The first case gives

$$X^2 = \frac{s^4 + 21t^4}{2}, \ Z = \frac{s^4 - 21t^4}{2}, \ Y = st, \tag{18}$$

or

$$X^2 = \frac{3s^4 + 7t^4}{2}, \ Z = \frac{3s^4 - 7t^4}{2}, \ Y = st, \tag{19}$$

with $(s,t) = 1$, respectively. Reducing the first equations of (18) and (19) modulo 3 gives contradiction. If $\gcd(X^2 - Z, X^2 + Z) = 2$, then $2|Y$, a contradiction.

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(iii) Finally consider the case $(d, C) = (3, -1)$ for equation (16). So, we have

$$-Z^2 + 7Y^2 = 3X^4. \tag{20}$$

Put $K = X^2$. Then we obtain

$$Z^2 + 3K^2 = 7Y^2. \tag{21}$$

Set $2Z \pm 3K = 7L$ and $Z \mp 2K = 7M$ with $\gcd(L, M) = 1$. So, equation (21) becomes

$$L^2 + 3M^2 = Y^2 \tag{22}$$

and one gets

$$(K, Z) = (\pm(L - 2M), \pm(2L + 3M)). \tag{23}$$

By (22), we have $\gcd(Y + L, Y - L) = 1$ or 2. Now, L is odd (otherwise $K = X^2$ is even, a contradiction), hence necessarily $\gcd(Y + L, Y - L) = 2$. In this case we have

$$\begin{cases} Y \pm L = 2^{2a-1}\alpha^2 \\ Y \mp L = 2 \cdot 3\beta^2 \end{cases} \quad \text{or} \quad \begin{cases} Y \pm L = 3 \cdot 2^{2a-1}\alpha^2 \\ Y \mp L = 2\beta^2 \end{cases} \tag{24}$$

with odd α, β satisfying $\gcd(\alpha, \beta) = 1$ and $a \geq 1$. It follows that

$$Y = 2^{2a-2}\alpha^2 + 3\beta^2, \ L = 2^{2a-2}\alpha^2 - 3\beta^2, \ M = 2^a\alpha\beta \tag{25}$$

or

$$Y = 3 \cdot 2^{2a-2}\alpha^2 + \beta^2, \ L = 3 \cdot 2^{2a-2}\alpha^2 - \beta^2, \ M = 2^a\alpha\beta, \tag{26}$$

respectively. By (21), (23), (25) and (26), all solutions of (20) are given by

$$\begin{aligned} X^2 &= \pm(2^{2a-2}\alpha^2 - 3\beta^2 - 2^{a+1}\alpha\beta), \\ Y &= 2^{2a-2}\alpha^2 + 3\beta^2 \\ Z &= \pm(2^{2a-1}\alpha^2 - 6\beta^2 + 3 \cdot 2^a\alpha\beta) \end{aligned} \tag{27}$$

or

$$\begin{aligned} X^2 &= 3 \cdot 2^{2a-2}\alpha^2 - \beta^2 - 2^{a+1}\alpha\beta, \\ Y &= 3 \cdot 2^{2a-2}\alpha^2 + \beta^2, \\ Z &= \pm(3 \cdot 2^{2a-1}\alpha^2 - 2\beta^2 + 3 \cdot 2^a\alpha\beta). \end{aligned} \tag{28}$$

If $a = 1$ or $a \geq 3$, then we get a contradiction for both (27) and (28).

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If $a = 2$, then (28) gives $X^2 = 12\alpha^2 - \beta^2 - 8\alpha\beta$. Since α, β and X are odd, we obtain $LHS \equiv 1 \pmod{8}$ and $RHS \equiv 3 \pmod{8}$, a contradiction. The remaining case is to consider (27) with $a = 2$ which corresponds the equations $X^2 + 4(\alpha - \beta)^2 = 7\beta^2$ and $X^2 + 7\beta^2 = 4(\alpha - \beta)^2$. The first one has no solutions: taking reduction modulo 8 we obtain a contradiction. The second one has infinitely many solutions, giving two 2-parameter families of solutions of (20)

$$
\begin{aligned}
X &= \pm(21s^2 - 14ts - 3t^2), \\
Y &= 1911s^4 + 1260ts^3 + 378t^2s^2 + 12t^3s + 7t^4, \\
Z &= \pm(4998s^4 + 3528ts^3 + 756t^2s^2 + 168t^3s - 10t^4)
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
X &= \pm(21s^2 - 14ts - 3t^2), \\
Y &= 343s^4 - 84ts^3 + 378t^2s^2 - 180t^3s + 39t^4, \\
Z &= \pm(-490s^4 - 1176ts^3 + 756t^2s^2 - 504t^3s + 102t^4).
\end{aligned}
\tag{30}
$$

Now the families (8) and (9) follow immediately. $\qquad\square$

Appendix B. Proof of Lemma 11

Let $p \geq 7$ be a prime. Suppose that (a, b, c) is a solution in coprime odd integers to the equation (11). Let $E = E(a, b, c)$ be the associated Frey type curve. The Galois representation $\overline{\rho}_{E,p}$ arises from a cuspidal newform of level $N = 588$ or 1176, weight 2 and trivial nebentypus character.

There are 6 Galois conjugacy-classes of newforms in $S_2(588)$, and 15 in $S_2(1176)$. Let $f_i \in S_2(588)$ ($i = 1, ..., 6$) and $g_j \in S_2(1176)$ ($j = 1, ..., 15$) denote the first newform in the i-th respectively j-th class. The numbering coincides with those in Magma.

We start elimination of the newforms by applying [3, Prop. 4.3] (and its improvement resulting from [22, Prop. 3]). We do this with Magma using a function whose code is attached below. The function returns true if a given newform can be eliminated on base of this result (for all but finitely many primes p) and false otherwise. In the first case it also returns a finite set of primes $p > 7$ for which the elimination is not possible.

```
IsEliminable := function(newform)
  N := Level(newform);
```

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```
  mu:= N;
  for k in [1..#PrimeDivisors(N)] do
    mu *:= 1+1/PrimeDivisors(N)[k];
  end for;
  NormsDivisors := [ ];
  for l in [x: x in [1..Floor(mu/6)]  |  IsPrime(x) and
                          GCD(N, x) eq 1 ] do
      cl := Coefficient(newform, l);
      normProduct := Norm(cl-l-1) * Norm(cl+l+1);
      if Degree(newform) ne 1 then
        normProduct *:= l;
      end if;
      for r in [-Floor(Sqrt(l))..Floor(Sqrt(l))] do
        normProduct *:= Norm(cl - 2*r);
      end for;
      if normProduct ne 0 then
        Pl := PrimeDivisors(Integers() ! normProduct);
        Append(~NormsDivisors, Pl);
      end if;
  end for;
  if not IsEmpty(NormsDivisors) then
    Pf := Set(NormsDivisors[1]) diff {2, 3, 5};
    for k in [2..#NormsDivisors] do
      Pf meet:= Set(NormsDivisors[k]);
    end for;
    return true, Pf;
  else
    return false, _;
    end if;
end function;
```

Remark. The function `IsEliminable` eliminates each newform that either has non-rational coefficients or corresponds to an isogeny class of elliptic curves over $\mathbb{Q}$ with trivial 2-torsion. So the code may be applied to other Diophantine problems if an attached Frey type curve is defined over $\mathbb{Q}$, has at least one rational point of order 2 and the corresponding level N is small enough (see [5]).

Using the above code we eliminate the following newforms:

$$f_1, f_4 \in S_2(588) \quad \text{for } p \geq 11,$$
$$g_2, g_4, g_{10}, g_{11}, g_{12}, g_{13}, g_{14}, g_{15} \in S_2(1176) \quad \text{for } p \geq 11,$$
$$g_1, g_6 \in S_2(1176) \quad \text{for } p \geq 13.$$

The newforms $f_2, f_6 \in S_2(588)$ and $g_3, g_5 \in S_2(1176)$ correspond to the isogeny classes of elliptic curves with Cremona label $588B$, $588F$ and $1176C$, $1176E$ respectively. All curves in these classes have j-invariant, whose denominator is divisible by 7. If F is such a curve, then from [3, Prop. 4.4] it follows that $\overline{\rho}_{E,p} \not\cong \overline{\rho}_{F,p}$ for $p \geq 11$.

The newform $g_9 \in S_2(1176)$ corresponds to the isogeny class $1176I$. For each curve F in this class the semistability defect at 7 is equal 2, while the semistability defect of E at 7 equals 4 (see Subsection 9.1). Hence from [17, Proposition 13] we have $\overline{\rho}_{E,p} \not\cong \overline{\rho}_{F,p}$ for $p \geq 11$.

If $p \geq 13$, there are four newforms to eliminate left: $f_3, f_5 \in S_2(588)$ and $g_7, g_8 \in S_2(1176)$ or equivalently four isogeny classes of elliptic curves: $588C$, $588E$ and $1176G$, $1176H$. These curves are the main obstacle in modular approach to the equation (11). In case $p = 11$, there are two more newforms which we couldn't eliminate, namely $g_1, g_6 \in S_2(1176)$. These newforms correspond to the isogeny classes $1176A$ and $1176F$. This ends the proof of Lemma 11.

Appendix C. Proof of Corollary 15

To prove Corollary 15 we perform calculations in Magma using the following function.

```
AreCriterionConditionsMet := function(p,k,F)
  q := k*p+1;
  if not IsPrime(q) then
    return false;
  end if;
  frob := FrobeniusTraceDirect(F,q);
  if frob^2 mod p eq 4 then
    return false;
  end if;
```

```
    Fq := FiniteField(q);
    mu_k := AllRoots(One(Fq), k);
    for xi in mu_k do
      if IsSquare((1-108*xi)/189) then
        delta := AllRoots((1-108*xi)/189, 2)[1];
        E_xi := EllipticCurve([ Fq | 0, 7*delta, 0, -7*xi, 0]);
        frob_xi := q+1-#E_xi;
        if frob^2 mod p eq frob_xi^2 mod p then
          return false;
        end if;
      end if;
    end for;
    return true;
end function;
```

The function `AreCriterionConditionsMet` checks the conditions (1)–(3) of Theorem 14 for a given prime p, integer k, and an elliptic curve F. In order to prove Corollary 15 we had to consider the following curves:

$$588C1, 1176G1 \quad \text{for } 11 < p < 10^9,$$
$$588C1, 1176A1, 1176G1 \quad \text{for } p = 11.$$

For fixed p and F we looked for the least (even) $k \leq 500$ such that the above function returns true. In the table below we list the values of k obtained for $p < 60$.

p	11	13	17	19	23	29	31	37	41	43	47	53	59
k ($588C1$)	6	-	6	10	2	2	12	6	18	22	6	2	12
k ($1176G1$)	2	12	-	22	2	2	22	16	2	4	6	2	14

Moreover, for $F = 1176A1$ and $p = 11$ we obtained $k = 2$.

The largest value of k found was 372 for the prime $p = 458121431$ (for both curves). Most of the values obtained are small, e.g. in both cases: $F = 588C1$ and $F = 1176G1$ we have $k \leq 16$ for more than half of the primes p checked.

We found no such k in the following cases:

$$(F, p) \in \{(588C1, 13), (1176G1, 17)\}.$$

Hence Corollary 15 follows.

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The computations took about 270 hours for each of the curves: $588C1$ and $1176G1$. For the calculations we used two desktop computers, each containing an Intel Pentium G4400 (3.3GHz) CPU and 8GB of RAM.

Appendix D. Proof of Corollary 17

Corollary 17 follows from the following combination of Theorem 14 and Theorem 16.

Let $p > 11$ be a prime. Suppose that for each $F \in \{588C1, 1176G1\}$ there exists an integer k satisfying the conditions (1) – (3) of Theorem 14 or the conditions (1) – (5) of Theorem 16. Then the equation (11) has no solution in coprime odd integers.

For the computation based on Theorem 16 we use the following function.

```
AreRefinedConditionsMet := function(p, k, F)
  q := k*p+1;
  if not IsPrime(q) then
    return false;
  end if;
  O<w> := IntegerRing(QuadraticField(21));
  if Norm((2+w)^k-1) mod q ne  0 then
    return false;
  end if;
  if #Decomposition(O,q) ne 2 then
    return false;
  end if;
  frob := FrobeniusTraceDirect(F,q);
  if frob^2 mod p eq 4 then
    return false;
  end if;
  I := Decomposition(O,q)[1][1];
  Fq, h := ResidueClassField(O, I);
  mu_k := AllRoots(One(Fq), k);
  S1 := { d: d in Fq | 1/h(2+2*w)*(1/3+h(2*w-1)*d) in mu_k }
    meet { d: d in Fq | 1/h(4-2*w)*(1/3-h(2*w-1)*d) in mu_k };
  S2 := { d: d in Fq | 1/h(4-2*w)*(1/3+h(2*w-1)*d) in mu_k }
    meet { d: d in Fq | 1/h(2+2*w)*(1/3-h(2*w-1)*d) in mu_k };
```

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```
  for delta in S1 join S2 do
    xi := 1/(4*27)-7/4*delta^2;
    E_delta := EllipticCurve([ Fq | 0, 7*delta, 0, -7*xi, 0]);
    frob_delta := q+1-#E_delta;
    if frob^2 mod p eq frob_delta^2 mod p then
      return false;
    end if;
  end for;
  return true;
end function;
```

For a given prime p, an integer k and an elliptic curve F the function `AreRefinedConditionsMet` checks if the conditions (1) – (5) of Theorem 16 are satisfied. The function returns true for $p = 17$, $F = 1176G1$ and $k = 374$.

Recall from Appendix C that if $p = 17$ and $F = 588C1$ then $k = 6$ satisfies conditions (1) – (3) of Theorem 14. This completes the proof of Corollary 17.

If $p = 13$ and $F = 588C1$, the above function returns false for all $k \leq 10^5$. The computation took about 51 hours (Intel Pentium CPU G4400, 8GB RAM).

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