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Permutation Tests for Reflected Symmetry

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The paper presents a permutation procedure for testing reflected (or diagonal) symmetry of the distribution of a multivariate variable. The test statistics are based in empirical characteristic functions. The resulting permutation tests are strictly distribution free under the null hypothesis that the underlying variables are symmetrically distributed about a center. Furthermore, the permutation tests are strictly valid if the symmetric center is known and are asymptotic valid if the center is an unknown point. The equivalence, in the large sample sense, between the tests and their permutation counterparts are established. The power behavior of the tests and their permutation counterparts under local alternative are investigated. Some simulations with small sample sizes ($\leqslant 20$) are conducted to demonstrate how the permutation tests works. © 1998 Academic Press

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1. INTRODUCTION

Testing for symmetry of a random variable has received considerable attention in the literature. In univariate cases, many statistics have been proposed. For example, Butler (1969), Rothman and Woodroofe (1972), Doksum, Fenstad, and Aaberge (1977), Antille, Kersting and Zucchini (1982), Shorack and Wellner (1986, Section 22), Aki (1987), Csörgö and Heathcote (1987), and Schuster and Barker (1987). There are two different but related issues in multivariate cases. One is to test for spherical (elliptical) symmetry. For example, Kariya and Eaton (1977), Beran (1979), Blough (1989), Baringhaus (1991), Baringhaus and Henze (1991), Fang, Zhu and Bentler (1993), and Zhu, Fang, and Zhang (1994). Another one is to test for reflected symmetry (or diagonal symmetry), which will be involved in the present paper, such as Aki (1993). Ghosh and Ruymgaart (1992) extended the statistic proposed by Feuerverger and Mureika (1972)

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to the multivariate case, which rests upon an integrated empirical characteristic function. Heathcote, Rachev, and Cheng (1995) used, among others, a maximized empirical characteristic function to investigate this problem.

Since little is known about the sampling and limiting null distributions of the test statistics (Ghosh and Ruymgaart, 1992, p. 439; Heathcote, Rachev, and Cheng, 1995, p. 99), some approximation procedures including Bootstrap were suggested for practical use of the tests. It is not clear, however, whether the approximations have good performance.

The purpose of the present paper is to develop a permutation procedure for testing the reflected symmetry of multivariate random variables. We will show that the permutation tests are, respectively, strictly (asymptotically) valid number reflected symmetry about a known (unknown) center. It will turn out that the tests and their associated permutation ones will be asymptotically equivalent. The permutation tests are conditionally distribution-free under the null hypothesis.

Section 2 will contain a review of the tests proposed by Ghosh and Ruymgaart (1992) and Heathcote, Rachev, and Cheng (1995). The permutation tests will be defined in Section 3 and the validity and asymptotic validity of the tests will be presented in the same section. A power study under local alternatives will be made in Section 4. Section 5 will contain some simulation experiments. Section 6 will present some concluding remarks. All proofs are postponed to the Appendix.

2. REVIEW OF TESTS

As mentioned in the previous section, a d-variate random variable x is said to be reflectedly symmetric about a center μ if

$$(x-\mu) \text{ and } -(x-\mu) \text{ have the same distribution,} \tag{2.1}$$

or equivalently, if the imaginary part of the characteristic function of $x-\mu$ equals zero, i.e.,

$$E\{\sin(t'(x-\mu))\}=0 \qquad \text{for} \quad t \in R^d, \tag{2.2}$$

where t' stands for the transpose of t. Let $x_1, ..., x_n$ be i.i.d. copies of x and $P_n(\cdot)$ the corresponding empirical measure. Based on (2.2), Ghosh and Ruymgaart (1992) and Heathcote, Rachev, and Cheng (1995) constructed the tests, respectively,

$$n \int_{B_r} \{P_n(\sin(t'(x-\mu)))\}^2 \, dw(t) \tag{2.3}$$

and

$$\sup_A |\sqrt{n}\, P_n\{\sin(t'(x-\hat{\mu}))\}|, \tag{2.4}$$

where both B_r (a sphere with the radius r) and A (a general region) are working regions, $w(\cdot)$ is a distribution function on R^d, and $P_n(f(x))$ stands for $(1/n)\sum_{j=1}^{n} f(x_j)$. We now use other notations to represent them. Define an empirical process,

$$\{V_n(X_n, t) = \sqrt{n}\, P_n\{\sin(t'(x-\mu))\} : t \in A\}, \tag{2.5}$$

and an estimated empirical process,

$$\{V_{n1}(X_n, \hat{\mu}, t) = \sqrt{n}\, P_n\{\sin(t'(x-\hat{\mu}))\} : t \in A\}, \tag{2.6}$$

where $X_n = (x_1, ..., x_n)$, $\hat{\mu}$ is an estimate of μ when μ is unknown, and A is a working region as Heathcote, Rachev, and Cheng (1995) mentioned. The test statistics are defined as

$$Q_1(X_n) = \int_A (V_n(X_n, t))^2\, dw(t), \tag{2.7}$$

$$M_1(X_n) = \sup_A |V_n(X_n, t)|, \tag{2.8}$$

$$Q_2(X_n, \hat{\mu}) = \int_A [V_{n1}(X_n, \hat{\mu}, t)]^2\, dw(t), \tag{2.9}$$

and

$$M_2(X_n, \hat{\mu}) = \sup_A |V_{n1}(X_n, \hat{\mu}, t)|, \tag{2.10}$$

where Q_1 and M_2 are the ones in (2.3) and (2.4). When μ is known Ghosh and Ruymgaart (1992) derived some asymptotic properties of the test based on (2.3), while Heathcote, Rachev, and Cheng (1995) investigated the limiting behavior of the test related to (2.4) for the case where the center μ is unknown.

3. PERMUTATION TESTS

Let $a \cdot b$ mean that every component of the vector b is multiplied by a common univariate variable a. Here P_n is the empirical measure of (e_i, x_i), $i = 1, ..., n$, where $e_1, ..., e_n$ are i.i.d. univariate variables, $e_i = \pm 1$, $i = 1, ..., n$, with probability values one half; define $E_n = (e_1, ..., e_n)$. We use a generic notation where P_n stands for a probability measure which may rest upon different sets of variables for each appearance. For the known center case define an empirical permutation process, given X_n, by

$$\{V_n(E_n, X_n, t) = \sqrt{n}\, P_n\{\sin(t'e \cdot (x-\mu))\} : t \in A\}. \tag{3.1}$$

Comparing this process with that defined in (2.5), the versions are the same excepts that the inserted permutation variables appear in the permutation process. We will derive, as stated in Theorem 3.3, the (asymptotic) equivalence of the processes, that is, almost surely, both processes converge in distribution to the same limit process.

For the unknown center case, the situation is not so simple. In order to ensure the equivalence between the empirical permutation process, which will be defined below, and its unconditional counterpart in (2.6), both versions cannot be the same. The definition of our permutation process is motivated by the following fact which will be proved in the Appendix, Proof of Theorem 3.4: For an unknown center μ, an estimate $\hat{\mu}$ is needed replacing μ in (2.6); here $\hat{\mu} = \bar{x}$, the sample mean, is applied. It can be proved that, uniformly on $t \in A$,

$$\begin{aligned}
\sqrt{n}\, P_n(\sin(t'(x-\bar{x}))) &= \sqrt{n}\, P_n(\sin(t'(x-\mu)) \cos(t'P_n(x-\mu))) \\
&\quad - \sqrt{n}\, (P_n((t'(x-\mu)) \sin(t'P_n(x-\mu)))) \\
&= \sqrt{n}\, P_n(\sin(t'(x-\mu)) \\
&\quad - \sqrt{n}\, \sin(t'P_n(x-\mu)))(P_n((t'(x-\mu)))) + o_p(1).
\end{aligned}$$

Accordingly, we define an estimated empirical permutation process $\{V_{n1}(E_n, X_n, \bar{x}, t) \colon t \in A\}$ given X_n by

$$\begin{aligned}
V_{n1}(E_n, X_n, \bar{x}, t) &= \sqrt{n}\, P_n(\sin(t'e \cdot (x-\bar{x}))) \\
&\quad - \sqrt{n}\, \sin(t'P_n(e \cdot (x-\bar{x})))\, P_n(\cos(t'e \cdot (x-\bar{x}))).
\end{aligned} \tag{3.2}$$

The resulting permutation test statistics given X_n are, for a known center μ,

$$Q_1(E_n, X_n) = \int_A (V_n(E_n, X_n, t))^2\, dw(t) \tag{3.3}$$

and

$$M_1(E_n, X_n) = \sup_A |V_n(E_n, X_n, t)|. \tag{3.4}$$

The unknown center counterparts are defined in (3.10). When the working region A is a cube $[-a, a]^d$ and the weight function $w(\cdot)$ is the uniform distribution on this cube, $Q_1(E_n, X_n)$ will have a specific form which will be easy to compute. In fact,

$$Q_1(E_n, X_n) = \int_{[-a, a]^d} (V_n(E_n, X_n, t))^2 \, dw(t) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} e_i e_j I(i, j), \tag{3.5}$$

where

$$I(i, j) = \frac{1}{2} \left(\prod_{k=1}^{d} \frac{\sin(a(x_i - x_j)_k)}{a(x_i - x_j)_k} - \prod_{k=1}^{d} \frac{\sin(a(x_i + x_j - 2\mu)_k)}{a(x_i + x_j - 2\mu)_k} \right),$$

and $(x)_k$ means the kth component of x. We will prove this formula in the Appendix. The following states the validity of the tests in (3.3) and (3.4).

THEOREM 3.1. *Assume that $x_1, ..., x_n$ are i.i.d. d-variate variables which are reflectedly symmetric about a known center. Let $E_n^{(1)}, ..., E_n^{(m)}$ be independent copies of E_n. Then for any $0 < \alpha < 1$ and $T = Q_1$ or M_1*

$$P_{n,m}^{(i)}(\alpha) = P\{T(X_n) > m - [m\alpha] \text{ of } T(E_n^{(j)}, X_n)'s\} \leqslant \frac{[m\alpha] + 1}{m + 1}, \tag{3.6}$$

where $[z]$ stands for the largest integer part of z.

Remark 3.2. Inequality (3.6) is strict only if T fails to resolve certain X_n, which can happen because of discreteness of X_n or because of $E_n^{(j)}$ (e.g., $T(X_n) = T(E_n^{(k)}, X_n)$, if $E_n^{(k)} = (1, ..., 1)$. But if m is reasonably large, and x satisfies some regularity conditions like continuity, (3.6) will be close. In fact under some conditions on the distribution of x,

$$\lim_{n \to \infty} \lim_{m \to \infty} P_{n,m}^{(i)}(\alpha) = \alpha \tag{3.7}$$

for any $0 < \alpha < 1$, which is a consequence of Theorem 3.3 below.

THEOREM 3.3. *Assume, in addition to the conditions of Theorem 3.1, that the distribution of x is continuous with third absolute moment. Then the empirical permutation process $\{V_n(E_n, X_n, t): t \in A\}$ given X_n in (3.1) converges weakly to a Gaussian process $\{V(t): t \in A\}$ for almost all sequences $\{x_1, ..., x_n, ...\}$, which is likewise the limit of the empirical process*

$\{V_n(X_n, t)\colon t \in A\}$ in (2.5). Then $Q_1(E_n, X_n)$ and $M_1(E_n, X_n)$ given X_n (in (3.3) and (3.4)) and $Q_1(X_n)$ and $M_1(X_n)$ (in (2.7) and (2.8)) have almost surely the same limit, say $Q_1 = \int (V(t))^2 \, dw(t)$ and $M_1 = \sup_A |V(t)|$.

Now, the convergence of the associated quantiles can be established.

Denote by $\lambda_n(\alpha)$, $\lambda_n(\alpha, X_n)$, and $\lambda(\alpha)$ the $1-\alpha$ quantiles of the distributions of $Q_1(X_n)$, $Q_1(E_n, X_n)$, given X_n and Q_1, respectively.

COROLLARY 3.4. *Under the conditions in Theorem 3.3 for almost all sequences $\{x_1, ..., x_n, ...\}$,*

$$\lambda_n(\alpha, X_n) \to \lambda(\alpha) \qquad \text{in Probab.} \tag{3.8}$$

$$\lambda_n(\alpha) \to \lambda(\alpha) \qquad \text{in Probab.} \tag{3.9}$$

as $n \to \infty$. A similar result holds for $M_1(X_n)$ and $M_1(E_n, X_n)$ given X_n.

For the symmetry testing about an unknown center, define permutation test statistics as

$$Q_2(E_n, X_n, \bar{x}) = \int_A (V_{n1}(E_n, X_n, \bar{x}, t))^2 \, dw(t), \tag{3.10}$$

$$M_2(E_n, X_n, \bar{x}) = \sup_{t \in A} |V_{n1}(X_n, \bar{x}, t)|, \tag{3.11}$$

where $V_{n1}(\cdot)$ is defined in (3.2). The following theorem states the asymptotic validity of the permutation test based on Q_2 and M_2.

THEOREM 3.5. *Assume that $x_1, ..., x_n, ...$ are i.i.d. univariate variables which are reflectedly symmetric about an unknown center μ. Let $E_n^{(1)}, ..., E_n^{(m)}, ...$ be independent copies of E_n. Then for any $0 < \alpha < 1$,*

$$\begin{aligned} &\lim_{n\to\infty} P\{Q_2(X_n, \bar{x}) > m - [m\alpha] \text{ of } Q_2(E_n^{(j)}, X_n, \bar{x})'\,s\} \\ &\quad = \lim_{n\to\infty} P\{Q_2(E_n^0, X_n, \mu) + O_p(1/\sqrt{n}) > \\ &\qquad m - [m\alpha] \text{ of } (Q_2(E_n^{(j)}, X_n, \mu) + O_p(1/\sqrt{n}))'\,s\} \\ &\quad \leqslant \frac{[m\alpha]+1}{m+1}, \end{aligned} \tag{3.12}$$

and similarly for M_2, where $Q_2(X_n, \bar{x})$ and $M_2(X_n, \bar{x})$ are defined in (2.9) and (2.10). In addition, assume that the x's have a common distribution with third absolute moment. Then the assertions of Theorem 3.3 and of Corollary 3.4 *continue to hold for processes in* (2.6), (3.2), *and a certain Gaussian process* $\{V_1(t) = t \in A\}$.

Remark 3.6. For performing the above tests, one has to choose a working region A and a weight function $w(\cdot)$ (for Q_1 and M_1). This issue was discussed by Heathcote, Rachev, and Cheng (1995). In our simulations in Section 5 below $A = [-1, 1]^d$ and $w(\cdot)$ is uniform distribution on A. The Gaussian processes $\{V(t): t \in A\}$ in Theorem 3.3 and $\{V_1(t): t \in A\}$ in Theorem 3.5 are just the ones in Ghosh and Ruymgaart (1992) and Heathcote, Rachev, and Cheng (1995).

4. POWER STUDY

Heathcote, Rachev, and Cheng (1995, Theorem 3.2) show that the test defined in (2.4) is consistent against any fixed alternative. We here investigate the behavior of the tests and the permutation tests for local alternatives. For convenience, let $\sin^{(i)}(t'x)$ be the ith derivative of $\sin(\cdot)$ at the point $t'x$.

Suppose that i.i.d. d-variate variables have the representation $x_i + y_i/n^\alpha$, $i = 1, ..., n$, for some $\alpha > 0$. This means that the distribution of x is the convolution of a symmetric distribution and a distribution converging to the degenerate one. The following theorem reveals the power behavior of the tests for such local alternatives.

THEOREM 4.1. *Assume that the following conditions hold*:

(1) *Both distributions of x and of y are continuous and, in addition, x is reflectedly symmetric about a known center* μ.

(2) *Let l denote the smallest integer, such that*

$$\sup_{t \in A} |B_l(t)| := \sup_{t \in A} |E((t'(y - Ey)^l \sin^{(l)}(t'(x - Ex))))| \neq 0,$$

$$E(\|y\|^{2l}) < \infty, \qquad \textit{and} \quad E(\|y\|^{2(l-1)} \|x\|^2) < \infty. \tag{4.1}$$

Then

$$\begin{aligned}\{\sqrt{n}\, P_n\{\sin(t'(x + y/n^{1/(2l)} - Ex - Ey/n^{1/(2l)}))\} : t \in A\} \\ = \{\sqrt{n}\, P_n\{\sin(t'(x - Ex)) + (1/l!)\, B_l(t)\} : t \in A\} + o_p(1).\end{aligned} \tag{4.2}$$

This leads to convergence in distribution ($\Rightarrow$)

$$\int_A \{\sqrt{n}\, P_n(\sin(t'(x+y/n^{1/(2l)}-Ex-Ey/n^{1/(2l)})))\}^2\, dw(t)$$

$$\Rightarrow \int_A (V(t)+(1/l!)\, B_l(t))^2\, dx(t), \tag{4.3}$$

$$\sup_{t\in A} |\sqrt{n}\, P_n(\sin(t'(x+y/n^{1/(2l)}-Ex-Ey/n^{1/(2l)})))|$$

$$\Rightarrow \sup_{t\in A} |V(t)+(1/l!)\, B_l(t)|, \tag{4.4}$$

where $V(t): t\in A\}$ *is a Gaussian process defined in Theorem* 3.3.

Remark 4.2. This conclusion means that the tests can detect local alternatives converging to the null hypothesis at $n^{1/(2l)}$-rate or slower (the test statistics will converge in distribution to infinity under the local alternative with slower convergence rate). In some cases, this rate can reach a parametric rate, that is, $l=1$. For example, if x has a uniform distribution on $[-\sqrt{3}, \sqrt{3}]^d$ and if $y=(x_1^2-1, ..., x_d^2-1)$, we can see easily that, via a little elementary calculation, $\sup_{t\in[-1,1]^d} |E(t'y\cos(t'x))| \neq 0$. Hence, l will be one. On the other hand, when x and y are independent of each other, l is at least three, and the tests can detect, at most, alternatives converging to the null hypothesis at $n^{1/6}$-rate. In fact, it is clear that for $l=1, 2$

$$\sup_{t\in[-1,1]^d} |E((t'y)^l \sin^{(l)}(t'x))| = 0.$$

We also note that our tests are omnibus because of the absolute and square values in the test statistics, and that therefore the tests are asymptotically unbiased for all shapes of the functions B_l.

Remark 4.3. Let us discuss the meaning of condition (4.1). It should be easier to understand the implication of this condition in the case where x and y are independent. Clearly, any d-variate variable, w say, can be decomposed into two components x and y say, where x is symmetric. Assume further that the moment generating function of w exists. In the case where these two components are independent, w is symmetric if and only if y is symmetric which, in turn, is equivalent to the fact that all odd centered moments of $t'y$ equal zero for any t. In other words, the larger l is, the more symmetric the variable y is in a certain sense and then the harder the alternative is detected. In view of $\sup_{t\in A} |B_l(t)|$, we can see that $\sup_{t\in A} |B_l(t) \neq 0$ is equivalent to $\sup_{t\in A} |E(t'(y-E_y))^l| \neq 0$ under the independence of x and y and the symmetry of x. Consequently, $\sup_{t\in A} |B_l(t)|$ is an index measuring the symmetry extent of the variable. The case where both components are dependent is more complicated, but the implication is similar.

We now study the permutation procedures for symmetry about a known center μ. Of course, we hope that the tests are sensitive to alternatives. In contrast, it is hoped that the permutation procedure is not sensitive to the underlying distribution. The reason is as follows. As described in Section 3, the permutation procedures are applied merely to determine the critical values. Therefore, it is important that the permutational distribution, serving for computing the critical value, is not affected by an asymmetry of the underlying distribution. The following theorem indicates that the critical values determined by the permutation tests, under local alternatives, equals in fact approximately the ones under the null hypothesis. Hence the critical values remain unaffected, in the sense of large sample, by the underlying distribution of the sample with small perturbation for symmetry.

THEOREM 4.4. *Assume* $E\,\|y\|^2<\infty$. *Then for any* $\theta>0$ *the empirical permutation processes* $\{\sqrt{n}\,P_n\{\sin(t'e\cdot(x+y/n^{\theta}-Ex-Ey/n^{\theta}))\}:\ t\in A\}$, *given the* (x_i, y_i)*'s, and* $\{\sqrt{n}\,P_n\{\sin(t'e\cdot(x-Ex))\}:\ t\in A\}$, *given the* x_i'*'s have, almost surely, the same limiting Gaussian process* $\{V(t)\colon t\in A\}$ *as the empirical process* $\{\sqrt{n}\,P_n\{\sin(t'(x-Ex))\}:t\in A\}$. *Hence the quadratic or maximum functionals of these processes have, almost surely, the same limiting distribution as the random variables* $\int_A (V(t))^2\,dw(t)$ *or* $\sup_{t\in A}|V(t)|$.

For the unknown center case, there also exists, similar to the known center case, a nonrandom shift function $t\to(1/l!)\,B_l(t)$ in the limiting process of $\{\sqrt{n}\,P_n\{\sin(t'(x+y/n^{1/(2l)}-\bar{x}-\bar{y}/n^{1/(2l)}))\}:\ t\in A\}$ under local alternatives. The following result describes this.

THEOREM 4.5. *Assume the same conditions as in Theorem* 4.1. *Then*

$$\begin{aligned}\{\sqrt{n}\,P_n\{&\sin(t'(x+y/n^{1/(2l)}-\bar{x}-\bar{y}/n^{1/(2l)}))\}:t\in A\}\\ &=\{\sqrt{n}\,P_n(\sin(t'(x-Ex)))-\sqrt{n}\sin(t'P_n(x-Ex)\,E(\cos(t'(Ex))))\\ &\quad+(1/l!)\,B_l(t)\colon t\in A\}+o_p(1)\end{aligned}\tag{4.5}$$

and consequently,

$$\begin{aligned}&\int_A\{\sqrt{n}\,P_n(\sin(t'(x+y/n^{1/(2l)}-\bar{x}-\bar{y}/n^{1/(2l)})))\}^2\,dw(t)\\ &\qquad\Rightarrow\int_A (V_1(t)+1/l!)\,B_l(t))^2\,dw(t)\end{aligned}\tag{4.6}$$

and

$$\sup_{t \in A} |\sqrt{n}\, P_n(\sin(t'(x + y/n^{1/(2l)} - \bar{x} - \bar{y}/n^{1/(2l)})))|$$

$$\Rightarrow \sup_{t \in A} |V_1(t) + (1/l!)\, B_l(t)|, \tag{4.7}$$

where $\{V_1(t) : t \in A\}$ *is a Gaussian process mentioned in Theorem* 3.5.

For the estimated empirical permutation process, there is a parallel conclusion to Theorem 4.4.

THEOREM 4.6. *Under the conditions of Theorem* 4.4, *for any* $\theta > 0$, *the estimated empirical permutation process, given* (x_i, y_i)*'s*,

$$\begin{aligned}\{\sqrt{n}\, P_n(\sin(t'e \cdot (x + y/n^{\theta} - \bar{x} - \bar{y}/n^{\theta})))\\ - \sqrt{n}\, P_n(\cos(t'e \cdot (x + y/n^{\theta} - \bar{x} - \bar{y}/n^{\theta})))\\ \times \sin(t' P_n(e \cdot (x + y/n^{\theta} - \bar{x} - \bar{y}/n^{\theta}))) : t \in A\}\end{aligned}$$

and the empirical permutation process, given x_i*'s*,

$$\begin{aligned}\{\sqrt{n}\, P_n(\sin(t'e \cdot (x - \bar{x})))\\ + \sqrt{n}\, P_n(\cos(t'e \cdot (x - \bar{x})))(\sin(t' P_n(e \cdot (x - \bar{x})))) : t \in A\},\end{aligned}$$

as well as the unconditional process

$$\{\sqrt{n}\, P_n(\sin(t'e \cdot (x - \bar{x}))) : t \in A\},$$

have, almost surely, a common limiting Gaussian process $\{V_1(t) : t \in A\}$ *mentioned in Theorem* 3.5. *Then the quadratic or maximum functionals of these processes converge weakly to* $\int_A (V_1(t))^2\, dw(t)$ *or* $\sup_{t \in A} |V_1(t)|$ (*almost surely for the permutation ones*).

5. SIMULATIONS

In order to demonstrate the performance of the permutation tests, some small-sample simulation experiments have been performed. In the simulation results reported in the tables below the sample sizes are $n = 10$ and $n = 20$, the dimensions of random variable are $x\, di = 2$, 4, and 6 and the following distributions of the variable have been taken:

$\mathcal{N}$—x has standard multivariate normal distribution $N(0, I_d)$;

$\mathcal{N}+\chi^2$—x has standard normal distribution $N(0, I_d)$ and $y=\{x_1^2-1, ..., x_d^2-1\}$; the resulting random variable is $w=x+y$;

$\mathcal{N}+\mathcal{N}_s$—$x$ has standard multivariate normal distribution $N(0, I_d)$ and the independent y has multivariate normal distribution $N(1/\sqrt{n}, I_d/n)$; the resulting random variable is $w=x+y$.

In order to get a critical value for fixed $\{(x_1, y_1), ..., (x_n, y_n)\}$, 2000 pseudo-random numbers $E_n=(e_1, ..., e_n)$ of size $n=10$ and $n=20$ are generated by the Monte Carlo method. The basic experiment was replicated 3000 times for each combination of sample size, dimension of random variable, and the underlying distribution of the variable. The nominal level is 0.05. The proportion of times that the values of the statistics exceeded the critical values are recorded as the estimated power of the tests. In order to judge the performance of the permutation tests, we compute the "accurate" power of the test using the Monte Carlo method. Here "accurate" power means that the null distribution of the variable is assumed to be the standard normal $N(0, I_d)$, a completely known distribution. So we can get by the Monte Carlo method the exact null distribution of the test statistics and the critical values (not approximated ones) as long as the number of replications is large enough. Based on these critical values, the obtained power of the tests is called "accurate" power here. The replication number of each single experiment is 3000. Let us explain why we consider the alternative $\mathcal{N}+\mathcal{N}_s$ in the known center case. Clearly $w-1/\sqrt{n}$ is symmetric, but we regard $N(1/\sqrt{n}, I_d/n)$ as an *unknown* small perturbation of the null distribution $N(0, I_d)$. Hence, we do not center the variable w in the simulations presented in the Tables I and II.

TABLE I

Estimated Power in the Known Center Case, $n=10$

		$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}+\chi^2$	$\mathcal{N}+\chi^2$	$\mathcal{N}+\mathcal{N}_s$	$\mathcal{N}+\mathcal{N}_s$
		Q_1	M_1	Q_1	M_1	Q_1	M_1
Accurate	$n=10$	0.0517	0.0487	0.2970	0.5130	0.1927	0.1803
Permutation	$di=2$	0.0410	0.0433	0.2897	0.0967	0.1867	0.1500
Accurate	$n=10$	0.0497	0.0500	0.3323	0.5147	0.2477	0.2010
Permutation	$di=4$	0.0623	0.0633	0.2700	0.0700	0.2333	0.1767
Accurate	$n=10$	0.0490	0.0470	0.1660	0.2643	0.2163	0.1793
Permutation	$di=6$	0.0400	0.0167	0.2023	0.0733	0.2467	0.1400

TABLE II

Estimated Power in the Known Center Case, $n = 20$

		$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}+\chi^2$	$\mathcal{N}+\chi^2$	$\mathcal{N}+\mathcal{N}_s$	$\mathcal{N}+\mathcal{N}_s$
		Q_1	M_1	Q_1	M_1	Q_1	M_1
Accurate	$n=20$	0.0490	0.0487	0.5847	0.7867	0.2083	0.1713
Permutation	$di=2$	0.05400	0.0300	0.5900	0.1877	0.1900	0.1543
Accurate	$n=20$	0.0500	0.0483	0.64376	0.7863	0.2517	0.2190
Permutation	$di=4$	0.0633	0.0500	0.6423	0.1933	0.2900	0.2000
Accurate	$n=20$	0.0503	0.0493	0.5497	0.6357	0.2990	0.2220
Permutation	$di=6$	0.0310	0.0600	0.5933	0.1133	0.2653	0.1933

Let us first look at Tables I and II. We see that, under the null hypothesis, the simulated level of the permutation tests Q_1 and M_1 is close to the nominal one in most cases, even if the sample size is quite small, such as $n = 10$. However, if the dimension is large and the sample size is too small as compared with the dimension, M_1 does not seem to be as good as we expected. This is the case when the dimension is 6 and the sample size is 10. With increasing sample size, the situation becomes better. Under the alternative considered here, Q_1 still has quite good performance. The results in Table II show that the power is very close to the "accurate" one. This means that the permutation test Q_1 behaves as a test being based on a known null distribution. Hence Q_1 is distribution-free, not only in theory but also in practice. On the other hand, the performance of M_1 is discouraging, although theoretically it is also a conditional distribution-free test. The estimated power under alternatives is considerably lower than the "accurate" one. Hence M_1 is presumably applicable in large sample cases, since, comparing Tables I and II the power of M_1 is increasing with increasing the sample size.

For the $\mathcal{N} + \mathcal{N}_s$ case, the tests cannot detect such an alternative. However, this is reasonable since the theory in Section 4 has told us that the tests hardly detect this kind of alternative, since the mean of y is $1/\sqrt{n}$, a too small shift. From Tables I and II, we can see that, even if the critical value is based on the sampling null distribution (the "accurate" one), $\mathcal{N} + \mathcal{N}_s$ is also hard to detect. This means that one may need to define a more efficient test for detecting such kinds of local alternatives. We will discuss this problem further in the next section.

In the cases where the symmetric center is unknown, the permutation test Q_2 can still hold the level, but M_2 cannot, especially in high-dimensional cases. Under alternatives Q_2 is still applicable especially in lower

TABLE III

Estimated Power in Unknown Center Case, $n = 10$

		$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N} + \chi^2$	$\mathcal{N} + \chi^2$
		Q_2	M_2	Q_2	M_2
Accurate	$n = 10$	0.0527	0.0527	0.7957	0.7593
Permutation	$di = 2$	0.0343	0.0367	0.4133	0.1177
Accurate	$n = 10$	0.0487	0.0503	0.9000	0.7430
Permutation	$di = 4$	0.0387	0.0167	0.2553	0.0143
Accurate	$n = 10$	0.0523	0.0477	0.9407	0.6100
Permutation	$di = 6$	0.0343	0.0033	0.877	0.0000

dimension cases. M_2 does not seem to be recommendable in small sample cases. However, comparing Tables III and IV, there is information supporting M_2. That is, with increasing the sample size, the power of M_2 is increasing. Hence, M_2 may be applicable in large sample cases.

Summarizing, the permutation tests have good performance in the case where the center of symmetry is known but are worse in the situation where it is unknown. This is easy to explain. As shown in the theorems and in Remark 3.6, the tests are asymptotically valid at $\sqrt{n}$-rate in the sense of Theorem 3.5. When the sample points are too sparse in the high-dimensional space, a $O(1/\sqrt{n})$ perturbation for the test statistic cannot be ignored in small sample cases. Hence, they cannot be expected to have a satisfying performance.

TABLE IV

Estimated Power in Unknown Center Case, $n = 20$

		$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N} + \chi^2$	$\mathcal{N} + \chi^2$
		Q_2	M_2	Q_2	M_2
Accurate	$n = 20$	0.0520	0.0513	0.9640	0.9550
Permutation	$di = 2$	0.0600	0.0567	0.9000	0.1600
Accurate	$n = 20$	0.0490	0.0503	0.9947	0.9523
Permutation	$di = 4$	0.0533	0.0267	0.7100	0.0467
Accurate	$n = 20$	0.0490	0.0487	0.9967	0.9193
Permutation	$di = 6$	0.0443	0.0167	0.3400	0.0133

6. CONCLUDING REMARKS

We have developed a permutation procedure for testing reflected symmetry of a multivariate variable, which is based on the empirical characteristic function. Under some regularity conditions on the distribution of the variable, we have investigated the validity of the permutation tests and the power behavior of the tests and their permutation counterparts under local alternatives.

Remark 6.1. In principle, other test statistics may be found for the above testing problems. When the symmetry center is assumed to be known, a permutation test would be based on its unconditional counterpart without any modification following our approach in Section 3 in (3.3) or (3.4). It is worthwhile to mention, that when the symmetry center has to be estimated, the permutation test will generally not have the same form as its unconditional counterpart as the one in (3.5) or (3.6) which is based on (3.2). The modification will guarantee an equivalence, in the large sample sense, between the test and the associated permutation one. On the other hand, if we want to have a test whose permutation counterpart is strictly valid for the unknown center case, it seems to us that such a test has to be location-invariant.

Remark 6.2. Although the choice of the working region and of the weight function have been under consideration in the present paper, it is necessary to explore how these choices affect the performance of the tests. On the other hand, in some cases, the choice of working regions is not very important. We now show an example in which the fact that the imaginary part of the characteristic function equals zero in a compact subset of R^d such as $[-1, 1]^d$ is equivalent to reflected symmetry of the variable. Suppose that the moment generating function of a multivariate variable x, say, exists in a cube $[-a, a]^d$, $a > 0$. Then the moment generating function of $\gamma' x$, the linear projector of x on R^1, exists in an interval $[-a_1, a_1]$ for any γ being on the unit sphere in R^d, where a_1 does not depend on γ. If the imaginary part of the characteristic function of x equals zero in a cube $[-a_2, a_2]^d$, so does the one of $\gamma' x$ in an interval $[-a_3, a_3]$. It is easy to see than then all moments of $\gamma' x$ with odd orders equal zero. This means that the characteristic function of $\gamma' x$ is real; hence, $\gamma' x$ is symmetric about the origin for any γ. This conclusion implies, in turn, that x is reflectedly symmetric. Consequently, the choice of the working region is not very important in such a case. For example, we could choose $\prod_{i=1}^{d} [-a_i, a_i]$ as a working region, where a_i is the variance of the ith component of x.

Remark 6.3. In the simulation of Section 5, the permutation test, for the known center case, is a distribution-free test. The power is close to the "accurate" one obtained by assuming the null distribution to be known. One may use the same principle, as mentioned in Remark 6.1, to define further tests. Hence, it remains to compare the different proposals.

Remark 6.4. For the unknown center case, the permutation tests defined in the present paper do not have a good performance in the small-sample-high-dimension case. One of the main reasons is that the test is only asymptotically valid. It would be interesting to construct a test being strict validity even in the unknown center case. We defer this to future research.

Remark 6.5. In this paper we use the sample mean as an estimate of the unknown center. If the underlying distribution is heavy-tailed, its use may be questionable due to its lack of robustness, and furthermore, according to the discussion in Heathcote, Rachev, and Cheng (1995) and Remark 6.4, it could not be expected that the tests have good performance in small sample cases we conducted. The sample size may be considerably larger for having good performance of the tests. We will study this question in connection with robustness considerations.

Remark 6.6. We name the conditional test procedure as the permutation test, although it is not exactly like the classical permutation procedure. Actually, it is a random symmetrization procedure. The same idea could be applied to some other setting in the statistical inference.

APPENDIX: PROOFS OF THEOREMS

We first prove the formulae in (3.5). Note that $\sin(x)\cdot\sin(y) = 1/2(\cos(x-y)-\cos(x+Y)$. Then

$$
\begin{aligned}
&Q_1(E_n, X_n)\\
&\quad = (2a)^{-d}\int_{[-a,a]^d}\left\{1/\sqrt{n}\sum_{i=1}^n \sin(t'e_i\cdot(x_i-\mu))\right\}^2 dt\\
&\quad = 1/n\sum_{i=1}^n\sum_{j=1}^n\left\{(2a)^{-d}\int_{[-a,a]^d}\sin(t'(x_i-\mu))\sin(t'(x_i-\mu))\,dt\right\}e_ie_j\\
&\quad = 1/n\sum_{i=1}^n\sum_{j=1}^n\left\{(2)^{-d-1}\int_{[-1,1]^d}\cos(t'a\cdot(x_i-x_j))\right.\\
&\qquad\quad \left.-\cos(t'a\cdot(x_i+x_j-2\mu))\,dt\right\}e_ie_j\\
&\quad := 1/n\sum_{i=1}^n\sum_{j=1}^n e_ie_jI(i,j).
\end{aligned}
$$

Since the uniform distribution on $[-1, 1]^d$ is symmetric, we have for u having uniform distribution on $[-1, 1]^d$,

$$
\begin{aligned}
I(i, j) &= 2^{-d-1}E(\cos(u'a \cdot (x_i - x_j)) - \cos(u'a \cdot (x_i + x_j - 2\mu))) \\
&= 2^{-d-1}(\mathrm{Re}\, E(e^{(u'a \cdot (x_i - x_j))}) - \mathrm{Re}\, E(e^{(u'a \cdot (x_i + x_j - 2\mu))})) \\
&= 2^{-d-1}\left(\mathrm{Re} \prod_{k=1}^{d} E(e^{(u_k a(x_i - x_j)_k)}) - \mathrm{Re} \prod_{k=1}^{d} E(e^{(u_k a(x_i + x_j - 2\mu)_k)})\right) \\
&= 1/2\left(\mathrm{Re} \prod_{k=1}^{d} \frac{\sin(a(x_i - x_j)_k)}{a(x_i - x_j)} - \prod_{k=1}^{d} \frac{\sin(a(x_i + x_j - 2\mu)_k)}{a(x_i + x_j - 2\mu)}\right).
\end{aligned}
$$

The proof is completed.

We now start to prove theorems. For convenience of the notations, let $a \circ b$ mean the vector with every component of a is multiplied by the corresponding component of b, and c denotes a generic constant which may change its meaning, even in the same formula.

Proof of Theorem 3.1. Recall the definition of $a \cdot b$ at the beginning of Section 3. First, we prove that x is reflectedly symmetric if and only if $x = e \cdot x^*$, where $e = \pm 1$ with probability one-half and is independent of x^* and, furthermore, x^* has the same distribution as x. Sufficiency is clear since

$$
\begin{aligned}
P\{e \cdot x^* \leqslant t\} &= \tfrac{1}{2}P\{x^* \leqslant t\} + \tfrac{1}{2}P\{-x^* \leqslant t\} \\
&= \tfrac{1}{2}P\{x \leqslant t\} + \tfrac{1}{2}P\{-x \leqslant t\} = P\{x \leqslant t\}.
\end{aligned}
$$

Similarly $P\{-e \cdot x^* \leqslant t\} = P\{-x \leqslant t\}$; hence

$$P\{x \leqslant t\} = P\{-x \leqslant t\}.$$

For necessity, let $x^* := e \cdot x$, where $e = \pm 1$ with probability one half and is independent of x. Then x and x^* have the same distribution, $x = e \cdot x^*$. Furthermore, x^* and e are independent since for any $s \in R^d$,

$$
\begin{aligned}
P\{e = \pm 1, e \cdot x \leqslant s\} &= \tfrac{1}{2}P\{e \cdot x^* \leqslant s \mid e = \pm 1\} \\
&= \tfrac{1}{2}P\{\pm x \leqslant s\} \\
&= P\{e = \pm 1\}\ P\{e \cdot x \leqslant s\}.
\end{aligned}
$$

Let $X_n^* = (x_1^*, ..., x_n^*)$. The set $\{Q_i(X_n) > m - [m\alpha]$ of $Q_i(E_n^{(j)}, X_n)'\,s\}$ equals exactly the set $\{Q_i(E_n, X_n^*) > m - [m\alpha]$ of $Q_i(E_n \circ E_n^{(j)}, X_n^*)$, $j = 1, ..., m\}$. It is easy to check that $\{E_n, E_n \circ E_n^{(j)}, j = 1, ..., m\}$ are i.i.d.

n-dimensional variables. Indeed, for any t in the set consisting of all n-dimensional variables of the form $(\pm 1, ..., \pm 1)$, say $\{t_1, ..., t_{2^n}\}$,

$$P\{E_n \circ E_n^{(1)} = t, E_n \circ E_n^{(2)} = s\}$$

$$= \frac{1}{2^n} \sum_{j=1}^{2^n} P\{E_n^{(1)} \circ t_j = t, E_n^{(2)} \circ t_j = s\}$$

$$= \frac{1}{2^n} \sum_{j=1}^{2^n} \frac{1}{2^n} \frac{1}{2^n}$$

$$= P\{E_n \circ E_n^{(1)} = t\} \; P\{E_n \circ E_n^{(2)} = s\}.$$

The independence between E_n and $E_n \circ E_n^{(j)}$ can be checked in the same way. Hence, given X_n^*, the $m+1$ variables $Q_i(E_n, X_n^*)$ and $Q_i(E_n \circ E_n^{(j)}, X_n^*)$, are i.i.d., which implies that

$$P\{Q_i(E_n, X_n^*) > m - [m\alpha] \text{ of } Q_i(E_n \circ E_n^{(j)}, X_n^*)' \, s \mid X_n^*\} \leqslant \frac{[m\alpha]+1}{m+1}.$$

The proof is conclude by integrating X_n^*.

Proof of Theorem 3.3. In the following, we first show that the process $\{V_n(E_n, X_n, t) \colon t \in A\}$, given X_n, converges almost surely to the process $\{V(t) \colon t \in A\}$ which is the limiting process of $\{V_n(X_n, t) \colon t \in A\}$. The conclusion of Theorem 3.3 will then hold.

Define sets $D_1 = \{\lim_{n \to \infty} 1/n \sum_{j=1}^n \|x_j - \mu\|^2 = E \, \|x - \mu\|^2\}$,

$$D_{t,s} = \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n \sin(t'(x_j - \mu)) \sin(s'(x_j - \mu)) \right.$$

$$\left. = E(\sin(t'(x - \mu)) \sin(s'(x - \mu))) \right\},$$

and $D = D_1 \cap \{\bigcap_{t, s \in A_0} D_{t,s}\}$, where A_0 is any countable dense set of A. D is a subset of the sample space having probability measure one. Then by the Lipschitz continuity of the sine function, it is clear that $D = D_1 \cap \{\bigcap_{t, s \in A} D_{t,s}\}$. We describe the convergence of the process in a lemma.

Lemma 7.1. *Under the conditions of Theorem* 3.3, *given any sequences* $\{x_1, ..., x_n, ...\} \in D$, *the process* $\{V_n(E_n, X_n, t) \colon t \in A\}$ *converges weakly to a centered Gaussian process* $\{V(t) \colon t \in A\}$ *with the covariance kernel* $E(\sin(t'(s_j - \mu)) \sin(s'(x - \mu)))$ *for* $t, s \in A$.

Proof. In the following, we always assume without further mentioning that the given $\{x_1, ..., x_n, ...\}$ belongs to D. We need to prove the fidis convergence and the uniform tightness of the process.

(1) *Fidis convergence.* This part of the proof is standard, so we only give an outline. For any integer k, $t_1, ..., t_k \in A$. Let

$$V^{(k)} = (E(\sin(t_i'(x-\mu)) \sin(t_l'(x-\mu))))_{1 \leqslant i, l \leqslant k}.$$

We have to show that

$$V_n^{(k)} = \{ V_n(E_n, X_n, t_i) : i = 1, ..., k\} \Rightarrow N(0, V^{(k)}). \tag{7.1}$$

It suffices to show that for any unit k-dimensional vector γ

$$\gamma' V_n^{(k)} \Rightarrow N(0, \gamma' V^{(k)} \gamma) \tag{7.2}$$

Note that the variance of the LHS in (7.2) is

$$\gamma' (P_n(\sin(t_i'(x-\mu)) \sin(t_l'(x-\mu))))_{1 \leqslant i, l \leqslant k} \gamma \to \gamma' V^{(k)} \gamma. \tag{7.3}$$

Hence, if $\gamma' V^{(k)} \gamma = 0$, (7.2) is trivial. Assume $\gamma' V^{(k)} \gamma > 0$. Invoking the Lyapunov condition,

$$\gamma' V_n^{(k)} / \sqrt{\gamma' V^{(k)} \gamma} \to N(0, 1)$$

The fidis convergence holds via combining with (7.3).

(2) *Uniform tightness.* All we need to do is to show that for any $\eta > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\limsup_{n \to \infty} P\{\sup_{[\delta]} |V_n(E_n, X_n, t) - V_n(E_n, X_n, s)| \geqslant \eta \mid X_n\} < \varepsilon, \tag{7.4}$$

where $[\delta] = \{(t, s) : \|t - s\| \leqslant \delta\}$. Since the limiting property is investigated for $n \to \infty$, n is always considered to be large enough below which simplifies some arguments of the proof.

Note that

$$V_n(E_n, X_n, t) = \sqrt{n}\, P_n(\sin(t'e \cdot (x-\mu))) = \sqrt{n}\, P_n(e \sin(t'(x-\mu))).$$

Write P_n° for the signed measure that places mass e_i/n at $(x_i-\mu)$. We then write the LHS of (7.4) in another form

$$\limsup_{n\to\infty} P\{\sup_{[\delta]} \sqrt{n}\, |P_n^\circ(\sin(t'(x-\mu))) - (\sin(s'(x-\mu))\ X_n, s)| > \eta \mid X_n\}. \tag{7.5}$$

Note that when $\|t-s\| \leqslant \delta$ for large n

$$P_n(\sin(t'(x-\mu)) - \sin(s'(x-\mu)))^2$$
$$\leqslant \|t-s\| \frac{1}{n} \sum_{j=1}^{n} \|x_j-\mu\|^2 \leqslant 2\,\|t-s\|\ E\,\|x-\mu\|^2 = c\,\|t-s\|. \tag{7.6}$$

Then applying the Hoeffding inequality for any $t, s \in A$,

$$P\{\sqrt{n}\, |(P_n^\circ(\sin(t'(x-\mu)) - \sin(s'(x-\mu)))) > \eta c\,\|t-s\| \mid X_n, E_n\}$$
$$\leqslant 2\exp(-\eta^2/32). \tag{7.7}$$

In order to apply the chaining lemma (e.g., Pollard, 1984, p. 144), we need to check that the covering integral

$$J_2(\delta, \|\cdot\|, A) = \int_0^\delta \{2\log\{(N_2(u, \|\cdot\|, A))^2/u\}\}^{1/2}\, du \tag{7.8}$$

is finite for small $\delta > 0$, where $\|\cdot\|$ is the Euclidean norm in R^d and the covering number $N_2(u, \|\cdot\|, A)$ is the smallest m for which there exist m points $t_1, ..., t_m$ with $\min_{1\leqslant i\leqslant m} \|t-t_i\| \leqslant u$ for every $t \in A$. It is clear that

$$N_2(u/c, \|\cdot\|, A) \leqslant cu^{-d}. \tag{7.9}$$

Consequently, for small $\delta > 0$

$$J_2(\delta, \|\cdot\|, A) \leqslant c \int_0^\delta (\log(1/u))^{1/2}\, du \leqslant c\delta^{1/2}. \tag{7.10}$$

Applying now the chaining lemma, there exists a countable dense subset $[\delta]^*$ of $[\delta]$ such that

$$P\{\sup_{[\delta]^*} \sqrt{n}\, |(P_n^\circ(\sin(t'e\cdot(x-\mu)) - \sin(s'-\mu)))| > 26cJ_2(\delta, \|\cdot\|, A) \mid X_n\}$$
$$\leqslant 2c\delta. \tag{7.11}$$

The countable dense subset $[\delta]^*$ can be replaced by $[\delta]$ itself because $\sqrt{n}\, P_n^\circ\{\sin(t'(x-\mu)) - \sin(s'(x-\mu))\}$ is a continuous function with respect to t and s for each fixed X_n. Hence, choosing δ smaller than $1/e^2$ in (7.10), $\varepsilon/(8C)$, and $(\eta/(26c))^2$ in (7.11) Eq. (7.4) is proved. The proof of the lemma is completed.

On the other hand, the weak convergence of the process $\{\sqrt{n}\, P_n(\sin(t'(x-\mu)): t \in A\}$ to the process $\{V(t): t \in A\}$ can follow from our results. The sequence $e_1 \cdot x_1, e_2 \cdot x_2, \ldots$ has the same distribution as the sequence $x_1, x_2, \ldots$. Hence, the limit process is the same for the $\{e_i \cdot x_i\}$ as for the $\{x_i\}$. The above convergence of conditional process implies immediately the convergence of unconditional process, which has been derived by Ghosh and Ruymgaart (1992) and Heathcote, Rachev, and Cheng (1995). The proof of the Theorem 3.3 is concluded from noticing that Q_i are continuous functionals of the process $\{V_n(E_n, X_n, t): t \in A\}$.

Proof of Theorem 3.5. Note that sine and cosine functions are, respectively, odd and even functions, then

$$\begin{aligned}\sqrt{n}\, P_n(\sin(t'e\cdot(x-\bar{x}))) &= \sqrt{n}\, P_n(\sin(t'e\cdot(x-\mu))\cos(t'P_n(x-\mu)))\\ &\quad -\sqrt{n}\,(P_n(e\cdot\cos(t'(x-\mu))\sin(t'P_n(x-\mu))))\\ &=: I_{n1}(t)-I_{n2}(t) \qquad (7.12)\end{aligned}$$

and

$$\begin{aligned}\sqrt{n}\sin(t'P_n(e\cdot(x-\bar{x}))) &= \sqrt{n}\sin(t'P_n e\cdot(x-\mu))\cos(t'\bar{e}\cdot P_n(x-\mu))\\ &\quad -\sqrt{n}\cos(t'P_n e\cdot(x-\mu))\sin(t'\bar{e}\cdot P_n(x-\mu))\\ &=: I_{n3}(t)-I_{n4}(t), \qquad (7.13)\end{aligned}$$

where $\bar{e}=(1/n)\sum_{j=1}^n e_j$. By the central limit theorem we have $\sqrt{n}\, P_n(x-\mu) = O_p(1)$. It is then easy to see that

$$\begin{aligned} I_{n1}(t) &= \sqrt{n}\, P_n(\sin(t'e\cdot(x-\mu))) + O_p(1/\sqrt{n}),\\ I_{n2}(t) &= O_p(1/\sqrt{n}),\\ I_{n3}(t) &= \sqrt{n}\sin(t'P_n(e\cdot(x-\mu))) + O_p((1/\sqrt{n})^2),\\ I_{n4}(t) &= O_p(1/\sqrt{n}), \end{aligned} \qquad (7.14)$$

uniformly over $t \in A$, as long as we notice that $\{\sqrt{n}\, P_n(\sin(t'e\cdot(x-\mu))): t \in A\}$ and $\{\sqrt{n}\, P_n(e\cdot\cos(t'(x-\mu))): t\in A\}$ both converge weakly to Gaussian processes. Consequently, for almost all sequences $\{x_1, \ldots, x_n, \ldots\}$

$$
\begin{aligned}
V_{n1}&(E_n, X_n, \bar{x}, t)\\
&= \sqrt{n}\, P_n(\sin(t'e\cdot(x-\mu)))\\
&\quad - \sqrt{n}\, P_n(\cos(t'e\cdot(x-\mu))\sin(t'P_n e\cdot(x-\mu))) + O_p(1/\sqrt{n})\\
&= V_{n1}(E_n, X_n, \mu, t) + O_p(1/\sqrt{n}),
\end{aligned} \tag{7.15}
$$

uniformly over $t \in A$. The proof of (3.12) can be based on (7.15). Moreover, following the argument used in the proof of Theorem 3.3 above, we see that the process $\{V_{n1}(E_n, X_n, \bar{x}, t): t \in A\}$, given X_n, in (3.2) converges weakly to a Gaussian process for almost all sequences $\{x_1, ..., x_n, ...\}$, which is likewise the limit of the process $\{V_{n1}(X_n, \bar{x}, t): t \in A\}$ defined in (2.6); see Theorem 3.1 in Heathcote, Rachev, and Cheng (1995). Hence, the conclusions in Theorem 3.3 and Corollary 3.4 hold. The proof of Theorem 3.5 is complete.

Proof of Theorem 4.1. Without loss of generality, assume $Ex = Ey = 0$. It is known that $\max_{1 \leqslant j \leqslant n} \|y_j\|/n^{1/(2l)} \to 0$, a.s. Hence, by the Taylor expansion of *sine* function for any $t \in A$,

$$
\begin{aligned}
\sqrt{n}\, &P_n\{\sin(t'(x + y/n^{1/(2l)}))\}\\
&= \sqrt{n}\, P_n\{\sin(t'x)\} + \sum_{i=1}^{l-1} (1/i!)\, n^{-i/(2l)} \sqrt{n}\, P_n\{(t'y)^i \sin^{(i)}(t'x)\}\\
&\quad + (1/l!)\, P_n\{(t'y)^l \sin^{(l)}(t'x)\}\\
&\quad + (1/l!)\, n^{-1} \sum_{j=1}^{n} \{(t'y_j)^l \sin^{(l)}(t'(x_j + (t'y_j)^*/n^{1/(2l)})) - \sin^l(t'x_j)\},
\end{aligned} \tag{7.16}
$$

where $(t'y_j)^*$ is a value between 0 and $t'y_j$. We have to show that the second and fourth summands in the RHS of the equality tend to zero in probability for $n \to \infty$ and that the third summand converges in probability to $E\{(t'y)^l) \sin^{(l)}(t', x)\}$. First, consider the second summand. It is enough to show that for each $1 \leqslant i \leqslant l-1$, $\{\sqrt{n}\, P_n((t'y)^i \sin^{(i)}(t'x)): t \in A\}$ converges weakly to a centered Gaussian process. By Theorem VII.21 and the equicontinuity lemma (Pollard, 1984, p. 157, p. 150), all we need to do is to check that for any $\eta > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ for which

$$
\limsup P\{J_2(\delta, P_n, \Omega_i) > \eta\} < \varepsilon, \tag{7.17}
$$

where $\Omega_i = \{(t'y)^i \sin^{(i)}(t'x)) : t \in A\}$. The covering integral $J_2(\delta, P_n, \Omega)$ is similar to that in (7.8) and the seminorm in $L^2(P_n)$ is $\sqrt{P_n(f-g)^2}$. Note that

$$\sqrt{P_n\{(t'y)^i \sin^{(i)}(t'x) - (s'y)^i \sin^{(i)}(s'x)\}^2}$$
$$\leqslant c \, \|t-s\| \, \sqrt{P_n \, \|y\|^{2i} + P_n(\|y\|^{2i} \, \|x\|^2)} =: c \, \|t-s\| \, C_{n1}, \tag{7.18}$$

where $C_{n1} \to c = \sqrt{E \, \|y\|^{2i} + E \, \|y\|^{2i} \, \|x\|^2}$, a.s. For the case $C_{n1} < 2c$, we can bound $J_2(\delta, P_n, \Omega_i)$ by $c\delta^{1/2}$, similar to (7.10). Hence,

$$P\{J_2(\delta, P_n, \Omega_i) > \eta\} < P\{C_{n1} \geqslant 2c\}$$
$$+ P\{J_2(\delta, P_n, \Omega_i) > \eta\} \to 0 \tag{7.19}$$

as $n \to \infty$. The convergence of the third summand in (7.16) can be derived by applying Theorem II. 24 (Pollard, 1984, p. 26). The fourth summand tends to zero, since for some constant $c > 0$,

$$\sup_{t \in A} |(t'y_j)^l \;\; (\sin^{(l)}(t'(x_j + (t'y_j)^*/n^{1/(2l)})) - \sin^l(t_j'))|$$
$$\leqslant c \, \|y_j\|^l \, (\max_j \, \|y_j\|/n^{1/(2l)}),$$

$\max_{1 \leqslant j \leqslant n} \|y_j\|/n^{1/(2l)} \to$ a.s., and $E \, \|y\|^l < \infty$; Eq. (4.2) in Theorem 4.1 is proved. Both (4.3) and (4.4) are consequences of (4.2). The proof is completed.

Proof of Theorem 4.4. Without loss of generality, assume $Ex = Ey = 0$. Since

$$\sqrt{n} \, P_n(\sin(t'e \cdot (x + y/n^\theta)))$$
$$= \sqrt{n} \, P_n(\sin(t'e \cdot x)(\cos(t'y/n^\theta))) - \cos(t'x) \sin(t'e \cdot y/n^\theta).$$

Hence, all we need to do is to show that for almost all sequences $\{(x_1, y_1), ...\}$

$$\sup_{t \in A} |\sqrt{n} \, P_n^\circ(\sin(t'x)(1 - \cos(t'y/n^\theta)))| \to 0 \qquad \text{in Probab.} \tag{7.20}$$

and

$$\sup_{t \in A} |\sqrt{n} \, P_n^\circ(\cos(t'x) \sin(t'y/n^\theta))| \to 0 \qquad \text{in Probab.}, \tag{7.21}$$

where P_n° is a signed measure that places mass e_i/n at (x_i, y_i). We show (7.20). A similar way can be applied to show (7.21). Let $Y_n = (y_1, ..., y_n)$.

If P is any probability, the seminorm in $L^1(P)$ is $P\,|f-g|$ and $\Omega_n = \{\sin(t'\cdot)(1-\cos(t'\cdot/n^\theta)\colon t\in A\}$. Then it is easy to show that, by the Lipschitz continuity of the function $\sin(t'\cdot)(1-\cos(t'\cdot/n^\theta))$, the covering number $N_1(u, P, \Omega_n)$ can be bounded by Bu^{-W} for some B and W uniformly over P. Precisely, we have for $0<u<1$

$$\sup_P N_1(u, P, \Omega_n) \leqslant Bu^{-W}. \tag{7.22}$$

Furthermore, note that

$$\sup_{t\in A} P_n(\sin(t'e\cdot x)(1-\cos(t'e\cdot y/n^\theta)))^2 \mid (X_n, Y_n)$$

$$< cP_n\,\|y\|^2/n^{2\theta} \leqslant cn^{-2\theta} \tag{7.23}$$

for some $c>0$. Applying the formula (31) of Pollard (1984, p. 31), we have

$$P\{\sup_{t\in A} |\sqrt{n}\, P_n^\circ(\sin(t'x)(1-\cos(t'y/n^\theta)))| > \varepsilon \mid (X_n, Y_n)\}$$

$$\leqslant 2B\left(\frac{\varepsilon}{\sqrt{n}}\right)^W \exp(\varepsilon^2/(2cn^{-2\theta})) \to 0 \tag{7.24}$$

for $n\to\infty$; (7.20) is proved. The proof of (7.21) is similar. This finishes the proof of Theorem 4.4.

Proof of Theorem 4.5. Note that

$$\sup_{t\in A} |P_n(\cos(t'(x-Ex)+(y-Ey)/n^{1/(2l)})) - \cos(t'(x-Ex))|$$

$$< cP_n\,\|y-Ey\|/n^{1/(2l)} = O(n^{-1/(2l)}) \qquad \text{a.s.},$$

$$\sup_{t\in A} |1-\cos(t'P_n((x-Ex)+(y-Ey)/n^{1/(2l)}))|$$

$$< c(\|P_n x - Ex\|^2 + \|P_n y - Ey\|^2/n^{1/l}) = O_p(n^{-1}),$$

and

$$\sup_{t\in A} |\sqrt{n}\,(\sin(t'P_n(x-Ex)+P_n(y-Ey)/n^{1/(2l)})) - \sin(t'P_n(x-Ex))|$$

$$< c\sqrt{n}\,\|P_n y - Ey\|/n^{1/(2l)} = O_p(n^{-1/(2l)}).$$

Based on these inequalities, it is easy to see that

$$\begin{aligned}
&\sqrt{n}\, P_n(\sin(t'(x+y/n^{1/(2l)}-(\bar{x}+\bar{y}/n^{1/(2l)}))))\\
&\quad=\sqrt{n}\, P_n(\sin(t'((x-Ex)+(y-Ey)/n^{1/(2l)}))\\
&\qquad\times\cos(t'P_n((x-Ex)+(y-Ey)/n^{1/(2l)})))\\
&\qquad-\sqrt{n}\, P_n(\cos(t'((x-Ex)+(y-Ey)/n^{1/(2l)}))\\
&\qquad\times\sin(t'P_n((x-Ex)+(y-Ey)/n^{1/(2l)})))\\
&\quad=\sqrt{n}\, P_n(\sin(t'(x+y/n^{1/(2l)}-(Ex+Ey/n^{1/(2l)}))))\\
&\qquad-\sqrt{n}\, P_n(\cos(t'(x-Ex))\sin(t'P_n(x-Ex))+O_p(n^{-1/(2l)}))\\
&\quad=\sqrt{n}\, P_n(\sin(t'(x-Ex)))+(1/l!)\, E\{(t'(y-Ey))^l\sin^{(l)}(t'(x-Ex))\}\\
&\qquad-\sqrt{n}\sin(t'P_n(x-Ex))\, E(\cos(t'(x-Ex))+O_p(n^{-1/(2l)})).
\end{aligned}$$

This is just the formula in (4.5), completing the proof.

Proof of Theorem 4.6. By arguments similar to those in Theorems 4.5 and 4.4, the conclusion can be derived, so we omit the details.

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