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Sample Path Properties of Stochastic Processes Represented as Multiple Stable Integrals

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This paper studies the sample path properties of stochastic processes represented by multiple symmetric α-stable integrals. It relates the "smoothness" of the sample paths to the "smoothness" of the (non-random) integrand. It also contains results about the behavior of the distribution of suprema and zero–one laws. © 1991 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

In this paper we study stochastic processes of the form

$$X(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_t(x_1, ..., x_n)\, M(dx_1)\cdots M(dx_n), \qquad t\in T, \quad (1.1)$$

where M is a symmetric α-stable ($S\alpha S$), $0<\alpha<2$, independently scattered

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random measure on $(\mathbf{R}, \mathscr{B})$ with a Radon control measure m (i.e., m is finite on compact subsets of $\mathbf{R}$), and $\{f_t, t \in T\}$ is a family of real measurable functions $\mathbf{R}^n \to \mathbf{R}$ symmetric with respect to permutations of their arguments and vanishing on the diagonals. Such processes can be regarded as an extension of both $S\alpha S$ processes (to which they reduce when $n = 1$), and multiple Gaussian integrals, which corresponds to the case $\alpha = 2$.

Stochastic processes of the form (1.1) can exhibit long range dependence and high variability, and they are useful for modeling of various natural phenomena (see Taqqu [Taq87] and references therein). It is therefore of interest to study properties of their sample path. This paper is a first step in that direction.

Multiple stable intergrals defining the stochastic process $\{X(t), t \in T\}$ have been a focus of many studies in recent years (see, for example, [RW86, MT86, KW87, KS88b]). Samorodnitsky and Szulga [SS89] proposed a series representation for multiple stable integrals, later improved and generalized by Samorodnitsky and Taqqu [ST91, ST90].

Let M be a $S\alpha S$ random measure on $(\mathbf{R}, \mathscr{B})$ with a Radon control measure m. The product random measure $M^{(n)}$ on $(\mathbf{R}^n, \mathscr{B}^n)$ is defined as the product of the marginal random measures on measurable rectangles, and it can be extended to an L^p-valued, $0 < p < \alpha$, vector measure on symmetric measurable subsets of $\mathbf{R}^n$ which either do not intersect the diagonals, or include them fully (see Krakowiak and Szulga [KS88b] and Samorodnitsky and Taqqu [ST91]). Let now $\mathbf{f}$ be a symmetric, vanishing on the diagonals, separable Banach-space valued Borel function on $\mathbf{R}^n$. We sat that $\mathbf{f}$ is $M^{(n)}$-integrable if there is a sequence of simple functions of the type

$$\mathbf{f}^{(k)} = \sum_{i=1}^{N_k} \mathbf{a}_i(k)\, \mathbf{1}_{A_i(k)}, \tag{1.2}$$

where

1. $A_1(k), ..., A_{N_k}(k)$ are disjoint symmetric Borel subsets of $\mathbf{R}^n$ with finite $m^{(n)} = m \times m \times \cdots \times m$ measure and which do not intersect the diagonals;

2. the $\mathbf{a}_i(k)$'s are Banach-valued coefficients, such that $\mathbf{f}^{(k)} \to \mathbf{f}$ as $k \to \infty$ in measure $m^{(n)}$;

3. the sequence $\mathbf{I}_n(\mathbf{f}^{(k)} \mathbf{1}_C)$, $k = 1, 2, ...$, converges in probability for any symmetric Borel subset C of $\mathbf{R}^n$ which does not intersect the diagonals, where, as usual, for simple functions,

$$\mathbf{I}_n(\mathbf{f}^{(k)} \mathbf{1}_C) = \sum_{i=1}^{N_k} \mathbf{a}_i(k)\, M^{(n)}(A_i(k) \cap C).$$

In this case we define

$$\mathbf{I}_n(\mathbf{f}) := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{f}(x_1, ..., x_n)\, M(dx_1) \cdots M(dx_n) = \operatorname*{plim}_{k \to \infty} \mathbf{I}_n(\mathbf{f}^{(k)}), \tag{1.3}$$

where plim denotes limit in probability. The integral $\mathbf{I}_n(\mathbf{f})$ is uniquely defined, i.e., the above limit does not depend on the choice of simple functions satisfying 1–3. Because we have not found a proof of this fact in the literature, we shall give an argument to that effect in the Appendix.

We now quote two results from Samorodnitsky and Szulga [SS89] and Samorodnitsky and Taqqu [ST90] which play a major role in the present work.

Let S be a real separable Banach space. S is said to be of the *Rademacher*-type (R-type) p if the random series $\sum_{j=1}^{\infty} \varepsilon_j \mathbf{x}_j$ converges a.s. for every sequence $\{\mathbf{x}_j\}_{j=1}^{\infty}$ of elements of S satisfying $\sum_{j=1}^{\infty} \|\mathbf{x}_j\|^p < \infty$, where $\varepsilon_1, \varepsilon_2, ...$ are i.i.d. Rademacher random variables, i.e., $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = \frac{1}{2}$. Recall that every Banach space has R-type at least 1 and no Banach space can have R-type $p > 2$; every Hilbert space is of R-type 2, implying that $\mathbf{R}^n$ equipped with the maximum norm, $n < \infty$, is of R-type 2 as well, since the R-type does not change when the norm is replaced by an equivalent one.

Given a separable Banach space S of R-type p, let $\mathbf{f}: \mathbf{R}^n \to S$ be a symmetric, vanishing on the diagonals, strongly measurable function, and let m be a *Radon* Borel measure on $\mathbf{R}$. This ensures that m is σ-finite. The following notation will be used throughout the paper. We denote by ψ a measurable function: $\mathbf{R} \to (0, \infty)$ satisfying:

- $$\int_{-\infty}^{+\infty} \psi(x)^\alpha\, m(dx) = 1, \tag{1.4}$$

- $\varepsilon_1, \varepsilon_2, ...$ is an i.i.d. Rademacher sequence,

- $\Gamma_1, \Gamma_2, ...$ is the sequence of jump times of a Poisson process with unit rate,

- $Y_1, Y_2, ...$ are i.i.d. real-valued random variables with common distribution $m_\psi(dx) = \psi(x)^\alpha\, m(dx)$.

All three sequences of random variables are always assumed independent. For an $x > 0$ we denote

$$\ln_+ x = \begin{cases} \ln x & \text{if } x \geq 1, \\ 0 & \text{if } x < 1. \end{cases}$$

THEOREM 1.1 (i) *Let M be a $S\alpha S$ random measure on $(\mathbf{R}, \mathscr{B})$ with a Radon control measure m, $0<\alpha<p$. Suppose that the function $\mathbf{f}$ satisfies*

$$\int_{\mathbf{R}^n} \|\mathbf{f}(x_1, ..., x_n)\|^\alpha \left(\ln_+ \frac{\|\mathbf{f}(x_1, ..., x_n)\|}{\psi(x_1)\cdots\psi(x_n)}\right)^{n-1} m(dx_1)\cdots m(dx_n) < \infty. \tag{1.5}$$

Then the series

$$\mathbf{S}_n(\mathbf{f}) = C_\alpha^{n/\alpha} \sum_{j_1=1}^{\infty} \sum_{j_n=1}^{\infty} \varepsilon_{j_1}\cdots\varepsilon_{j_n} \Gamma_{j_1}^{-1/\alpha}\cdots\Gamma_{j_n}^{-1/\alpha}\psi(Y_{j_1})^{-1}\cdots$$
$$\times\psi(Y_{j_n})^{-1}\mathbf{f}(Y_{j_1}, ..., Y_{j_n}) \tag{1.6}$$

converges a.s., where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin x\, dx\right)^{-1}, \tag{1.7}$$

the multiple integral

$$\mathbf{I}_n(\mathbf{f}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathbf{f}(x_1, ..., x_n)\, M(dx_1)\cdots M(dx_n)$$

exists and

$$\mathbf{I}_n(\mathbf{f}) \stackrel{d}{=} \mathbf{S}_n(\mathbf{f}).$$

(ii) *If $S=\mathbf{R}$, then $I_n(f)$ exists if and only if $S_n(f)$ converges, and $I_n(f) \stackrel{d}{=} S_n(f)$.*

Remarks. • By convergence of a multiple series $\sum_{j_1=1}^{\infty}\cdots\sum_{j_n=1}^{\infty} a(j_1, ..., j_n)$, we always mean that the limit $\lim_{K\to\infty}\sum_{j_1=1}^{K}\cdots\sum_{j_n=1}^{K} a(j_1, ..., j_n)$ exists.

• Since $p\leqslant 2$, the Gaussian case $\alpha=2$ is explicitly excluded from the theorem. In fact the series representation (1.6) does not hold when $\alpha=2$ even if $n=1$.

• Bold letters denote vectors. We write $I_n(f)$ if f is real, and $\mathbf{I}_n(\mathbf{f})$ if $\mathbf{f}$ is a vector.

• The function ψ in (1.4) plays a double role in Theorem 1.1 and throughout the paper. First, it effectively reduces a σ-finite control measure m to a probability measure $m_\psi(dx)=\psi(x)^\alpha m(dx)$, from which we can generate the random vectors needed in the series expansion (1.6). Second, and more important: Note that for a given ψ condition (1.5) is only a sufficient condition for existence of the integral $\mathbf{I}_n(\mathbf{f})$ and for convergence of the series $\mathbf{S}_n(\mathbf{f})$. An appropriate choice of ψ can weaken the restrictions on

$\mathbf{f}$ even in the case where m is the Lebesgue measure on $[0, 1]$. It has been demonstrated in Samorodnitsky and Taqqu [ST90] that in the case $n=2$ an appropriate choice of ψ makes the sufficient condition (1.5) also necessary; and the right choice of ψ depends on the function f!

To shorten the notation we will write

$$\mathbf{N}_n = \{1, 2, ...\}^n,$$
$$[a_{\mathbf{j}}] = a_{j_1} a_{j_2} \cdots a_{j_n},$$
$$\mathbf{f}(\mathbf{Y}_{\mathbf{j}}) = \mathbf{f}(Y_{j_1}, ..., Y_{j_n})$$

for $\mathbf{j} = (j_1, ..., j_n)$. Also, $\mathbf{I}_n(\mathbf{f}) = \int_{\mathbf{R}_n} \mathbf{f} dM^{(n)}$ and

$$\mathbf{S}_n(\mathbf{f}) = C_\alpha^{n/\alpha} \sum_{\mathbf{j} \in \mathbf{N}_n} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} [\psi(Y_{\mathbf{j}})]^{-1} \mathbf{f}(\mathbf{Y}_{\mathbf{j}}).$$

Thus, with a real-valued stochastic process $\{X(t), t \in T\}$ as in (1.1), we conclude immediately that

$$\{X(t), t \in T\} \stackrel{d}{=} \{S_n(f_t), t \in T\}. \tag{1.8}$$

We call $\{S_n(f_t), t \in T\}$ the series representation of the stochastic process $\{X(t), t \in T\}$. The series representation is very important in our study of the sample path properties of the process $\{X(t), t \in T\}$ for it shows that the properties of the integrands $\{f_t(x_1, ..., x_n), t \in T\}$, $x_1, ..., x_n \in R$, as functions on T, have impact on the sample path properties of $\{X(t), t \in T\}$. This phenomenon has been observed and studied by Rosinski [Ros86, Ros89] in the case of stable and infinitely divisible processes; some of the ideas used in the present paper originate from the papers of Rosinski.

In Section 2 we find conditions for a stochastic process of the form (1.1) to have "smooth" sample paths, more generally, for the sample paths of the process to belong to a given vector space. The case of bounded sample paths is handled in Section 3, in which we also study the tail behavior of the distribution of $\sup_{t \in T} |X(t)|$ for bounded stochastic processes of the form (1.1). Finally, Section 4 states zero–one laws for stochastic processes of the form (1.1) with $n=2$. These zero–one laws complement the results of Section 2.

2. Processes with Sample Paths in a Vector Space

Let $\{X(t), t \in T\}$ be a stochastic process of the form (1.1) and let V be a vector space of real-valued functions on T. We study, in this section, whether $\{X(t), t \in T\}$ has a version with all sample paths belonging to V.

This question is of interest because path properties can be typically formulated in terms of vector subspaces V of $\mathbf{R}^T$. Much is known in the case of Gaussian and stable processes. Our results shed some light in the case of multiple stable integrals (1.1).

In order to make our discussion meaningful and to avoid obvious measurability problems, we introduce some assumptions.

From now on, the parameter space T is assumed to be a separable metric space. We extend the notion of *separable representation*, introduced by Rosinski [Ros89] to multiple stochastic integrals. Let $m^{(n)} = m \times \cdots \times m$. An integral representation (1.1) of $\{X(t),\ t \in T\}$ is said to be separable if there is a countable subset $T_0 \subset T$ and a Borel measurable symmetric set $N_0 \subset \mathbf{R}^n$ which does not include the diagonals, such that $m^{(n)}(N_0) = 0$, and for every $(x_1, ..., x_n) \notin N_0$ and for every $t \in T$, they is a sequence $\{t_j\}_{j=1}^{\infty} \subset T_0$ such that $f_t(x_1, ..., x_n) = \lim_{j \to \infty} f_{t_j}(x_1, ..., x_n)$. Separability of the integral representation (1.1), as pointed out in [Ros89] for the case $n = 1$, is an assumption which can always be made, although at this point we shall have to allow the integrand to take values in the extended real line $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$. Since the infinite values will be taken on a set of measure 0 for each fixed $t \in T$, this will not change the validity of the equality (1.1).

Indeed, $m^{(n)}$, if restricted to the Borel subsets of the lower-triangular space $L^n = \{(x_1, ..., x_n) \in \mathbf{R}^n: x_1 < x_2 < \cdots < x_n\}$ is a σ-finite measure, so letting $\hat{m}^{(n)}$ be a probability measure on the Borel subsets of L^n, equivalent to the measure $m^{(n)}$, we may regard $\{f_t,\ t \in T\}$ as a stochastic process indexed by T with $\hat{m}^{(n)}$ as underlying probability measure. By Doob's theorem (see [Doo53, Theorem 2.4, Chap. II]), there is a countable subset $T_0 \subset T$, a measurable set $M_0 \subset L^n$ with $\hat{m}^{(n)}(M_0) = 0$ (and, therefore, with $m^{(n)}(M_0) = 0$,) and a family of measurable functions $\{g_t,\ t \in T\}$, $g_t: L^n \to \bar{\mathbf{R}}$ such that for every $t \in T$,

$$m^{(n)}\{(x_1, ..., x_n) \in L_n: f_t(x_1, ..., x_n) \neq g_t(x_1, ..., x_n)\} = 0, \tag{2.1}$$

and for every $t \in T$ and every $(x_1, ..., x_n) \notin M_0$ there is a sequence $\{t_j\}_{j=1}^{\infty} \subset T_0$ such that

$$g_t(x_1, ..., x_n) = \lim_{j \to \infty} g_{t_j}(x_1, ..., x_n);$$

g_t, at this point, is defined only on L^n. We further extend g_t to the whole of $\mathbf{R}^n$ by setting $g_t(x_1, ..., x_n) = g_t(x_{(1)}, ..., x_{(n)})$, where $x_{(1)}, ..., x_{(n)}$ is an increasing rearrangement of $x_1, ..., x_n$ if the numbers $x_1, ..., x_n$ are all different, otherwise we define $g_t(x_1, ..., x_n) = 0$. Then (2.1) extends to $\mathbf{R}^n$, so that we obtain

$$X(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_t(x_1, ..., x_n)\, M(dx_1) \cdots M(dx_n), \qquad t \in T. \tag{2.2}$$

The integral representation (2.2) is separable by construction, with

$$N_0 = \{(x_1, ..., x_n) \in \mathbf{R}^n: x_i \neq x_j \quad \text{if } i \neq j, (x_{(1)}, ..., x_{(n)}) \in M_0\}.$$

This completes the argument.

We may and will assume, therefore, that the integral representation (1.1) is, to start with, separable.

As in [CR73, Ros89], we consider the following vector spaces V of functions on T:

(a) space of bounded functions on T,
(b) space of continuous functions on T,
(c) space of uniformly continuous functions on T,
(d) space of Lipschitz continuous functions on T,

and, if $T = \mathbf{R}$,

(e) space of functions without oscillatory discontinuities on T,
(f) space of functions of locally bounded variation on T,
(g) space of absolutely continuous functions on T,
(h) space of everywhere differentiable functions on T.

This list can be continued. In fact, we consider any function space V which satisfies the following condition:

Condition 2.1. There exists a linear measurable subspace $\hat{V}$ of $\mathbf{R}^\infty$ such that for every separable stochastic process $\{Y(t), t \in T\}$, there is an event Ω_1 with $P(\Omega_1) = 1$, such that for the countable subset $T_0 \subset T$ in the definition of separability,

$$\{\omega: \{Y(t), t \in T\} \in V\} \,\Delta\, \{\omega: \{Y(t), t \in T_0\} \in \hat{V}\} \subset \Omega_1^c.$$

For example, if $T = [0, 1]$ and $V =$ space of continuous functions on T, then we can take $T_0 =$ rationals and $\hat{V} =$ space of uniformly continuous functions on T_0. Cambanis and Rajput [CR73] showed that the function spaces V in (a)–(h) all satisfy Condition 2.1.

We are now ready to state our first theorem. It gives necessary conditions for a stochastic process of the type (1.1) to have a version with sample paths in a function space.

THEOREM 2.1 *Let $\{X(t), t \in T\}$ be a separable stochastic process with a separable representation* (1.1) *and let V be one of the vector spaces* (a)–(h) *above (or any other vector space satisfying Condition 2.1). If there is an event Ω_0 with $P(\Omega_0)=1$ such that, for every $\omega \in \Omega_0$,*

$$\{X(t, \omega), t \in T\} \in V,$$

then there is a Borel measurable set $S_0 \subset \mathbf{R}^n$ such that $m^{(n)}(\mathbf{R}^n \backslash S_0)=0$ and for every $(x_1, ..., x_n) \in S_0$,

$$\{f_t(x_1, ..., x_n), t \in T\} \in V.$$

Proof. It is sufficient to suppose Condition 2.1 holds. We apply first this condition to the stochastic process $\{X(t), t \in T\}$. Setting $\Omega_2=\Omega_0 \cap \Omega_1$, we obtain $P(\Omega_2)=1$ and for every $\omega \in \Omega_2$, $\{X(t, \omega), t \in T_0\} \in \hat{V}$. Using Theorem 1.1, we conclude that

$$Z(t)=\sum_{\mathbf{j} \in \mathbf{N}_n} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} [\psi(Y_{\mathbf{j}})]^{-1} f_t(\mathbf{Y}_{\mathbf{j}}), \qquad t \in T_0,$$

satisfies

$$P(\omega\colon \{Z(t, \omega), t \in T_0\} \in \hat{V})=1, \tag{2.3}$$

since $X \stackrel{d}{=} Z$ and T_0 is countable. Now let $\tilde{\varepsilon}_1=-\varepsilon_1$, $\tilde{\varepsilon}_j=\varepsilon_j$, $j \geqslant 2$. Clearly, $\{\tilde{\varepsilon}_j\}_{j=1}^{\infty}$ is a Rademacher sequence independent of the sequences $\Gamma_1, \Gamma_2, ...$ and $Y_1, Y_2, ...$ Therefore,

$$\tilde{Z}(t)=\sum_{\mathbf{j} \in \mathbf{N}_n} [\tilde{\varepsilon}_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} [\psi(Y_{\mathbf{j}})]^{-1} f_t(\mathbf{Y}_{\mathbf{j}}), \qquad t \in T_0,$$

is a version of $\{Z(t), t \in T_0\}$, and hence

$$P(\omega\colon \{\tilde{Z}(t, \omega), t \in T_0\} \in \hat{V})=1. \tag{2.4}$$

Since $\hat{V}$ is a linear space, we conclude by (2.3) and (2.4) that

$$P(\omega\colon \{Z(t, \omega)-\tilde{Z}(t, \omega), t \in T_0\} \in \hat{V})=1.$$

But for each $t \in T_0$,

$$Z(t)-\hat{Z}(t)=2\varepsilon_1 \Gamma_1^{-1/\alpha} \psi(Y_1)^{-1} \sum_{\substack{\mathbf{j} \in \mathbf{N}_{n-1} \\ j_i \geqslant 2}} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} [\psi(Y_{\mathbf{j}})]^{-1}$$
$$\times f_t(Y_1, Y_{j_1}, ..., Y_{j_{n-1}}).$$

Thus, with

$$Z_1(t)=\sum_{\substack{\mathbf{j} \in \mathbf{N}_{n-1} \\ j_i \geqslant 2}} [\varepsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} [\psi(Y_{\mathbf{j}})]^{-1} f_t(Y_1, Y_{j_1}, ..., Y_{j_{n-1}}), \qquad t \in T_0,$$

we obtain

$$P(\omega: \{Z_1(t, \omega), t \in T_0\} \in \hat{V}) = 1. \tag{2.5}$$

Repeating the procedure which led us from (2.3) to (2.5) $n-1$ times, we conclude

$$P(\omega: \{f_t(Y_1(\omega), Y_2(\omega), ..., Y_n(\omega)), t \in T_0\} \in \hat{V}) = 1. \tag{2.6}$$

Let

$$S_2 = \{(x_1, ..., x_n) \in \mathbf{R}^n: \{f_t(x_1, ..., x_n), t \in T_0\} \in \hat{V}\}.$$

Since each Y_i has distribution m_ψ, relation (2.6) implies $m_\psi^{(n)}(\mathbf{R}^n \backslash S_2) = 0$ and thus

$$m^{(n)}(\mathbf{R}^n \backslash S_2) = 0. \tag{2.7}$$

We now apply Condition 2.1 to $\{f_t(\cdot), t \in T\}$ regarded as a separable stochastic process on the probability space $(\mathbf{R}^n, \mathscr{B}^n, m_\psi^{(n)})$. We may assume without loss of generality, that the countable set T_0 here is the same as before (take the union of the two sets, if necessary), and we replace, in this case, Ω_1 by S_1. Set $S_0 = S_1 \cap S_2$. Then $m^{(n)}(\mathbf{R}^n \backslash S_0) = 0$, and since $S_0 \subset S_2$, we conclude that for every $(x_1, ..., x_n) \in S_0$, $\{f_t(x_1, ..., x_n), t \in T\} \in V$. This completes the proof of the theorem. ▮

Remarks. • Since $\mathbf{f}$ is symmetric and vanishes on diagonals, the measurable set S_0 in Theorem 2.1 can always be chosen to be symmetric and to include all the diagonals.

• In the case $n = 2$ the statement of Theorem 2.1 remains true if we replace the assumption $P(\Omega_0) = 1$ by $P(\Omega_0) > 0$, since our process satisfies an appropriate zero–one law. See Section 4 for more details.

• Note the relation between the two notions of separability appearing in Theorem 2.1. We should understand it as follows: $\{X(t), t \in T\}$ is separable, $\{X(t), t \in T\} \stackrel{d}{=} \{I_n(f_t), t \in T\}$, where the equality is in terms of finite-dimensional distribution, and the integral representation defined by the functions $\{f_t, t \in T\}$ is separable.

Theorem 2.1 provides a *necessary* condition for a stochastic process of the type (1.1) to have almost all sample paths in a vector space V. We now focus on *sufficient* conditions and assume that the space V satisfies the following:

Condition 2.2. V is a normed space of real-valued functions on T such that all evaluations $\pi_t: V \to \mathbf{R}$ defined by $\pi_t(\mathbf{x}) = (\mathbf{x})_t$ are continuous.

The following result ensures that the multiple integral $\mathbf{I}_n(\mathbf{f})$ of a function $\mathbf{f}$ taking values in V may be regarded as the vector of the multiple integrals of the evaluations $I_n(f_t)$ of this function.

PROPOSITION 2.1. *Let* $\mathbf{f}$ *be a symmetric, vanishing on the diagonals, measurable function from* $\mathbf{R}^n$ *to a vector space* V *satisfying Condition 2.2 and suppose that the multiple integral* $\mathbf{I}_n(\mathbf{f})$ *exists. Then for each* $t \in T$ *the evaluation* $f_t := \pi_t(\mathbf{f})$ *is* $M^{(n)}$*-integrable, and for each* $t \in T$,

$$I_n(f_t) = (\mathbf{I}_n(\mathbf{f}))_t, \qquad a.s., \tag{2.8}$$

where $(\mathbf{I}_n(\mathbf{f}))_t = \pi_t(\mathbf{I}_n(\mathbf{f}))$ *for each* $t \in T$.

Proof. Suppose first that $\mathbf{f}$ is a simple function, i.e.,

$$\mathbf{f}(x_1, ..., x_n) = \sum_{i=1}^{N} \mathbf{a}(i)\, \mathbf{1}((x_1, ..., x_n) \in A_i), \tag{2.9}$$

where $\mathbf{a}(1), ..., \mathbf{a}(N) \in V$, and $A_1, ..., A_N$ are disjoint symmetric Borel sets in $\mathbf{R}^n$ with finite $m^{(n)}$ measure and which do not include the diagonals. Then

$$I_n(f_t) = \sum_{i=1}^{N} (\mathbf{a}(i))_t\, M^{(n)}(A_i) = \left(\sum_{i=1}^{N} \mathbf{a}(i)\, M^{(n)}(A_i) \right)_t = (\mathbf{I}_n(\mathbf{f}))_t.$$

Thus (2.8) holds for simple functions $\mathbf{f}$.

Now let $\mathbf{f}$ be $M^{(n)}$-integrable. Then, by definition, there is a sequence of simple functions $\{\mathbf{f}^{(k)}\}_{k=1}^{\infty}$ as in (2.9) converging to $\mathbf{f}$ in measure $m^{(n)}$, such that $\{\mathbf{I}_n(\mathbf{f}^{(k)}\mathbf{1}_C), k = 1, 2, ...\}$ converges to $\mathbf{I}_n(\mathbf{f}\mathbf{1}_C)$ in probability for each symmetric Borel set C of $\mathbf{R}^n$.

Now, for every $t \in T, m^{(n)}\{\mathbf{x}: \|\mathbf{f}^{(k)}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\|_V > \varepsilon_1\} \to 0, \forall \varepsilon_1$, implies $m^{(n)}\{\mathbf{x}: |f_t^{(k)}(\mathbf{x}) - f_t(\mathbf{x})| > \varepsilon_2\} \to 0, \forall \varepsilon_2$, by Condition 2.2, and hence the sequence $\{f_t^{(k)}\}_{k=1}^{\infty}$ converges to f_t in measure $m^{(n)}$. Similarly, the sequence $\{(\mathbf{I}_n(\mathbf{f}^{(k)}\mathbf{1}_C))_t, k = 1, 2, ...\}$ converges in probability to $(\mathbf{I}_n(\mathbf{f}\mathbf{1}_C))_t$.

Since $\mathbf{f}^{(k)}\mathbf{1}_C$ is a simple function, (2.8) holds for each $\mathbf{f}^{(k)}\mathbf{1}_C$ and $t \in T$. Letting $k \to \infty$, we infer that for each $t \in T$, f_t is $M^{(n)}$-integrable and $I_n(f_t) = (\mathbf{I}_n(\mathbf{f}))_t$ a.s. ∎

The following result gives *sufficient* conditions for a stochastic process of the type (1.1) to have almost all its sample paths in a Banach space with special properties.

THEOREM 2.2. *Let* V *be a separable Banach space with norm* $\|\ \|_V$ *of R-type* p *satisfying Condition 2.2, and let* $\{X(t), t \in T\}$ *be a stochastic process given in the form of a multiple* $S\alpha S$ *integral with a separable representation* (1.1), $0 < \alpha < p$. *Suppose that there is a Borel measurable set* $S_0 \subset \mathbf{R}^n$ *such that* $m^{(n)}(\mathbf{R}^n \backslash S_0) = 0$, *and for every* $(x_1, ..., x_n) \in S_0$, $\{f_t(x_1, ..., x_n), t \in T\} \in V$, *and that there is a function* ψ *as in* (1.4) *such that*

$$\int_{\mathbf{R}^n} \|\mathbf{f}(x_1, ..., x_n)\|_V^{\alpha} \left(\ln + \frac{\|\mathbf{f}(x_1, ..., x_n)\|_V}{\psi(x_1) \cdots \psi(x_n)} \right)^{n-1} m(dx_1) \cdots m(dx_n) < \infty, \tag{2.10}$$

where

$$\mathbf{f}(x_1, ..., x_n) = \begin{cases} \{f_t(x_1, ..., x_n), t \in T\} & \textit{if } (x_1, ..., x_n) \in S_0, \\ 0 & \textit{otherwise}. \end{cases} \tag{2.11}$$

Then there is a version of $\{X(t), t \in T\}$ *with all sample paths in* V.

Proof. Let $\mathbf{f}$ be as in (2.11). Separability of V and Condition 2.2 imply that $\mathbf{f} = \mathbf{R}^n \to V$ is a Borel measurable function (see [Ros86, p. 6]). Applying Theorem 1.1, we conclude that $\mathbf{f}$ is $M^{(n)}$-integrable. Therefore, $\mathbf{I}_n(\mathbf{f})$ is well defined and since $\mathbf{f}$ is V-valued, so is $\mathbf{I}_n(\mathbf{f})$. Thus $\mathbf{I}_n(\mathbf{f})$ is the version we are looking for, since, by Proposition 2.1, $\{(\mathbf{I}_n(\mathbf{f}))_t, t \in T\}$ is a version of $\{X(t), t \in T\} \equiv \{I_n(f_t), t \in T\}$. This completes the proof. ∎

Remarks. • The restriction that the space V must be of R-type p, $0 < \alpha < p$, disappears if $0 < \alpha < 1$, since every Banach space is of R-type 1. On the other hand, if V is of R-type 1, then the theorem applies only when $0 < \alpha < 1$. Unfortunately, many Banach spaces of interest are of R-type 1. This is the case for example of the Banach spaces that are among the spaces (a)–(h) listed in the beginning of this section.

• The results of this section do not provide a condition which is both necessary and sufficient for the sample paths of a process of type (1.1) to belong to a vector space.

• There are many unsolved questions even in the case of single stable integrals (i.e., $n = 1$ in (1.1)). Although much is known for $\alpha < 1$, general conditions for regularity are largely unknown when $\alpha \geqslant 1$: one has results only for specific f's and specific path properties. The multiple integration case (i.e., $n > 1$ in (1.1)) which we are considering here, is naturally even more complicated.

3. Boundedness

Since the space of bounded functions is not separable (even on a countable set), Theorem 2.2 cannot be used to study stochastic processes of the type (1.1) with a.s. bounded sample paths. Nevertheless, one has

Theorem 3.1. *Let* $\{X(t), t \in T\}$ *be a separable stochastic process represented in the form of a multiple* $S\alpha S$ *integral with a separable representation* (1.1).

(i) *Suppose that* $\{X(t), t \in T\}$ *is a.s. bounded. Then*

$$\int_{\mathbf{R}^n} f^*(x_1, ..., x_n)^\alpha \, m(dx_1) \cdots m(dx_n) < \infty, \tag{3.1}$$

where $f^*(x_1, ..., x_n) := \sup_{t \in T} |f_t(x_1, ..., x_n)|$.

(ii) *Let* $0<\alpha<1$, *and suppose that there is a function* ψ *satisfying* (1.4) *such that*

$$\int_{\mathbf{R}^n} f^*(x_1, ..., x_n)^\alpha \left(\ln_+ \frac{f^*(x_1, ..., x_n)}{\psi(x_1)\cdots\psi(x_n)}\right)^{n-1} m(dx_1)\cdots m(dx_n) < \infty. \tag{3.2}$$

Then $\{X(t), t \in T\}$ *is a.s. bounded.*

Proof. By the separability assumptions, we may and will assume that the set T is countable. We identify it with the set of positive integers and denote our process $\{X(k), k=1, 2, ...\}$.

(i) We know that $\sup_k |X(k)| < \infty$ a.s. But we shall, at first, make a stronger assumption, namely $\lim_{k\to\infty} X(k) = 0$ a.s. We view then $\mathbf{X} = \{X(k), k=1, 2, ...\}$ as random vector in the separable Banach space c_0 of sequences converging to zero, equipped with the supremum norm $\|\cdot\|_\infty$. Clearly, as $m \to \infty$ $\mathbf{X}^{(m)} \to \mathbf{X}$ in c_0 a.s., where

$$\mathbf{X}^{(m)} = \{X(1), ..., X(m), 0, 0, ...\}, \qquad m=1, 2, ...$$

Obviously, $\mathbf{X}^{(m)} = \mathbf{I}_n(\mathbf{f}^{(m)})$, $m=1, 2, ...$, where

$$\mathbf{f}^{(m)} = (f_1, ..., f_m, 0, 0, 0, ...), \qquad m=1, 2, ...$$

is regarded as a c_0-valued function.

It follows from [KS88a] that the random vectors $\mathbf{X}^{(m)}$, $m=1, 2, ...$ belong to the same Marcinkiewicz–Paley–Zygmund class, and so for this sequence of random vectors, convergence in probability implies convergence in L^r for every $0<r<\alpha$. We conclude that for any $0<r<\alpha$, $\lim_{m\to 0} E\,\|\mathbf{X}-\mathbf{X}^{(m)}\|_\infty^r = 0$ and $E\,\|\mathbf{X}\|_\infty^r < \infty$. Moreover, it follows by Proposition 5.1(ii) of [KS88b] that for any $m=1, 2, ...$,

$$E\,\|\mathbf{X}^{(m)}\|_\infty^r \geqslant C_{\alpha,r}\left(\int_{\mathbf{R}^n} \max_{i<m} |f_i(x_1, ..., x_m)|^\alpha\, m(dx_1)\cdots m(dx_n)\right)^{r/\alpha}, \tag{3.3}$$

where $C_{\alpha,r}$ is a positive constant depending only on α and r. Letting $m\to\infty$, we obtain

$$\infty > E\,\|\mathbf{X}\|_\infty^r \geqslant C_{\alpha,r}\left(\int_{\mathbf{R}^n} f^*(x_1, ..., x_n)^\alpha\, m(dx_1)\cdots m(dx_n)\right)^{r/\alpha}. \tag{3.4}$$

Let us now return to our original assumption $\sup_k |X(k)| < \infty$ a.s. and drop the requirement $\lim_{k\to\infty} X(k) = 0$ a.s. Let $\{a_k, k=1, 2, ...\}$ belong to c_0, and $\sup_k |a_k| \leqslant 1$. Then $\{a_k X(k), k=1, 2, ...\}$ belongs to c_0 a.s., so (3.4) gives

$$E(\sup_k |X_k|)^r \geqslant E(\sup_k |a_k X_k|)^r$$
$$\geqslant C_{\alpha,r}\left(\int_{\mathbf{R}^n} \sup_k |a_k f_k(x_1, ..., x_n)|^\alpha \, m(dx_1) \cdots m(dx_n)\right)^{r/\alpha} \tag{3.5}$$

and we know also that

$$E(\sup_k |a_k X(k)|)^r < \infty, \qquad \forall r \in (0, \alpha). \tag{3.6}$$

By taking $a_k = 1$ for $k \leqslant N$ and 0 otherwise, and letting $N \to \infty$, (3.5) implies

$$E(|\sup_k | X(k)|)^r \geqslant C_{\alpha,r}\left(\int_{\mathbf{R}^n} f^*(x_1, ..., x_n)^\alpha \, m(dx_1) \cdots m(dx_n)\right)^{r/\alpha}. \tag{3.7}$$

Now, (3.1) will follow if we establish $E(\sup_k |X(k)|)^r < \infty$. Assume, to the contrary, that for some $0 < r < \alpha$, $E(\sup_k |X(k)|)^r = \infty$. Choose $0 < K_1 < K_2 < \cdots$ such that

$$E(\max_{k \leqslant K_j} |X(k)|)^r \geqslant j, \qquad j = 1, 2, ...$$

Choose now $a_k = j^{-1/(2r)}$ if $K_{j-1} < k \leqslant K_j$, $j = 1, 2, ..., K_0 = 0$. Then $E(\max_{k \leqslant K_j} |a_k X(k)|)^r \geqslant j^{1/2}$ for every $j = 1, 2, ...$, so that $E(\sup_k |a_k X(k)|)^r = \infty$, contradicting (3.6). This proves that $E(\sup_k |X(k)|)^r < \infty$ for every $0 < r < \alpha$, and thus the proof of part (i) is complete.

(ii) Let again $\{a_k, k = 1, 2, ...\}$ belong to c_0, and $\sup_k |a_k| \leqslant 1$. Set

$$\mathbf{g}(x_1, ..., x_n) = \begin{cases} \{a_k f_k(x_1, ..., x_n),\ k = 1, 2, ...\} & \text{if } f^*(x_1, ..., x_n) < \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

Clearly $\mathbf{g}$ is a measurable function $\mathbf{R}^n \to c_0$ and by (3.2), for every $k = 1, 2, ..., \pi_k(\mathbf{g}) = a_k f_k$ $m^{(n)}$—almost everywhere. Since $\|\mathbf{g}(x_1, ..., x_n)\|_\infty \leqslant f^*(x_1, ..., x_n)$ for every $(x_1, ..., x_n)$, we obtain

$$\int_{\mathbf{R}^n} \|\mathbf{g}(x_1, ..., x_n)\|^\alpha \left(\ln_+ \frac{\|\mathbf{g}(x_1, ..., x_n)\|}{\psi(x_1) \cdots \psi(x_n)}\right)^{n-1} m(dx_1) \cdots m(dx_n) < \infty.$$

Theorem 1.1 applies since $0 < \alpha < 1$ and every Banach space is of R-type $p = 1$. Hence $\mathbf{g}$ is $M^{(n)}$ integrable and $\mathbf{I}_n(\mathbf{g})$ is a well-defined c_0-valued variable. By Proposition 2.1 we have $\{a_k X(k), k = 1, 2, ...\} \stackrel{d}{=} \{\mathbf{I}_n(\mathbf{g})_k, k = 1, 2, ...\}$, and it follows as in the proof of part (i) that $E \sup_k |a_k X(k)|^r < \infty$ for any $0 < r < \alpha$, and, thus, also $E \sup_k |X(k)|^r < \infty$. Hence $\{X(k), k = 1, 2, ...\}$ is a.s. bounded. This completes the proof. ∎

COROLLARY 3.1. *Under the conditions of Theorem* 3.1, *Part* (i), *for every* $0<r<\alpha$,

$$(E\sup_{t\in T}|X(t)|^r)^{1/r}\geqslant C_{\alpha,r}\left(\int_{\mathbf{R}^n} f^*(x_1,...,x_n)^\alpha\, m(dx_1)\cdots m(dx_n)\right)^{1/\alpha},$$

where $C_{\alpha,r}$ *is a positive constant depending only on* α *and* r.

Proof. Follows immediately from the proof of part (i) of Theorem 3.1. ∎

Remark. In the same spirit, we may give asymptotic lower bounds for the $P(\sup_{t\in T}|X(t)|>\lambda)$. Specifically, let $\{X(t), t\in T\}$ be a separable stochastic process represented in the form of a multiple $S\alpha S$ integral with a separable representation (1.1) satisfying the following property: there is a function ψ satisfying (1.4) such that, for every $t\in T_0$ (T_0 is the countable subset of T appearing in the definition of a separable representation), we have

$$\int_{\mathbf{R}^n}|f_t(x_1,...,x_n)|^\alpha\left(\ln_+\frac{|f_t(x_1,...,x_n)|}{\psi(x_1)\cdots\psi(x_n)}\right)^{n-1} m(dx_1)\cdots m(dx_n)<\infty, \quad (3.9)$$

if $n\geqslant 3$, or

$$\int_{\mathbf{R}^2}|f_t(x_1,x_2)|^\alpha \ln_+\frac{|f_t(x_1,x_2)|}{\psi(x_1)\,\psi(x_1)}\ln_+\left|\ln_+\frac{|f_t(x_1,x_2)|}{\psi(x_1)\,\psi(x_2)}\right| m(dx_1)\, mdx_2)<\infty, \quad (3.10)$$

if $n=2$. Then

$$\liminf_{\lambda\to\infty}\frac{\lambda^\alpha}{(\ln\lambda)^{n-1}}P(\sup_{t\in T}|X(t)|>\lambda)$$
$$\geqslant n(n!)^{\alpha-2}\alpha^{n-1}C_\alpha^n\int_{\mathbf{R}^n} f^*(x_1,...,x_n)^\alpha\, m(dx_1)\cdots m(dx_n), \quad (3.11)$$

where C_α is given by (1.7) and f^* is defined as $\sup_{t\in T}|f_t(x_1,...,x_n)|$. Moreover, when $0<\alpha<1$, then

$$\lim_{\lambda\to\infty}\frac{\lambda^\alpha}{(\ln\lambda)^{n-1}}P(\sup_{t\in T}|X(t)|>\lambda)$$
$$=n(n!)^{\alpha-2}\alpha^{n-1}C_\alpha^n\int_{\mathbf{R}^n} f^*(x_1,...,x_n)^\alpha\, m(dx_1)\cdots m(dx_n). \quad (3.12)$$

These statements follow firectly from Samorodnitsky and Taqqu [ST90].

4. Zero-One Laws

Sample path properties of some stochastic processes can satisfy various zero–one laws: see [Kal70] for Gaussian processes, [DK74] for stable processes, and [Jan84] and [Ros90] for infinitely divisible processes. In this section we establish some zero–one laws for stochastic processes of the form (1.1) with $n=2$. We restrict ourselves to the case $n=2$ because we use here a result of de Acosta [DeA76] on quadratic forms in Gaussian vectors. We believe that similar zero–one laws hold for $n \geqslant 3$.

THEOREM 4.1. *Let $\{X(t), t \in T\}$ be a separable stochastic process with a separable representation* (1.1) *and with $n=2$. Let V be a vector space of functions over T satisfying Condition* 2.1. *Then*

$$P\{\{X(t), t \in T\} \in V\} = 0 \text{ or } 1.$$

Before proving this theorem, we collect a number of facts which will be used in the proof.

Focus first on the $S\alpha S$ random measure M in (1.1). We assume for simplicity $m((-\infty, 0))=0$, and by denoting $M(t) := M([0, t])$, we may regard M as a $S\alpha S$ process with independent increments on $\mathbf{R}^+$. (The case $m((-\infty, 0))>0$ can be treated similarly, by considering M as consisting of two independent components $\{M_1(t)=M([0, t]), t \geqslant 0\}$ and $\{M_2(t)=M([-t, 0)), t \geqslant 0\}$.) It is well known that $\{M(t), t \geqslant 0\}$ has a version in the separable space $D[0, +\infty)$ equipped with the Skorokhod topology. It follows from [LeP80, Kal73], that such a version is given in particular by the series

$$M(t) = (\mathscr{C}_\alpha C_\alpha)^{1/2} \sum_{j=1}^{\infty} G_j \Gamma_j^{-1/\alpha} \psi(Y_j)^{-1} 1_{[0,t]}(Y_j), \qquad t \geqslant 0, \tag{4.1}$$

which converges uniformly in t on finite intervals. Here the Γ_j's and Y_j's are as in Section 1, the G_j's are i.i.d. standard normal random variables independent of the Γ_j's and Y_j's and $\mathscr{C}_\alpha = (E|\mathscr{C}_1|^\alpha)^{-1}$.

Thus, $\{M(t), t \geqslant 0\}$ can be regarded as a random vector taking values in the space $D[0, +\infty)$ equipped with the Skorohod topology. Moreover, all finite-dimensional projections of $\{M(t), t \geqslant 0\}$ are $S\alpha S$. Therefore, $\{M(t), t \geqslant 0\}$ is a $S\alpha S$ vector in $D[0, +\infty)$, as the Skorohod Borel σ-algebra coincides with the cylindrical σ-algebra.

Assume now that the random vector $\{M(t), t \geqslant 0\}$ is defined on the product of two probability spaces, $(\Omega_1, \mathscr{F}_1, P_1)$ and $(\Omega_2, \mathscr{F}_2, P_2)$, and let the Gaussian sequence $G_1, G_2, \ldots$ live on $(\Omega_1, \mathscr{F}_1, P_1)$, while the sequences

$\Gamma_1, \Gamma_2, \ldots$ and $Y_1, Y_2, \ldots$ live on $(\Omega_2, \mathscr{F}_2, P_2)$. Arguing as above, we conclude that, for a fixed $\omega_2 \in \Omega_2$, $\{M(t), t \geqslant 0\}$ is a zero-mean Gaussian random vector on $D[0, +\infty)$, defined on the probability space $(\Omega_1, \mathscr{F}_1, P_1)$.

We will need the following extension of the above mentioned result of deAcosta [DeA76].

LEMMA 4.1. *Let $(E, \mathscr{B})$ be a measurable vector space, and let G be a zero-mean Gaussian vector in E. Let $\{A^{(i)}, i = 1, 2, \ldots\}$, be a sequence of measurable bilinear forms on E^2 taking values in a topological vector space E_1. Let S be a measurable subspace of E_1. Then*

$$P(\lim_{i \to \infty} A^{(i)}(G, G) \text{ exists and belongs to } S) = 0 \text{ or } 1.$$

Proof. Mimic the proofs of Theorems 3.1 and 3.2 in [DeA76]. ∎

Proof of Theorem 4.1. In view of Condition 2.1, we must prove

$$P(\{X(t), t \in T_0\} \in \hat{V}) = 0 \text{ or } 1. \tag{4.2}$$

For each $t \in T_0$, there is a sequence $\{f_t^{(i)}, i = 1, 2, \ldots\}$ of simple symmetric functions as described in Section 1 such that, as $i \to \infty$, $I_2(f_t^{(i)}) \to X(t)$ in probability. As a matter of fact one can choose these functions to be supported by the finite unions of disjoint rectangles. Moreover, choosing, if necessary, a subsequence, we may and will assume that $I_2(f_t^{(i)}) \to X(t)$ a.s. as $i \to \infty$ for any $t \in T_0$.

Now, each $I_2(f_t^{(i)})$ is, clearly, a measurable bilinear form in $\{M(s), s \geqslant 0\}$. Applying Lemma 4.1 with $E_1 = \mathbf{R}^{T_0}$ we conclude

$$P(\{X(t), t \in T_0\} \in \hat{V} \mid \mathscr{F}_2) = 0 \text{ or } 1 \text{ a.s.} \tag{4.3}$$

which is a zero-one law for the (conditional) Gaussian measures.

To establish (4.2), we must remove the conditioning. Set

$$A = \{\omega_2 \in \Omega_2 \colon P(\{X(t), t \in T_0\} \in \hat{V} \mid \mathscr{F}_2) = 1\},$$

and observe that

$$P(\{X(t), t \in T_0\} \in \hat{V}) = P_2(A). \tag{4.4}$$

We want to apply the Hewitt–Savage zero–one law to the event A in order to show $P_2(A) = 0$ or 1. Recall that $\Gamma_1, \Gamma_2, \ldots$ and $Y_1, Y_2, \ldots$ live on $(\Omega_2, \mathscr{F}_2, P_2)$. In fact, $A \in \sigma((e_1, Y_1), (e_2, Y_2), \ldots)$, where $e_1, e_2, \ldots$, are i.i.d. exponential random variables such that $\Gamma_j = e_1 + e_2 + \cdots e_j$. Let π be an arbitrary permutation of the numbers $\{1, \ldots, k\}$ and

$$M_\pi(t) = (\mathscr{C}_\alpha C_\alpha)^{1/\alpha} \sum_{j=1}^{k} G_j (e_{\pi(1)} + \cdots + e_{\pi(j)})^{-1/\alpha} \psi(Y_{\pi(j)})^{-1} 1_{(-\infty, t]}(Y_{\pi(j)})$$
$$+ (\mathscr{C}_\alpha C_\alpha)^{1/\alpha} \sum_{j=k+1}^{\infty} G_j \Gamma_j^{-1/\alpha} \psi(Y_j)^{-1} 1_{(-\infty, t]}(Y_j)$$

for $t \geq 0$. For fixed $\omega_2 \in A$, $\{M_\pi(t), t \geq 0\}$ is again a Gaussian vector on $D[0, +\infty)$, and it is easy to check that the laws of $\{M(t), t \geq 0\}$ and $\{M_\pi(t), t \geq 0\}$ are equivalent. Therefore, $P(\{\lim_{i \to \infty} I_2(f_t^{(i)}), t \in T_0\} \in \hat{V} \mid \mathscr{F}_2) = 1$ implies $P(\{\lim_{i \to \infty} I_2^{(\pi)}(f_t^{(i)}), t \in T_0\} \in \hat{V} \mid \mathscr{F}_2) = 1$, where $I_2^{(\pi)}(f_t^{(i)})$ is obtained by replacing $\{M(t), t \geq 0\}$ by $\{M_\pi(t), t \geq 0\}$ in the bilinear form $I(f_t^{(i)})$. Thus, the event A is invariant under the permutations π of the above kind. By the Hewitt–Savage zero–one law, $P_2(A) = 0$ or 1, and hence by (4.4),

$$P(\{X(t), t \in T_0\} \in \hat{V}) = P_2(A) = 0 \text{ or } 1.$$

This completes the proof. ∎

The following proposition complements a result of Krakowiak and Szulga [KS88a, Theorem 2.11] about the equivalence of different modes of convergence of sequences to a double $S\alpha S$ integral. Its proof uses the techniques developed in the proof of Theorem 4.1.

PROPOSITION 4.1. *Let* $\mathbf{f}^{(m)}$, $m = 1, 2, \ldots$, *be a sequence of symmetric, vanishing on the diagonals, Banach space-valued simple functions as in* (1.2), *and assume that*

$$P(\mathbf{I}_2(\mathbf{f}^{(m)}), m = 1, 2, \ldots \text{ converges}) > 0.$$

Then

$$P(\mathbf{I}_2(\mathbf{f}^{(m)}), m = 1, 2, \ldots \text{ converges}) = 1.$$

Proof. Let $m = 1, 2, \ldots$ be arbitrary. We can choose a sequence of simple functions $\mathbf{g}^{(m,k)}$, $k = 1, 2, \ldots$, each one of the type sym $(\sum_{j=1}^{K} \mathbf{a}_j 1((x_1, x_2) \in I_1^{(j)} \times I_2^{(j)}))$, defined as in the proof of Theorem 4.1, such that $\mathbf{I}_2(\mathbf{g}^{(m,k)}) \to \mathbf{I}_2(\mathbf{f}^{(m)})$ a.s. as $k \to \infty$. Let now $\{k_m, m = 1, 2, \ldots\}$ be a sequence of positive integers such that $\sum_{m=1}^{\infty} d_m < \infty$, where for $m = 1, 2, \ldots$,

$$d_m = \inf\{\varepsilon > 0 \colon P(\|\mathbf{I}_2(\mathbf{g}^{(m,k_m)}) - \mathbf{I}_2(\mathbf{f}^{(m)})\| > \varepsilon) \leq \varepsilon\}.$$

Then, by the Borel–Cantelli lemma,

$$P(\lim_{m \to \infty} \|\mathbf{I}_2(\mathbf{g}^{(m,k_m)}) - \mathbf{I}_2(\mathbf{f}^{(m)})\| \to 0) = 1. \tag{4.5}$$

Now $P(\mathbf{I}_2(f^{(m)}), m = 1, 2, \ldots \text{ converges}) > 0$ implies $P(\mathbf{I}_2(\mathbf{g}^{(m,k_m)}), m = 1, 2, \ldots$

converges) > 0. But each $\mathbf{I}_2(\mathbf{g}^{(m,k_m)})$ is a measurable quadratic form in $\{M(t), t \geqslant 0\}$. Applying Lemma 4.1 and arguing as in the proof of Theorem 4.1, we conclude

$$P(\mathbf{I}_2(\mathbf{g}^{(m,k_m)}), m = 1, 2, ..., \text{converges}) = 1. \tag{4.6}$$

Now (4.5) and (4.6) imply

$$P(\mathbf{I}_2(\mathbf{f}^{(m)}), m = 1, 2, ..., \text{converges}) = 1,$$

proving the proposition. ∎

A. Appendix

As indicated in the introductory Section 1, we shall now show that $\mathbf{I}_n(\mathbf{f})$ is uniquely defined by (1.3). The argument, in the case $S = \mathbf{R}$, is similar to the one given in Dunford and Schwartz [DS58, Chap. IV.10], for the case of Banach space-valued vector measures. Simply, we treat $M^{(n)}$ as an $L^q(\Omega)$-valued vector measure, $0 \leqslant q < \alpha$. The fact that $L^q(\Omega)$ is not a Banach space if $q < 1$ is not relevant here, since all we need is the fact which follows from Theorem 5.4 of Krakowiak and Szulga [KS88c], that the semivariation $|M^{(n)}|_q$ defined by

$$|M^{(n)}|_q\,(C) = \sup\left\{\left[E\left|\sum_{j=1}^{k} a_j M^{(n)}(A_j \cap C)\right|^q\right]^{1/q}\right\}$$

is finite for every symmetric Borel set C of finite $m^{(n)}$ measure which does not intersect the diagonals. The above supremum is taken over all possible finite disjoint partitions $\{A_j\}_{j=1}^k$ of $\mathbf{R}^n$, A_j's being Borel symmetric sets, and all real sequences $\{a_j\}_{j=1}^k$ with $|a_j| \leqslant 1$, $k \geqslant 1$. We are now ready to give the proof of uniqueness of $\mathbf{I}_n(\mathbf{f})$.

Case $S = \mathbf{R}$. Suppose that $f^{(k)}$ and $g^{(k)}$, $k = 1, 2, ...$, satisfy conditions 1–3 of Section 1. Then as $k \to \infty$, $h^{(k)} - g^{(k)} \to 0$ in $m^{(n)}$ and $I_n(h^{(k)}\mathbf{1}_C)$ converges in probability to some random variable $X(C)$ for any symmetric set C of $\mathbf{R}^n$ which does not intersect the diagonals. We need to prove that $X(C) = 0$ a.s. To this end we may assume that $h^{(k)} \to 0$, $m^{(n)}$—a.e. By Egoroff's theorem there is a partition $C_0, C_1, ...$ of the set C into symmetric Borel sets of finite $m^{(n)}$ measure which do not intersect the diagonals and such that $h^{(k)} \to 0$ uniformly on each C_i, $i \geqslant 1$, and $m^{(n)}(C_0) = 0$. Since for every simple function $h^{(k)}$ and $i \geqslant 1$,

$$\left[E\left|\int_{C_i} h^{(k)} dM^{(n)}\right|^q\right]^{1/q} \leqslant |M^{(n)}|_q\,(C_i)\;\|h^{(k)}\mathbf{1}_{C_i}\|_\infty$$

and the right side tends to zero as $k \to \infty$, we get that $X(C_i) = 0$ for every $i \geq 1$. On the other hand, by the Vitali–Hahn–Saks theorem $X(C) = \sum_{i=0}^{\infty} X(C_i)$ in $L^q(\Omega)$, hence $X(C) = X(C_0) = \text{plim}_{k \to \infty} I_n(h^{(k)} \mathbf{1}_{C_0}) = 0$.

General case (S is a separable Banach space). Let $\mathbf{h}^{(k)} = \mathbf{f}^{(k)} - \mathbf{g}^{(k)}$, where, as before, $\mathbf{f}^{(k)}$ and $\mathbf{g}^{(k)}$ are two sequences of simple functions satisfying conditions 1–3 of Section 1. Then, as above, we have $\mathbf{h}^{(k)} \to 0$ in $m^{(n)}$ and $\mathbf{I}_n(\mathbf{h}^{(k)} \mathbf{1}_C) \to \mathbf{X}(C)$ in probability. Let $x' \in S'$. Then $\mathbf{I}_n(\langle x', \mathbf{h}^{(k)} \rangle \mathbf{1}_C) = \langle x', \mathbf{I}_n(\mathbf{h}^{(k)} \mathbf{1}_C) \rangle \to \langle x', \mathbf{X}(C) \rangle$, and since $\langle x', \mathbf{h}^{(k)} \rangle \to 0$ in $m^{(n)}$ we obtain by the case $S = \mathbf{R}$ that $\langle x', \mathbf{X}(C) \rangle = 0$ a.s. Hence $\mathbf{X}(C) = 0$ a.s. by the weak * separability of S' which concludes the proof.

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References

[CR73] Cambanis, S., and Rajput, B. S. (1973). Some zero–one laws for Gaussian processes. *Ann. Probab.* **1** 304–312.

[DeA76] DeAcosta, A. (1976). Quadratic zero–one laws for Gaussian measures and the distribution of quadratic forms. *Proc. Amer. Math. Soc.* **54** 319–325.

[DK74] Dudley, R. M., and Kanter, M. (1974). Zero–one laws for stable measures. *Proc. Amer. Math. Soc.* **45** 245–252.

[Doo53] Doob, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[DS58] Dunford, N., and Schwartz, J. T. (1958). *Linear Operators, Part 1: General Theory.* Wiley, New York.

[Jan84] Jansen, A. (1984). A survey about zero–one laws for probability measures on linear spaces and locally compact groups. In *Lecture Notes in Math.*, Vol. 1064, pp. 551–553. Springer-Verlag, New York/Berlin.

[Kal70] Kallianpur, G. (1970). Zero–one laws for Gaussian processes. *Trans. Amer. Math. Soc.* **149** 199–211.

[Kal73] Kallenberg, O. (1973). Series of random processes without discontinuities of the second kind. *Ann. Probab.* **2** 729–737.

[KS88a] Krakowiak, W., and Szulga, J. (1988). Hypercontraction principle and random multilinear forms. *Probab. Theory Related Fileds* **77** 325–342.

[KS88b] Krakowiak, W., and Szulga, J. (1988). A multiple stochastic integral with respect to a strictly p-stable measure. *Ann. Probab.* **16** 764–777.

[KS88c] Krakowiak, W., and Szulga, J. (1988). On a p-stable multiple integral, I. *Probab. Theory Related Fields* **78** 437–448.

[KW87] Kwapień, S., and Woyczyński, W. A. (1987). Double stochastic integrals, random quadratic forms and random series in Orlicz spaces. *Ann. Probab.* **15** 1072–1096.

[LeP80] LePage, R. (1980). Multidimensional infinitely divisible variables and processes. Part I. Stable case. Preprint.

[MT86] McCONNELL, T., AND TAQQU, M. S. (1986). *Double Integration with Respect to Symmetric Stable Processes*. Technical Report 618, Cornell University.

[Ros86] ROSIŃSKI, J. (1986). On stochastic integral representation of stable processes with sample paths in Banach spaces. *J. Multivariate Anal.* **20** 277–302.

[Ros89] ROSIŃSKI, J. (1989). On path properties of certain infinitely divisible processes. *Stochastic Process. Appl.* **33** 73–87.

[Ros90] ROSIŃSKI, J. (1990). An application of series representations to zero–one laws for infinitely divisible random vectors. In *Progress in Probability*, Vol. 21. Birkhäuser, Boston.

[RW86] ROSIŃSKI, J., AND WOYCZYŃSKI, W. A. (1986). On Itô stochastic integration with respect to p-stable motion: Inner clock, integrability of sample paths, double and multiple integrals. *Ann. Probab.* **14** 271–286.

[SS89] SAMORODNITSKY, G., AND SZULGA, J. (1989). An asymptotic evaluation of the tail of a multiple symmetric α-stable integral. *Ann. Probab.* **17** 1503–1520.

[ST90] SAMORODNITSKY, G., AND TAQQU, M. S. (1990). Multiple stable integrals of Banach-valued function. *J. Theoret. Probab.* **3** 267–287.

[ST91] SAMORODNITSKY, G., AND TAQQU, M. S. (1991). Construction of multiple stable measures and integrals using LePage representation. In *Stable Processes and Related Topics* (G. Samorodnitsky, S. Cambanis, and M. S. Taqqu, Ed.). Birkhäuser, Boston. To appear.

[Taq87] TAQQU, M. S. (1987). Random processes with long-range dependence and high variability. *J. Geophys. Res.* **92** 9683–9686.

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