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Pseudoisotropic Random Walks on Free Groups and Semigroups

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We present strong laws of large numbers, central limit theorems and laws of the iterated logarithm for the growth of some random walks on free groups and semigroups which generalize isotropic random walks. Our method to derive these results is based on corresponding limit theorems on polynomial hypergroups. If this method can be applied, then it has the advantage that expectation and variance appearing in the limit theorems can be computed explicitly.

INTRODUCTION

In this paper we present a method to derive limit theorems for the growth of certain random walks on free groups and semigroups by using corresponding limit theorems for polynomial hypergroups. In essence, the idea is the following: Suppose that there exists a Banach subalgebra M of the Banach algebra $M_b(\Gamma)$ of all bounded measures on a free group or semigroup Γ such that M admits a linear basis consisting of probability measures μ_n on Γ with $\operatorname{supp} \mu_n = \{w \in \Gamma \mid |w| = n\}$. It turns out that then M is commutative and isomorphic to a polynomial hypergroup algebra. We can consequently study the growth $|X_1 X_2 \cdots X_n|$ of the product of independent Γ-valued random variables X_n by regarding $(|X_1 X_2 \cdots X_n|)_{n \in \mathbb{N}}$ as a Markov chain on $\mathbb{N}_0$ associated with a suitable polynomial hypergroup.

Since this method depends entirely on the existence of a suitable M, it is clear that our method has the disadvantage that the results cannot be extended to arbitrary random walks on free groups or semigroups. Furthermore, it is not possible to obtain more general results such as that of Sawyer and Steger [15] who considered random walks for groups generated by a finite number of free involutions.

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On the other hand, when applicable, our method has some advantages in comparison to the results of Derriennic [3], Guivarc'h [6] and Sawyer and Steger [15], if we are interested in the behaviour of $|X_1 X_2 \cdots X_n|$. The first advantage is that the parameters in the limit theorems (which means expectation and variance) can be computed explicitly in a straightforward way and that the variance is in fact positive except for some trivial cases. The paper of Derriennic [3, p. 197] contains an expression for the expectation, but it seems impossible to immediately get explicit values (except for isotropic random walks). Sawyer and Steger (see [15, (6.5), (6.6)]) have explicit analytic expressions for expectation and variance of nearest neighbor random walks. But, only very few facts about the general case are known. For instance, Sawyer and Steger [15] do not give general conditions when the variance is positive. Further advantages of our method are that our limit theorems require very weak moment conditions, and that it is possible to derive limit theorems for random variables X_n which are not necessarily identically distributed.

This paper is organized as follows: In Section 1 we introduce pseudoisotropic random walks on free groups, formulate the associated limit theorems and give explicit formulas for expectation and variance of these random walks. Section 2 then contains the proofs of the results of Section 1 that are based on a reduction of the problem to limit theorems for random walks on polynomial hypergroups. A central step of this reduction is valid in a more general situation for random walks on discrete hypergroups. Since this reduction result is of independent interest, it will be discussed and proved in an appendix in the end of the paper. In Section 3 we transfer the methods used in Section 2 to a class of discrete semigroups and obtain limit theorems for pseudoisotropic random walks on such semigroups. These results generalize similar results of Soardi [16] slightly. Finally, in Section 4 we deal with the question to what free semigroups we can apply our method.

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1. Limit Theorems for Pseudoisotropic Random Walks on Free Groups

1.1. Let Γ be the free group generated by d generators $g_1, ..., g_d$. Denote the neutral element by e. Every word $w \in \Gamma$ can be written in a unique way as

$$w = g_{i_1}^{n_1} g_{i_2}^{n_2} \cdots g_{i_k}^{n_k} \quad \text{with} \quad k \in \mathbb{N}_0, i_1, ..., i_k \in \{1, ..., d\},$$
$$n_1, ..., n_k \in \mathbb{Z} - \{0\}, i_{l-1} \neq i_l \quad \text{for} \quad l \in \{2, ..., k\}. \tag{1.1}$$

Using this representation of $w \in \Gamma$, we define the length of w by $|w| := \sum_{l=1}^{k} |n_l|$. Let $M_b(\Gamma)$ be the Banach algebra of all bounded measures on Γ equipped with the convolution which is induced by the group multiplication on Γ.

Fix a constant $c > 0$ and numbers $s_1, ..., s_d, t_1, ..., t_d > 0$ such that

$$\sum_{k=1}^{d} (s_k + t_k) = 1 \qquad \text{and} \qquad s_1 t_1 = s_2 t_2 = \cdots = s_d t_d = c. \tag{1.2}$$

The arithmetic–geometric mean inequality implies that $0 < c \leqslant 1/(4d^2)$. Define positive measures $\mu_n \in M^+(\Gamma)$ $(n \in \mathbb{N}_0)$ as follows:

(a) μ_n is supported by $\{g \in \Gamma : |g| = n\}$.

(b) $\mu_n(\{g\}) = \prod_{l=1}^{k} s_{i_l}^{\max(n_l, 0)} \cdot t_{i_l}^{\max(-n_l, 0)}$ if $g \in \Gamma$ with $|g| = n$ has the representation (1.1) which determines the parameters $k, n_1, ..., n_k$ and $i_1, ..., i_k$ uniquely.

The closed subspace of $M_b(\Gamma)$ generated by the measures μ_n $(n \in \mathbb{N}_0)$ is denoted by $M_{s_1, ..., s_d, t_1, ..., t_d}$ or, in brief, M if there is no possible confusion with respect to the indices.

The purpose of this paper is to present limit theorems for the growth of random walks on Γ whose laws are contained in a common space M. This means the following: Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent Γ-valued random variables which are defined on a common probability space $(\Omega, \mathscr{A}, \mathbf{P})$ such that their distributions ν_n are contained in a common space M. We shall give limit theorems for $(|S_n|)_{n \in \mathbb{N}}$, where $S_n := X_1 X_2 \cdots X_n$ is the random walk of the partial products of the X_n. Following the notation of Soardi [16] who considers similar random walks on special semigroups (see also Section 3), we say that a random walk as introduced above is *pseudoisotropic*. In particular, if $s_1 = \cdots = s_d = t_1 = \cdots = t_d = 1/(2d)$, then we have the isotropic case which is considered in Sawyer [14] and Voit [19].

The restriction to investigating pseudoisotropic random walks is a result of our method to derive limit theorems. As shown in Section 2, the spaces $M_{s_1, ..., s_d, t_1, ..., t_d}$ are Banach subalgebras of $M_b(\Gamma)$ and $(|S_n|)_{n \in \mathbb{N}}$ is a Markov chain on $\mathbb{N}_0$. Using these facts, we shall forget the construction of $(|S_n|)_{n \in \mathbb{N}}$ and consider it as Markov chain on $\mathbb{N}_0$ whose transition probabilities are associated with a convolution structure which is connected with $M_{s_1, ..., s_d, t_1, ..., t_d}$ in a natural way. This convolution structure is a polynomial hypergroup (for details see Heyer [8], Lasser [11], or Voit [17–19]), and we can apply limit theorems for Markov chains on polynomial hypergroups (see Voit [17–21]) to derive the desired results. This method is also used in Soardi [16] and Voit [17, 19] in order to derive limit theorems for isotropic random walks on certain semigroups and infinite distance-transitive graphs. Soardi also studies some pseudoisotropic random walks.

In order to avoid a trivial case which has to be considered separately, we now assume that $d \geqslant 2$. In fact, $d=1$ means $\Gamma = \mathbb{Z}$, in which case the classical limit theorems for sums of independent, real valued random variables can be used. These theorems also work for $c=0$, which is formally excluded by our assumptions. This is a consequence of the fact that in this case no cancellation of symbols appears during our random multiplication, and therefore $|S_n| = \sum_{k=1}^{n} |X_k|$ holds.

Before we present the limit theorems, we define moment functions m_1 and m_2 which are needed to describe the parameters of these limit theorems explicitly.

1.2. For $n \in \mathbb{N}_0$ let

$$m_1(n) = n\left(1 - \frac{2(rw-1)}{(w-r)\,w^n + rw - 1}\right) - \frac{2(1-r)\,w(w^n-1)}{(w-1)[(w-r)\,w^n + rw - 1]} \tag{1.3}$$

and

$$m_2(n) = n^2 - 4n \cdot \frac{w(1-r)(w^n+1)}{(w-1)[(w-r)\,w^n + rw - 1]} + \frac{4(w+1)\,w(1-r)(w^n-1)}{(w-1)^2\,[(w-r)\,w^n + rw - 1]}, \tag{1.4}$$

where

$$r := \frac{1}{2d-1} \qquad \text{and} \qquad w := \frac{1+\sqrt{1-4c(2d-1)}}{1-\sqrt{1-4c(2d-1)}} > 1.$$

We consider two important special cases: For $n \in \{0, 1\}$ we have

$$m_1(0) = m_2(0) = 0, \qquad m_1(1) = \sqrt{1-4c(2d-1)}, \qquad m_2(1) = 1. \tag{1.5}$$

$m_1(1)$ and $m_2(1)$ are important for nearest neighbor random walks. Furthermore, for isotropic random walks on Γ we have $c = 1/(4d^2)$, $w = 1/r = 2d-1$ and thus

$$m_1(n) = n - \frac{2d-1}{2(d-1)d}(1 - (2d-1)^{-n}) \tag{1.6}$$

and

$$m_2(n)=n^2-\frac{2d-1}{(d-1)d}(1+(2d-1)^{-n})\cdot n$$
$$+\frac{2d-1}{(d-1)^2}(1-(2d-1)^{-n}). \tag{1.7}$$

Equations (1.6) and (1.7) are special cases of formulas for moments of isotropic random walks on infinite distance-transitive graphs (see Section 5.6 of Voit [19]). Equation (1.6) can be found also in Sawyer [14] and Derrienic [3].

The following theorems are proved in Section 2.

1.3. THEOREM. *Let the X_n be identically distributed with law $\nu \in M$. Then*

$$|S_n|/n \to \mu := E(m_1(|X_1|)) \qquad \textit{almost surely},$$

where $\mu \in [0, \infty]$ is possible.

Moreover, if in addition, $E(m_2(|X_1|))<\infty$ and $\nu \neq \delta_e$, then $\mu \in]0, \infty[$,

$$\sigma^2 := E(m_2(|X_1|)) - E(m_1(|X_1|))^2 \in]0, \infty[,$$

and $(|S_n| - n\mu)/\sqrt{n}$ tends in distribution to the normal distribution $N(0, \sigma^2)$.

Finally, if $E(|X_1|^3)<\infty$ and $\nu \neq \delta_e$, then $\sigma^2 \in]0, \infty[$,

$$\limsup_{n\to\infty} \frac{|\,|S_n|-n\mu|}{\sqrt{2\sigma^2 n\cdot \ln\ln n}}=1 \qquad \textit{almost surely},$$

and the set of cluster points of the sequence

$$(|\,|S_n|-n\mu|/\sqrt{2\sigma^2 n\cdot \ln\ln n})_{n\in\mathbb{N}}$$

is equal to $[-1, 1]$ with probability one.

1.4. THEOREM. *Assume that the distributions ν_n of X_n are not degenerate (i.e., $\neq \delta_e$). If $\sigma_n^2 := E(m_2(|X_n|)) - E(m_1(|X_n|))^2$, thrn $\sigma_n^2>0$, and we have the following results:*

(1) *If $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that $r_n \to \infty$ and $\sum_{n=1}^{\infty} \sigma_n^2/r_n^2<\infty$, then*

$$\lim_{n\to\infty} \frac{1}{r_n}\left(|S_n| - \sum_{k=1}^{n} E(m_1(|X_k|))\right)=0 \qquad \textit{almost surely}.$$

(2) *If* $s_n^2 := \sum_{k=1}^{n} \sigma_k^2$ *satisfies the Lindeberg condition*

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} E(m_2(|X_k|)\cdot \mathbf{1}_{\{|X_k|>\delta s_n\}})=0 \qquad \textit{for every} \quad \delta>0 \tag{1.8}$$

($\mathbf{1}_A$ *being the characteristic function of a measurable set* A), *and if*

$$\sup_{n\in\mathbb{N}} \frac{1}{s_n^2} \sum_{k=1}^{n} E(|X_k|^i)<\infty \qquad \textit{for} \quad i\in\{1,2\}, \tag{1.9}$$

then $(|S_n| - \sum_{k=1}^{n} E(m_1(|X_k|)))/s_n$ *converges in distribution to* $N(0, 1)$.

2. Pseudoisotropic Random Walks and Polynomial Hypergroups

In this section we shall show that, for every pseudoisotropic random walk $(S_n)_{n\in\mathbb{N}}$ on Γ, $(|S_n|)_{n\in\mathbb{N}}$ is a Markov chain on a suitable polynomial hypergroup. As a consequence, we shall derive Theorems 1.3 and 1.4 from limit theorems for polynomial hypergroups.

2.1. Proposition. *The measures* μ_n *of Section* 1 *satisfy the recursion formula*

$$\begin{aligned} &\mu_1 * \mu_1 = \mu_2 + 2dc\,\mu_0, \\ &\mu_1 * \mu_n = \mu_n * \mu_1 = \mu_{n+1} + (2d-1)c\,\mu_{n-1} \qquad (n\geqslant 2). \end{aligned} \tag{2.1}$$

In particular, M *is a commutative subalgebra of* $M_b(\Gamma)$.

Proof. We first show the recursion formula for $\mu_n * \mu_1$. To do this, we take $g\in\Gamma$ with $|g|=n+1$ and note that g has exactly one representation as $g=h\cdot w$ with $|h|=n$ and $|w|=1$. Therefore, by the definition of μ_n, it follows that

$$\mu_n * \mu_1(g) = \mu_{n+1}(g) \qquad \text{for} \quad n\in\mathbb{N}. \tag{2.2}$$

Now take $g\in\Gamma$ with $|g|=n-1$. Then g can be written as $g=h\cdot w$ with $|h|=n$ and $|w|=1$ if and only if h is of the form $h=gw^{-1}$, where the last character of g is not equal to w. Therefore, since $\mu_1(u)\,\mu_1(u^{-1})=c$ for all u with $|u|=1$, and since $\mu_n(gw^{-1})=\mu_{n-1}(g)\cdot\mu_1(w^{-1})$ in the case above, we conclude that

$$\mu_1 * \mu_1(g) = \sum_{w\in\Gamma, |w|=1} \mu_1(w)\,\mu_1(w^{-1}) = 2dc = 2dc\,\mu_0(g) \tag{2.3}$$

and, for $n \geqslant 2$,

$$\mu_n * \mu_1(g) = \sum_{w \in \Gamma, |w| = 1, w \text{ not last symbol of } g} \mu_1(w)\, \mu_n(gw^{-1})$$

$$= \sum_{\cdots} \mu_{n-1}(g)\, \mu_1(w^{-1})\, \mu_1(w) = (2d-1)c\, \mu_{n-1}(g). \qquad (2.4)$$

Equations (2.2)–(2.4) prove the recursion formula for $\mu_n * \mu_1$. Since the equations for $\mu_1 * \mu_n$ can be derived in a similar way, we omit details.

It follows from $\mu_0 = \delta_e$, (2.1), and induction that $\mu_m * \mu_n = \mu_n * \mu_m \in M$ for all $m, n \in \mathbb{N}$.

2.2. Using the basis $(\mu_n)_{n \in \mathbb{N}_0}$ of M, we define an isometric isomorphism π from M onto $M_b(\mathbb{N}_0)$, the Banach algebra of all bounded measures on $\mathbb{N}_0$, by $\pi(\mu_n) := \|\mu_n\| \cdot \delta_n$, δ_n being the point measure of $n \in \mathbb{N}_0$. π preserves probability measures, and, if $M_b(\mathbb{N}_0)$ is equipped with the convolution $\bullet$ inherited from M via π, $(M_b(\mathbb{N}_0), \bullet)$ becomes a Banach algebra isomorphic to M. The convolution product of probability measures on this algebra is always a probability measure.

Let us consider details of this convolution: Since $\|\mu_0\| = \|\mu_1\| = 1$, we have

$$\mu_1 * \mu_1 = \|\mu_2\| \cdot \frac{\mu_2}{\|\mu_2\|} + 2cd \cdot \mu_0$$

and

$$\mu_1 * \frac{\mu_n}{\|\mu_n\|} = \frac{\|\mu_{n+1}\|}{\|\mu_n\|} \cdot \frac{\mu_{n+1}}{\|\mu_{n+1}\|} + \frac{(2d-1)\, c\, \|\mu_{n-1}\|}{\|\mu_n\|} \cdot \frac{\mu_{n-1}}{\|\mu_{n-1}\|} \qquad (2.5)$$

for $n \geqslant 2$, and therefore,

$$\delta_1 \bullet \delta_1 = \|\mu_2\|\, \delta_2 + 2dc\delta_0$$

and

$$\delta_n \bullet \delta_1 = \frac{\|\mu_{n+1}\|}{\|\mu_n\|}\, \delta_{n+1} + \frac{\|\mu_{n-1}\|(2d-1)c}{\|\mu_n\|}\, \delta_{n-1} \qquad (n \geqslant 2). \qquad (2.6)$$

Let

$$\begin{aligned} &a_1 = \|\mu_2\|, \qquad c_1 = 2cd, \qquad a_n = \|\mu_{n+1}\|/\|\mu_n\|, \\ &c_n = (2d-1)c\, \|\mu_{n-1}\|/\|\mu_n\| \qquad \text{for} \quad n \geqslant 2. \end{aligned} \qquad (2.7)$$

We then have $a_n + c_n = 1$ for $n \in \mathbb{N}$, since the convolution product of probability measures is a probability measure. Define polynomials $(P_n)_{n\in\mathbb{N}_0}$ by

$$P_0(x):=1, \qquad P_1(x):=2x\cdot\sqrt{(2d-1)c},$$
$$P_1(x)\cdot P_n(x)=a_nP_{n+1}(x)+c_nP_{n-1}(x) \qquad (n\in\mathbb{N}). \tag{2.8}$$

Standard facts on orthogonal polynomials (see Chihara [2, Chap. I]) show that $(P_n)_{n\in\mathbb{N}_0}$ is a sequence of orthogonal polynomials. The choice of P_1 seems to be unnatural at first glance, but in Section 2.4 we shall see that this normalization ensures that the associated orthogonality measure is supported by $[-1, 1]$.

We shall next show that the study of pseudoisotropic random walks $|S_n|$ can be reduced to investigating Markov chains on a plynomial hypergroup.

2.3. Proposition. *The convolution $\bullet$ on $M_b(\mathbb{N}_0)$ defines a polynomial hypergroup which is associated with the orthogonal polynomials $(P_n)_{n\in\mathbb{N}_0}$. Moreover, if $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables on Γ with laws $\nu_n\in M$, and if $S_0=e$ and $S_n=X_1X_2\cdots X_n$ $(n\in\mathbb{N})$, then $(|S_n|)_{n\in\mathbb{N}_0}$ is a Markov chain on $\mathbb{N}_0$ with*

$$|S_0|=0, \qquad \mathbf{P}(|S_n|=k\,|\,|S_{n-1}|=l)=(\pi(\nu)\bullet\delta_l)(k) \qquad (k,l\in\mathbb{N}_0, n\in\mathbb{N}). \tag{2.9}$$

Proof. To check that $\bullet$ defines a polynomial hypergroup which is associated with $(P_n)_{n\in\mathbb{N}_0}$, we have to prove that, for $m,n\in\mathbb{N}_0$, the linearization formula $P_m\cdot P_n=\sum_{k=|m-n|}^{m+n} g_{m,n,k}P_k$ implies that $\delta_m\bullet\delta_n=\sum_{k=|m-n|}^{m+n} g_{m,n,k}\delta_k$. By (2.6) and (2.8), this is true for $m\in\{0,1\}$ and all $n\in\mathbb{N}_0$. Once this is true, it follows for $m>1$ by (2.6), (2.8), and induction.

The further assertion is a consequence of Proposition 2.1 and Theorem A.1 which is formulated and proved in the end of this paper.

2.4. We are now in the position to apply results for random walks on polynomial hypergroups (see [17–21]). However, in order to get explicit formulas for the moment functions which are defined in [17 or 19] in terms of derivations of the P_n, we first derive a suitable explicit representation of these polynomials. To do this, we set

$$Q_0(x):=1, \qquad Q_n(x):=\frac{a_1\cdot a_2\cdot\cdots a_{n-1}}{((2d-1)c)^{n/2}}\cdot P_n(x) \qquad (n\in\mathbb{N}).$$

Then, using $a_nc_{n+1}=(2d-1)c$ for $n\in\mathbb{N}$, we obtain

$$Q_0(x)=1,\ Q_1(x)=2x, \qquad Q_1(x)\,Q_1(x)=Q_2(x)+\frac{2d}{2d-1}\,Q_0(x)$$

and

$$Q_n(x)\,Q_1(x)=Q_{n+1}(x)+Q_{n-1}(x) \qquad (n\geqslant 2)$$

which implies

$$Q_n(\cos\theta)=\frac{\sin(n+1)\theta-(1/(2d-1))\sin(n-1)\theta}{\sin\theta} \qquad (\theta\in\mathbb{C},\theta\neq 0, n\geqslant 2) \tag{2.11}$$

(see, for example, Askey and Wilson [1, (4.29), (4.33)]). In particular, by [1, Eq. (4.31)], the orthogonality measure of the sequence $(P_n)_{n\in\mathbb{N}_0}$ is supported by $[-1,1]$ and has the Lebesgue density

$$(1-x^2)^{1/2}/(1-(2d-1)\,d^{-1}x^2). \tag{2.12}$$

We now use the Q_n to compute the a_n, c_n, and $\|\mu_n\|$ as well as the moment functions. For this, we define w as in Section 1.2 and observe that $\varphi_0:=\frac{1}{2}\ln w>0$ satisfies $\cosh\varphi_0=(2\sqrt{(2d-1)c})^{-1}>1$. Therefore, by (2.8), we have $P_n(\cosh\varphi_0)=1$ (for $n\in\mathbb{N}_0$) and

$$P_n(x)=\frac{1}{Q_n(\cosh\varphi_0)}\,Q_n(x) \qquad (n\in\mathbb{N}_0). \tag{2.13}$$

It follows that

$$\begin{aligned}
a_n&=\frac{Q_{n+1}(\cosh\varphi_0)}{Q_1(\cosh\varphi_0)\,Q_n(\cosh\varphi_0)}\\
&=\sqrt{(2d-1)c}\cdot\frac{\sinh(n+2)\,\varphi_0-r\sinh n\varphi_0}{\sinh(n+1)\,\varphi_0-r\sinh(n-1)\,\varphi_0}\\
&=\sqrt{(2d-1)c}\cdot e^{\varphi_0}+O(e^{-n})=\sqrt{(2d-1)c}\cdot\sqrt{w}+O(e^{-n})\\
&=\tfrac{1}{2}(1+\sqrt{1-4c(2d-1)})+O(e^{-n})
\end{aligned} \tag{2.14}$$

for $n\in\mathbb{N}$. Using the definition of a_n (see Section 2.1), we obtain

$$\|\mu_n\|=\frac{Q_n(\cosh\varphi_0)}{Q_1^n(\cosh\varphi_0)} \qquad (n\in\mathbb{N}). \tag{2.15}$$

The moment functions m_1 and m_2 were defined in Section 2.4 of Voit [19] by

$$m_i(n)=\left.\frac{d^i}{(d\varphi)^i}\,P_n(\cosh\varphi)\right|_{\varphi=\varphi_0} \qquad (i=1,2,\quad n\in\mathbb{N}_0).$$

If we use (2.11), (2.13), the definition of w and the abbreviation $r = 1/(2d-1)$, then a (longer) straightforward computation (or an application of a computer algebra program like MAPLE) leads to the explicit formulas (1.3) and (1.4).

We conclude the discussion of the P_n by noting that by (2.11) the Q_n and hence the P_n can be written as linear combinations of Tchebichef polynomials of the first kind with nonnegative coefficients. In other words, Property (T) required in Voit [17, 19, 21] is true in the situation here.

If we use Eq. (2.14) and Property (T), then Theorem 1.3 is a consequence of Theorem 3 in Voit [17], Theorem 2.9 in Voit [19], and Theorem 2.5(2) in Voit [21]. Part (1) of Theorem 1.4 follows from Corollary 2 in Voit [17] and part (2) from Theorem 2.9 in Voit [19]. In both cases we have to use the fact that $\sup_{n\in\mathbb{N}} |m_1(n)-n| < \infty$ (see Eq. (1.3)) which ensures that $m_1(|S_n|)$ can be replaced by $|S_n|$ in the limit results.

2.5. *Interpretation of the moment* m_1. Let $m\colon \Gamma \to \mathbb{R}$ be an additive function, i.e., $m(xy) = m(x) + m(y)$ for $x, y \in \Gamma$. The set of all additive functions is a vector space, and every additive function is determined uniquely by $m(g_1), ..., m(g_d)$. If we fix a Banach subalgebra $M = M_{s_1,...,s_d,t_1,...,t_d}$ as in Section 1, then the functions $\tilde{m}_k\colon \mathbb{N}_0 \to \mathbb{R}$, $\tilde{m}_k(n) := \sum_{|w|=n} m(w)^k \cdot \mu_n(w)/\|\mu_n\|$ satisfy the addition formula

$$\delta_i \bullet \delta_j(\tilde{m}_k) = \sum_{l=0}^{k} \binom{k}{l} \tilde{m}_l(i)\, \tilde{m}_{k-l}(j). \tag{2.16}$$

Since m_1 is determined uniquely by $m_1(1)$ and the addition formula $\delta_i * \delta_j(m_1) = m_1(i) + m_1(j)$ (see Voit [19, Section 2.4]), we have $\tilde{m}_1 = rm_1$, r being a constant. Therefore, up to the constant r, m_1 can be interpreted as the average of an arbitrary additive function on Γ. But, unfortunately, we do not have any good interpretation of r. Even worse, if M is the algebra of isotropic measures (i.e., $s_1 = \cdots = s_s = t_1 = \cdots = t_d$), then $\tilde{m}_1 \equiv 0$ and $\tilde{m}_2 = rm_1$ for every additive function $m \neq 0$ ($r > 0$, a constant depending on m). It is fairly unclear how this relation can be interpreted.

Additive functions on Γ can be used to derive further facts about random walks on free groups. For instance, if we define an additive function m by $m(g_i) = 1$ $(i = 1, ..., d)$, then $m(S_n)$ counts the difference of how often the generators $g_1, ..., g_d$ and their inverse elements, respectively, appear in S_n. Let $R_+(g)$ and $R_-(g)$ be the number of generators and their inverses, respectively, that appear in the reduced form of $g \in \Gamma$. Using $R_+(g) + R_-(g) = |g|$ and $R_+(g) - R_-(g) = m(g)$, we obtain the following strong law of large numbers from Theorem 1.3 and the strong law of large numbers for sums of independent real valued random variables.

2.6. THEOREM. *Let the random variables X_n be identically distributed on Γ with law $\nu \in M$. If $\mu := E(m_1(|X_1|)) < \infty$, then $\lambda := E(m(X_1)) \in \mathbb{R}$ and*

$$R_+(S_n)/n \to \tfrac{1}{2}(\mu+\lambda) \qquad \text{and} \qquad R_-(S_n)/n \to \tfrac{1}{2}(\mu-\lambda) \qquad \text{almost surely.}$$

3. PSEUDOISOTROPIC RANDOM WALKS ON SOME DISCRETE SEMIGROUPS

3.1. Let Γ be the discrete semigroup generated by the $D+E$ symbols $a_1, ..., a_D$ and $b_1, ..., b_E$ with F different (and consequently independent) relations

$$a_{i_k} b_{j_k} = e, \qquad k=1, ..., F,\ i_k \in \{1, ..., D\}, j_k \in \{1, ..., E\}, \tag{3.1}$$

where e is the neutral element of Γ. To avoid trivial cases, we assume that $E, D, F \geqslant 1$.

The purpose of this section is to derive limit theorems for random walks on Γ which are similar to that of Section 1. The family of semigroups we here investigate contains all semigroups which were considered by Soardi [16]. In fact, Soardi assumes $D=E=F$ and $a_i b_i = e$ $(i=1, ..., F)$.

Fix Γ as above. Every word $w \in \Gamma$ can be written as

$$\begin{aligned} w = g_1 \cdots g_k \qquad &\text{with} \quad k \in \mathbb{N}_0, g_i \in \{a_1, ..., a_D, b_1, ..., b_E\} \\ &(i=1, ..., k) \text{ and } g_i g_{i+1} \neq e \ (i=1, ..., k-1). \end{aligned} \tag{3.2}$$

If a word $w \in \Gamma$ has this unique reduced form, then $|w| := k$ is the length of w. Let $M_b(\Gamma)$ be the Banach algebra of all bounded measures on Γ equipped with the convolution which is induced by the semigroup multiplication.

Fix an arbitrary probability measure $\mu_1 \in M^1(\Gamma)$ with $\operatorname{supp} \mu_1 \subset \{w \in \Gamma : |w| = 1\}$ and define

$$c := \sum_{k=1}^{F} \mu_1(a_{i_k})\, \mu_1(b_{j_k}) \geqslant 0.$$

If $p := \sum_{i=1}^{D} \mu_1(a_i) \leqslant 1$, then

$$c \leqslant \sum_{i=1}^{D} \sum_{j=1}^{E} \mu_1(a_i)\, \mu_1(b_j) \leqslant p(1-p) \leqslant \frac{1}{4}$$

with equality if and only if $F = D \cdot E$ and $p = \frac{1}{2}$, in which case we in essentially have isotropic random walks on Soardi's semigroup S_1.

Now define positive measures $\mu_n \in M_b(\Gamma)$ $(n \in \mathbb{N}_0)$ with $\operatorname{supp} \mu_n \subset \{w \in \Gamma : |w| = n\}$ by $\mu_0 = \delta_e$ and

$$\mu_n(g_1 \cdot g_2 \cdots g_n) = \prod_{l=1}^{n} \mu_1(g_l), \tag{3.3}$$

where

$$|g_1 \cdot g_2 \cdots g_n| = n \qquad \text{and} \qquad g_1, ..., g_n \in \{a_1, ..., a_D, b_1, ..., b_E\}.$$

Using the methods of the proof of Proposition 2.1, we obtain the following result:

3.2. Proposition. The measures μ_n satisfy the recursion formula

$$\mu_1 * \mu_n = \mu_n * \mu_1 = \mu_{n+1} + c\mu_{n-1} \qquad (n \in \mathbb{N}). \tag{3.4}$$

In particular, the closed subspace M of $M_b(\Gamma)$ which is generated by the measures μ_n $(n \in \mathbb{N}_0)$ is a commutative subalgebra of $M_b(\Gamma)$.

3.3. Let (X_i) be a sequence of independent, Γ-valued random variables with distributions $\nu_n \in M$. Let $S_0 := e$ and $S_n := X_1 \cdot X_2 \cdot \cdots X_n$ $(n \in \mathbb{N})$ be the associated random walk. We say that such a random walk is pseudoisotropic with respect to M. We are interested in deriving limit theorems for $|S_n|$. If $c = 0$, then no cancellation can appear and $|S_n| = \sum_{l=1}^{n} |X_l|$ holds. Since this case can be completely treated by using classical results for sums of real independent random variables we shall now exclude it by assuming $c > 0$. We then can copy the methods of Section 2 and introduce the isometric isomorphism π from M onto $M_b(\mathbb{N}_0)$ by $\pi(\mu_n) := \|\mu_n\| \, \delta_n$ $(n \in \mathbb{N}_0)$. Let $\bullet$ be the convolution on $M_b(\mathbb{N}_0)$ such that π becomes a Banach algebra isomorphism. Using

$$\mu_1 * \frac{\mu_n}{\|\mu_n\|} = \frac{\|\mu_{n+1}\|}{\|\mu_n\|} \cdot \frac{\mu_{n+1}}{\|\mu_{n+1}\|} + \frac{c\,\|\mu_{n-1}\|}{\|\mu_n\|} \cdot \frac{\mu_{n-1}}{\|\mu_{n-1}\|} \qquad (n \in \mathbb{N}),$$

we obtain

$$\delta_n \bullet \delta_1 = a_n \delta_{n+1} + c_n \delta_{n-1}, \qquad \text{where} \quad a_n := \frac{\|\mu_{n+1}\|}{\|\mu_n\|},\; c_n := \frac{c\,\|\mu_{n+1}\|}{\|\mu_n\|} \tag{3.5}$$

for $n \in \mathbb{N}$. The associated orthogonal polynomials which are given by

$$P_0 := 1,\; P_1(x) := 2x \cdot \sqrt{c}, \qquad P_1 \cdot P_n = a_n P_{n+1} + c_n P_{n-1} \; (n \in \mathbb{N}) \tag{3.6}$$

are—up to normalization—Tchebichef polynomials of the second kind. More exactly, if

$$Q_0 = 1,\, Q_n(\cos\theta) = \sin(n+1)\,\theta/\sin\theta \qquad (\theta \in \mathbb{C},\, \theta \neq 0,\, n \in \mathbb{N}),$$

then $P_n(x) = Q_n(x)/Q_n((2\sqrt{c})^{-1})$ for $n \in \mathbb{N}_0$, where $(2\sqrt{c})^{-1} \geqslant 1$. Moreover, using the methods of the proof of Proposition 2.3, we obtain that $(M_b(\mathbb{N}_0), \bullet)$ is a polynomial hypergroup, that the $(P_n)_{n\in\mathbb{N}_0}$ are the associated orthogonal polynomials, and, finally, that $(|S_n|)_{n\in\mathbb{N}_0}$ is a Markov chain on $\mathbb{N}_0$ satisfying

$$\begin{gathered} |S_0| = 0, \\ \mathbf{P}(|S_n| = k \mid |S_{n-1}| = l) = (\pi(\nu_n) \bullet \delta_l)(k) \; (k, l \in \mathbb{N}_0, n \in \mathbb{N}). \end{gathered} \tag{3.7}$$

3.4. It turns out that we have to consider the cases $c \in [0, \frac{1}{4}[$ and $c = \frac{1}{4}$ separately in order to derive limit theorems for $|S_n|$ (see also Remark 5 in Soardi [16]). Let us first assume that $c \in\,]0, \frac{1}{4}[$. Then we obtain exactly the same limit theorems as stated in Theorems 1.3 and 1.4, where only the moments m_1 and m_2 have to be adapted to the present situation. Using the definitions of m_1 and m_2 in terms of derivatives of the P_n (see Voit [19, Section 2.4]), we obtain

$$m_1(n) = n\left(1 + \frac{2}{w^{n+1} - 1}\right) - \frac{2w(w^n - 1)}{(w-1)(w^{n+1} - 1)} \tag{3.8}$$

and

$$m_2(n) = n^2 - \frac{4(w^n + 1)n}{(w-1)(w^{n+1} - 1)} + \frac{4(w+1)\,w(w^n - 1)}{(w-1)^2\,(w^{n+1} - 1)}, \tag{3.9}$$

where $w := (1 + \sqrt{1-4c})/(2\sqrt{c})$. In particular, we have $m_1(1) = \sqrt{1-4c}$ and $m_2(1) = 1$. Therefore, if $(S_n)_{n\in\mathbb{N}_0}$ is an arbitrary nearest neighbor random walk with law μ_1, then the assertions of Theorem 1.3 hold with expectation $\sqrt{1-4c}$ and variance $4c$, where $c := \sum_{k=1}^{F} \mu_1(a_{i_k})\, \mu_1(b_{j_k})$.

We next present some limit results for $c = \frac{1}{4}$. In essence, this case was treated by Soardi [16]. While the central limit theorem agrees with Soardi's, our strong law of large numbers slightly improves the corresponding results in [16]. The law of the iterated logarithm is new.

3.5. Theorem. *Let $c = \frac{1}{4}$. Let X_n be independent, identically distributed Γ-valued random variables with law $\nu \in M$. If $E(|X_1|^\lambda) < \infty$ for a constant $\lambda \in [1, 2[$, then $|S_n|/n^{(3-\lambda)/2} \to 0$ for $n \to \infty$ almost surely. Moreover, if $\nu \neq \delta_e$, and if*

$$m_2(n) := \frac{n(n+2)}{3} \quad \textit{for} \quad n \in \mathbb{N}_0 \quad \textit{and} \quad \sigma^2 := E(m_2(|X_1|)) \in\,]0, \infty[,$$

then $S_n/\sqrt{\sigma^2 n}$ *converges in distribution to the Rayleigh distribution* $\rho_{1/2}$ *on* $[0, \infty[$ *which has the Lebesgue density* $\sqrt{2/\pi}\cdot x^2\cdot\exp(-x^2/2)$. *Finally, if in addition,* $E(|X_1|^4)<\infty$ *and* $\nu\neq\delta_e$, *then* $\sigma^2\in\,]0,\infty[$,

$$\limsup_{n\to\infty} S_n/\sqrt{2\sigma^2 n\ln\ln n}=1 \qquad \textit{almost surely},$$

and, with probability one, the set of cluster points of $(S_n/\sqrt{2\sigma^2 n\ln\ln n})_{n\in\mathbb{N}}$ *is equal to* $[0, 1]$.

Proof. The strong law of large numbers follows from Gallardo and Ries [5] (see also Theorem 5 in Voit [17]). The central limit theorem is a consequence of Theorem 3.1 in Gallardo [4] (see also Voit [18, Theorem 3.2]). Finally, the law of the iterated logarithm results from Theorem 1.5 in Voit [20]. A weaker version of this law was earlier stated in Gallardo [4].

We conclude this section by noting that the results of Mabrouki [13] yield an invariance principle for the central limit theorem in the situation of Theorem 3.5. We suggest that an analoguous principle exists for $c\in\,]0,\frac{1}{4}[$.

4. For Which Semigroups Do There Exist Associated Polynomial Hypergroups?

In this section we briefly discuss the limitations of the method to derive limit theorems on free groups or semigroups by using polynomial hypergroups. To do this, we consider a semigroup Γ which is generated by N symbols $a_1, ..., a_N$ with the relations

$$a_ia_j=e \qquad \text{where} \quad (i,j)\in H\subset\{1,...,N\}\times\{1,...,N\},\ H\neq\emptyset, \tag{4.1}$$

where e is as usual the neutral element of Γ. Assume without loss of generality that $a_i\neq a_j\neq e$ for all $i,j\in\{1,...,N\}$, $i\neq j$ (otherwise we could remove some generators).

To come to a classification of semigroups to which our methods apply, we ask for the existence of a sequence $(\mu_n)_{n\in\mathbb{N}_0}$ of probability measures on Γ with $\operatorname{supp}\mu_n=\{w\in\Gamma:|w|=n\}$ such that the Banach space $M\subset M_b(\Gamma)$ generated by the μ_n is closed under convolution. It follows by induction that M is generated by the n-fold convolution powers μ_1^n of μ_1 $(n\in\mathbb{N}_0)$. Consequently, the following proposition leads to the desired classification:

4.1. Proposition. *In the position above, there exists a probability measure* $\mu\in M^1(\Gamma)$ *with* $\operatorname{supp}\mu=\{a_1,...,a_N\}$ *such that the Banach space generated by the measures* $\mu^n|_{\{w\in\Gamma:|w|=n\}}$ $(n\in\mathbb{N}_0)$ *is a Banach subalgebra of*

$M_b(\Gamma)$ if and only if one of the following cases holds (after a possible permutation of the generators):

(1) $N = 2d + r$ $(d, r \geqslant 0)$ *and*

$$H = \{(1, d+1), (2, d+2), ..., (d, 2d), (d+1, 1), (d+2, 2), ..., (2d, d), (2d+1, 2d+1), (2d+2, 2d+2), ..., (N, N)\};$$

i.e., Γ is the free product of the free group F_d (which is generated by d elements; see Section 1) *and the group generated by r free involutions. Furthermore, the measure μ satisfies*

$$\mu(a_i)\,\mu(a_{i+d}) = \mu(a_j)^2 = c > 0 \qquad \text{for} \quad 1 \leqslant i \leqslant d,\ 2d+1 \leqslant j \leqslant N,$$

c being a constant.

(2) *Γ is a semigroup as considered in Section* 3 *with $N = D + E$. μ is an arbitrary probability measure with* supp $\mu = \{a_1, ..., a_N\}$.

Proof. Since $a_i a_j = a_j a_k = e$ $(i, j, k \in \{1, ..., N\})$ implies $a_i = a_i a_j a_k = a_k$ and therefore $i = k$, the set of generators splits into two parts: The first set S_1 consists of all elements which have a full (i.e., left and right) inverse element and which cannot satisfy any further relations of the type (4.1). The subsemigroup of Γ generated by S_1 is just a group as considered in part (1) of our proposition. The second set S_2 contains the generators which have no or no full inverse. Moreover, by our conclusion at the beginning of the proof, we obtain that no element of S_2 can be cancelled on the left side and on the right side at the same time. In orther words, the subsemigroup generated by S_2 is of the type considered in Section 3. In summary, the fundamental assumptions of the proposition yield that Γ is a free product of two subsemigroups which satisfy (1) and (2), respectively.

We next prove that the existence of a measure μ as claimed above implies that Γ is either of form (1) or of form (2). To do this, we first observe that $\mu^3|_{\{w \in \Gamma : |w| = 1\}} = c\mu$, $c > 0$ being a constant. Since

$$\begin{aligned}
\mu^3(a_i) &= ((\mu * \mu) * \mu)(a_i) \\
&= \mu(a_i) \cdot \mu * \mu(e) + \sum_{j,k,l \text{ with } a_l a_j \neq e,\, (a_l a_j) a_k = a_i} \mu * \mu(a_l a_j) \cdot \mu(a_k) \\
&= \mu(a_i) \cdot \mu * \mu(e) + \sum_{j,k \text{ with } a_i a_j \neq e,\, a_j a_k = e} \mu * \mu(a_i a_j) \cdot \mu(a_k) \\
&= \mu(a_i) \cdot \mu * \mu(e) + \mu(a_i) \cdot \sum_{j,k \text{ with } a_i a_j \neq e,\, a_j a_k = e} \mu(a_j) \cdot \mu(a_k)
\end{aligned}$$

for $i = 1, ..., N$, it follows that $H(a_i) := \sum_{a_i a_j \neq e, a_j a_k = e} \mu(a_j)\,\mu(a_k)$ is independent of i. If $S_2 \neq \emptyset$, then we can choose $a_i \in S_2$ such that $a_i a_j \neq e$ for all j which implies

$$H(a_i) = \sum_{j,k \text{ with } a_j a_k = e} \mu(a_j)\,\mu(a_k).$$

On the other hand, if $a_l \in S_1 \neq \emptyset$, then $H(a_l) = H(a_i) - \mu(a_l)\,\mu(a_l^{-1}) \neq H(a_i)$ which proves that either $S_1 = \emptyset$ or $S_2 = \emptyset$. Moreover, if $S_2 = \emptyset$, then the independence of $H(a_i)$ also ensures that $\mu(a_l)\,\mu(a_l^{-1})$ does not depend on l which concludes the proof of the necessity.

The converse statement follows from Proposition 3.2 and an obvious generalization of Proposition 2.3. We omit details.

4.2. *Remark.* Proposition 4.1 classifies random walks on semigroups which can be analyzed by using polynomial hypergroups. In view of our limit theorems for random walks, the class of groups contained in part (1) is a trivial extension of the class of free groups (the same orthogonal polynomials and the same moment functions appear). Therefore, we have, in essence, discussed all possible examples which can be treated using polynomial hypergroups. But this statement is only true under the assumption that there exist no further relations for the generators which are independent of the relations of type (4.1). For instance, if we permit relations of the type $a_i a_j = a_k$, then many further examples appear. Examples are discrete groups whose Cayley graphs are infinite distance-transitive graphs (see Voit [19]). Further examples are discussed in Section 5 of Voit [17]. Since we are still not able to give a complete list of possible examples, we omit a discussion here.

Appendix: Random Walks on Discrete Hypergroups

In this appendix we prove the following general result for random walks on a discrete hypergroup or discrete semihypergroup with neutral element. In Sections 2 and 3, this theorem is applied to free groups or semigroups, respectively. For details on (semi)hypergroups see Jewett [9] who uses the term (semi)convo.

A.1. Theorem. *Let K be a countable discrete hypergroup (with neutral element e and involution $^-$), and H a discrete space. Let $f: K \to H$ be a surjective mapping and $(\mu_x)_{x \in H}$ a family of probability measures on K such that*

(1) $f^{-1}(f(e)) = \{e\}$,

(2) $\operatorname{supp} \mu_x = f^{-1}(x)$ *and* $|f^{-1}(x)| < \infty$ *for all* $x \in H$,

(3) *if* $f(u)=f(v)$ *for* $u, v\in K$, *then* $f(\bar{u})=f(\bar{v})$, *and*

(4) *the Banach space* M *in* $M_b(K)$ *generated by* $\{\mu_x : x\in H\}$ *is closed under convolution.*

Then

$$\delta_x \bullet \delta_y := f(\mu_x * \mu_y)$$

defines a hypergroup structure on H. *The Banach algebra* $(M_b(H), \bullet)$ *is isomorphic to* M *via* f. *Moreover, if* $(S_n)_{n\in\mathbb{N}_0}$ *is a Markov chain on* K *with*

$$S_0=e, \qquad \mathbf{P}(S_{n+1}=u \mid S_n=v)=\delta_v * \nu_n(u) \qquad (n\in\mathbb{N}_0, u, v\in K),$$

the ν_n *being probability measures contained in* M, *then* $(f(S_n))_{n\in\mathbb{N}_0}$ *is a Markov chain on* H *with*

$$f(S_0)=f(e),$$
$$\mathbf{P}(f(S_{n+1})=x \mid f(S_n)=y)=\delta_y \bullet f(\nu_n)(x) \qquad (n\in\mathbb{N}_0, x, y\in H).$$

The assertions above remain valid when the term "hypergroup" is replaced by "semihypergroup with neutral element" and condition (3) is omitted.

Proof. Since the measures $\mu_x * \mu_y$ are finitely supported, and $e\in\operatorname{supp}(\mu_x * \mu_y)$ if and only if $f^{-1}(x)=(f^{-1}(y))^-$, $(H, \bullet)$ is obviously a hypergroup. The corresponding result for semihypergroups is trivial.

To check the assertion concerning Markov chains, we write the measures ν_n as $\nu_n=\sum_{z\in H} c_{n,z}\mu_z$ and note

$$\begin{aligned}
\sum_{u\in K} \mu_x(u)\cdot\delta_u * \nu_n(v) = \mu_x * \nu_n(v) &= \sum_{z\in H} c_{n,z}(\mu_x * \mu_z)(v) \\
&= \sum_{z\in H} c_{n,z}(\delta_x \bullet \delta_z)(f(v))\cdot\mu_{f(v)}(v) \\
&= (\delta_x \bullet f(\nu_n))(f(v))\cdot\mu_{f(v)}(v) \\
&= \sum_{x\in H} (\delta_x \bullet f(\nu_n))(y)\cdot\mu_y(v)
\end{aligned}$$

for all $v\in K$ and $x\in H$. The proof is completed by the following general result about Markov chains.

A.2. LEMMA. *Let* K *and* H *be countable discrete sets and* $f\colon K\to H$ *be a surjective mapping. Let* $(S_n)_{n\in\mathbb{N}_0}$ *be a Markov chain on* K *with transition matrices* $p_n(x, y)$ $(n\in\mathbb{N}_0)$ *and with* $S_0=u_0\in K$, u_0 *satisfying* $f^{-1}(f(u_0))=$

$\{u_0\}$. *Suppose there are probability measures* $(\mu_x)_{x \in H}$ *on* K *and constants* $\tilde{p}_n(x, y) \geqslant 0$ $(x, y \in H,\ n \in \mathbb{N}_0)$ *such that*

(1) $\operatorname{supp} \mu_x \subset f^{-1}(x)$ *for all* $x \in H$, *and*

(2) $\sum_{u \in K} \mu_x(u)\, p_n(u, v) = \sum_{y \in H} \tilde{p}_n(x, y)\, \mu_y(v)$ *for all* $x \in H$, $v \in K$, *and* $n \in \mathbb{N}_0$.

Then $(f(S_n))_{n \in \mathbb{N}_0}$ *is a Markov chain on* H *with transition matrices* $\tilde{p}_n(x, y)$.

Proof. Take $x_1, ..., x_n \in H$. Then, by assumption, $p_0(u_0, v_1) = \tilde{p}_0(f(u_0), x_1) \cdot \mu_{x_1}(v_1)$ whenever $v_1 \in f^{-1}(x_1)$, and

$$\sum_{v_i \in f^{-1}(x_i)} \mu_{x_i}(v_i) \cdot p_i(v_i, v_{i+1}) = \tilde{p}_i(x_i, x_{i+1}) \cdot \mu_{x_{i+1}}(v_{i+1})$$

for $i = 1, ..., n-1$ and $v_{i+1} \in f^{-1}(x_{i+1})$. Therefore,

$$\begin{aligned} \mathbf{P}(f(S_1) = x_1, f(S_2) = x_2, ..., f(S_n) = x_n) &= \sum_{v_i \in f^{-1}(x_i),\, 1 \leqslant i \leqslant n} p_0(u_0, v_1) \cdot p_1(v_1, v_2) \cdots p_{n-1}(v_{n-1}, v_n) \\ &= \tilde{p}_0(f(u_0), x_1) \cdot \tilde{p}_1(x_1, x_2) \cdots \tilde{p}_{n-1}(x_{n-1}, x_n) \end{aligned}$$

which ensures that $(f(S_n))_{n \in \mathbb{N}_0}$ is a Markov chain with transition matrices $\tilde{p}_n(x, y)$.

References

[1] Askey, R., and Wilson, J. A. (1985). Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Mem. Amer. Math. Soc.* **54**, No. 319.

[2] Chihara, T. S. (1978). *An Introduction to Orthogonal Polynomials.* Gordon & Breach, New York.

[3] Derriennic, Y. (1980). Quelques applications du théorème ergodique sousadditif. *Astérique* **74** 183–201.

[4] Gallardo, L. (1984). Comportement asymptotique des marches aléatoires associées aux polynomes de Gegenbauer. *Adv. Appl. Probab.* **16** 293–323.

[5] Gallardo, L., and Ries, V. (1979). La loi de grands nombres pour les marches aléatoires sur le dual de $SU(2)$. *Studia Math.* **66** 93–105.

[6] Guivarc'h, Y. (1980). Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire. *Astérique* **74** 47–98.

[7] Guivarc'h, Y., Keane, M., and Roynette, B. (1977). *Marches aléatoires sur les Groupes de Lie.* Lecture Notes in Math., Vol. 624, Springer-Verlag, Berlin/Heidelberg/New York.

[8] Heyer, H. (1984). Probability theory on hypergroups: A survey. *Proceedings, Conf., Oberwolfach, 1983*, pp. 481–550. Lecture Notes in Math., Vol. 1064. Springer-Verlag, Berlin/Heidelberg/New York.

[9] Jewett, R. I. (1975). Spaces with an abstract convolution of measures. *Adv. in Math.* **18** 1–101.

[10] Kingman, J. F. C. (1968). The ergodic theory of subadditive processes. *J. Roy. Statist. Soc. Ser. B* **30** 499–510.

[11] Lasser, R. (1983). Orthogonal polynomials and hypergroups. *Rend. Math. Appl.* **2** 185–209.

[12] Levit, B., and Molchanov, S. (1971). Invariant chains on free groups with a finite number of generators. *Vestnik Moscow Univ.* **6** 80–88. [Russian]

[13] Mabrouki, M. (1984). Principe d'invariance pour les marches aléatoires associées aux polynomes de Gegenbauer et applications. *C. R. Acad. Sci. Paris Ser. I Math.* **299** 991–994.

[14] Sawyer, S. (1978). Isotropic random walks in a tree. *Z. Wahrsch. Verw. Gebiete* **42** 279–292.

[15] Sawyer, S., and Steger, T. (1987). The rate of escape for an anisotropic random walk in a tree. *Probab. Theory Related Fields* **76** 207–230.

[16] Soardi, P. M. (1989). Limit theorems for random walks on discrete semigroups related to nonhomogeneous trees and Chebyshef polynomials. *Math. Z.* **200** 313–327.

[17] Voit, M. (1990). Laws of large numbers for polynomial hypergroups and some applications. *J. Theoret. Probab.* **3** 245–266.

[18] Voit, M. (1990). Central limit theorems for a class of polynomial hypergroups. *Adv. Appl. Probab.* **22** 68–87.

[19] Voit, M. (1990). Central limit theorems for random walks on $\mathbb{N}_0$ that are associated with orthogonal polynomials. *J. Multivariate Anal.* **34** 290–322.

[20] Voit, M. (1990). Laws of the iterated logarithm for a class of polynomial hypergroups. *Mh. Math.* **109** 311–327.

[21] Voit, M. (1991). A law of the iterated logarithm for Markov chains on $\mathbb{N}_0$ which are associated with a sequence of orthogonal polynomials. *Math. Ann.*, submitted.

[22] Zeuner, Hm. (1989). Laws of large numbers for hypergroups on $\mathbb{R}_+$. *Math. Ann.* **283** 257–678.

[23] Zeuner, Hm. (1989). The central limit theorem for Chebli–Trimeche hypergroups. *J. Theoret. Probab.* **2** 51–63.