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On the Distributions of Some Test Criteria for a Covariance Matrix under Local Alternatives and Bootstrap Approximations

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The asymptotic distribution of some test criteria for a covariance matrix are derived under local alternatives. Except for the existence of some higher moments, no assumption as to the form of the distribution function is made. As an illustration, a case of t distribution included normal model is considered and the power of the likelihood ratio test and Nagao's test for sphericity, as described in Srivastava and Khatri and Anderson, is computed. Also, the power is computed using the bootstrap method. In the case of t distribution, the bootstrap approximation does not appear to be as good as the one obtained by the asymptotic expansion method.

1. INTRODUCTION

Let $p \times 1$ vectors $X_1, ..., X_N$ be a random sample of size N on the vector X having a continuous multivariate distribution function with pdf $h(x)$, mean vector μ, and covariance matrix Σ. Suppose we are interested in testing the hypothesis $H_0: \Sigma \in \omega$ against the alternative $H_1: \Sigma \in \mathbf{\Omega} - \omega$, where $\mathbf{\Omega}$ stands for the set of all $p \times p$ positive definite matrices and ω is a subset of $\mathbf{\Omega}$ with $\frac{1}{2} p(p+1) - r(T)$ unknown parameters. This problem contains many tests of problems hypotheses on the covariance matrix Σ such as $H_0: \Sigma = I_p$, or $H_0: \Sigma = \sigma^2 I_p$, etc. For these problems, we consider test statistics of the form

$$T_n = nT(S_N), \tag{1.1}$$

where

$$nS_N = \sum_{i=1}^{N} (X_i - \bar{X})(X_i - \bar{X})', \ \bar{X} = N^{-1} \sum_{i=1}^{N} X_i, \text{ and } n = N - 1. \tag{1.2}$$

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For example, in the case of testing for sphericity $\Sigma = \sigma^2 I_p$, T_n could be taken as the likelihood ratio test or Nagao's test for the normal model as given in Srivastava and Khatri [13] or Anderson [1].

The asymptotic non-null distribution of T_n for fixed alternative H_1 was given by Chan and Srivastava [4] without assuming any form for the distribution function of the random vector X (except for the existence of some higher moments). The first term of this asymptotic distribution is a normal distribution. However, such asymptotic expansions do not provide good approximation for alternatives close to the null hypothesis, as was shown by Chan and Srivastava [4]. In this paper, we give an asymptotic distribution of T_n under local alternatives, the first term of which can be expressed as a weighted sum of independent noncentral χ^2 variates. As an illustration, we compute the power of the likelihood ratio test and Nagao's test for sphericity under local alternatives, assuming normality and t distribution. We also compute the power by the bootstrap method.

2. Stochastic Expansion of T_n and Distribution of Logarithmic Transformation of S_N

Without loss of generality, we can assume $E(S_N) = \Sigma_n = I + (1/\sqrt{n})\,\Theta$, as local alternatives, where Θ is fixed symmetric matrix and the rank of Θ is $r(T)$. To derive the asymptotic distribution, we assume

$$\begin{aligned} &\text{(i)} \quad T(I) = 0, \\ &\text{(ii)} \quad \partial T(I)/\partial s_{ab} = 0, \\ &\text{(iii)} \quad F = (\partial^2 T(I)/\partial s_{ab}\,\partial s_{cd}) \text{ is positive semi-definite} \\ &\qquad\quad \text{matrix with rank } r(T). \end{aligned} \tag{2.1}$$

In the above and in the rest of the paper the derivatives such as $\partial T(S)/\partial s_{\alpha\beta}$ evaluated at $S = I$ will be denoted by $\partial T(I)/\partial s_{\alpha\beta}$, etc. Let

$$\begin{aligned} S_N &= \Sigma_n^{1/2} \exp\left(\sqrt{2/n}\, Z\right) \Sigma_n^{1/2}, \\ T_n &= nT(S_N), \end{aligned} \tag{2.2}$$

where $A^{1/2}$ is a symmetric matrix such that $A = A^{1/2} A^{1/2}$. Such a matrix always exists for a positive semi-definite matrix. Expanding the exponential term in S_N and $\Sigma_n^{1/2} = (I + (1/\sqrt{n})\,\Theta)^{1/2}$, we find that, under assumptions (i) and (ii), T_n can be expanded as

$$
\begin{aligned}
T_N = nT(S_N) = & \sum_{a\leqslant b} \sum_{c \leqslant d} T_{ab:cd}(I)\left(z_{ab}+\frac{1}{\sqrt{2}}\theta_{ab}\right) \\
& \times\left(z_{cd}+\frac{1}{\sqrt{2}}\theta_{cd}\right)+\frac{1}{\sqrt{n}}\left\{\sqrt{2}\sum_{a\leqslant b}\sum_{c\leqslant d} T_{ab:cd}(I)\right. \\
& \times\left(z_{ab}+\frac{1}{\sqrt{2}}\theta_{ab}\right)\left[\sum_{\alpha=1}^{p}\left(z_{c\alpha}+\frac{1}{\sqrt{2}}\theta_{c\alpha}\right)\left(z_{\alpha d}+\frac{1}{\sqrt{2}}\theta_{\alpha d}\right)\right. \\
& \left.\left.-\frac{1}{2}\sum_{\alpha=1}^{p}\theta_{c\alpha}\theta_{\alpha d}\right)\right]+\frac{\sqrt{2}}{3}\sum_{a\leqslant b}\sum_{c\leqslant d}\sum_{e\leqslant f} T_{ab:cd:ef}(I) \\
& \times\left(z_{ab}+\frac{1}{\sqrt{2}}\theta_{ab}\right)\left(z_{cd}+\frac{1}{\sqrt{2}}\theta_{cd}\right) \\
& \left.\times\left(z_{ef}+\frac{1}{\sqrt{2}}\theta_{ef}\right)\right\}+O_p(n^{-1}),
\end{aligned}
\tag{2.3}
$$

where $Z=(z_{ab})$ and $\Theta=(\theta_{ab})$. To obtain the asymptotic distribution of Z which is a logarithmic transformation of S_N, first we give an Edgeworth expansion of the distribution of $V=(v_{ab})=\sqrt{n/2}\,(\Sigma_n^{-1/2} S_N \Sigma_n^{-1/2}-I)$. The characteristic function of $v=(v_{11}, ..., v_{pp}, v_{12}, ..., v_{p-1,p})'$ is given by

$$
\begin{aligned}
& E\left[\exp\left(i\sum_{a\leqslant b} t_{ab}v_{ab}\right)\right] \\
& \quad = 1+iE\left(\sum_{a\leqslant b} t_{ab}v_{ab}\right)-\frac{1}{2}E\left(\sum_{a\leqslant b} t_{ab}v_{ab}\right)^2-\frac{1}{3!}E\left(\sum_{a\leqslant b} t_{ab}v_{ab}\right)^3+\cdots \\
& \quad = 1-\frac{1}{2}E\left(\sum_{a\leqslant b} t_{ab}v_{ab}\right)^2-\frac{i}{3!}E\left(\sum_{a\leqslant b} t_{ab}v_{ab}\right)^3+\cdots,
\end{aligned}
\tag{2.4}
$$

where $t=(t_{11}, ..., t_{pp}, t_{12}, ..., t_{p-1,p})$ is a real vector and $i^2=-1$. To evaluate the above expressions, we need the following assumption on moments of v_{ab} (or s_{ab}), in addition to assumptions (i)–(iii) given above:

$$
\begin{aligned}
\text{(iv)}\quad & Ev_{ab}v_{cd}=c_{ab:cd}+O(n^{-1}) \\
& Ev_{ab}v_{cd}v_{ef}=\sqrt{2/n}\,d_{ab:cd:ef}+O(n^{-3/2}).
\end{aligned}
\tag{2.5}
$$

Then we have

$$
\begin{aligned}
E&\left\{\exp\left(i \sum_{a \leqslant b} t_{ab} v_{ab}\right)\right] \\
&= \exp\left(-\frac{1}{2} \sum_{a \leqslant b} \sum_{c \leqslant d} t_{ab} t_{cd} c_{ab:cd}\right) \\
&\quad \times \left\{1 - \frac{\sqrt{2}\, i}{6\sqrt{n}} \sum_{a \leqslant b} \sum_{c \leqslant d} \sum_{e \leqslant f} d_{ab:cd:ef} t_{ab} t_{cd} t_{ef} + O(n^{-1})\right\}. \qquad (2.6)
\end{aligned}
$$

Thus the first term of the asymptotic distribution of v is a $\frac{1}{2}p(p+1)$ variate normal distribution with mean 0 and covariance matrix $C = (c_{ab:cd})$.

In order to obtain the distribution of v having the above characteristic function, let $h_0(v)$ be a distribution corresponding to the characteristic function $M_0(t) = \exp(-\frac{1}{2}\sum_{a \leqslant b}\sum_{c \leqslant d} t_{ab} t_{cd} c_{ab:cd})$:

$$
M_0(t) = \int \exp\left(i \sum_{a \leqslant b} t_{ab} v_{ab}\right) h_0(v) \prod_{a \leqslant b} dv_{ab}. \qquad (2.7)
$$

Integrating by parts, we obtain

$$
-it_{ab} M_0(t) = \int \exp\left(i \sum_{a \leqslant b} t_{ab} v_{ab}\right)\left(\frac{\partial}{\partial v_{ab}} h_0(v)\right) \prod_{a \leqslant b} dv_{ab}. \qquad (2.8)
$$

Similarly, we have

$$
it_{ab} t_{cd} t_{ef} M_0(t) = \int \exp\left(i \sum_{a \leqslant b} t_{ab} v_{ab}\right)\left(\frac{\partial^3 h_0(v)}{\partial v_{ab}\, \partial v_{cd}\, \partial v_{ef}}\right) \prod_{a \leqslant b} v_{ab}. \qquad (2.9)
$$

Now we have

$$
\begin{aligned}
\frac{\partial^3 h_0(v)}{\partial v_{ab}\, \partial v_{cd}\, \partial v_{ef}} = h_0(v) &\left\{ (g_{ab:ef}) \left(\sum_{g \leqslant h} g_{cd:gh} v_{gh}\right) \right. \\
&+ (g_{cd:ef}) \left(\sum_{g \leqslant h} g_{ab:gh} v_{gh}\right) + (g_{ab:cd}) \left(\sum_{g \leqslant h} g_{ef:gh} v_{gh}\right) \\
&\left. - \left(\sum_{g \leqslant h} g_{ab:gh} v_{gh}\right)\left(\sum_{g \leqslant h} g_{cd:gh} v_{gh}\right)\left(\sum_{g \leqslant h} g_{ef:gh} v_{gh}\right)\right\},
\end{aligned}
\qquad (2.10)
$$

with $G=(g_{ab:cd})=C^{-1}$. Thus the asymptotic distribution of v is given by

$$h_0(v)\Bigg[1+\frac{\sqrt{2}}{6\sqrt{n}}\sum_{a\leqslant b}\sum_{c\leqslant d}\sum_{e\leqslant f}d_{ab:cd:ef}$$
$$\times\Bigg\{\Bigg(\sum_{g\leqslant h}g_{ab:gh}v_{gh}\Bigg)\Bigg(\sum_{g\leqslant h}g_{cd:gh}v_{gh}\Bigg)\Bigg(\sum_{g\leqslant h}g_{ef:gh}v_{gh}\Bigg)$$
$$-(g_{ab:ef})\Bigg(\sum_{g\leqslant h}g_{cd:gh}v_{gh}\Bigg)-(g_{cd:ef})\Bigg(\sum_{g\leqslant h}g_{ab:gh}v_{gh}\Bigg)$$
$$-(g_{ab:cd})\Bigg(\sum_{g\leqslant h}g_{ef:gh}v_{gh}\Bigg)\Bigg\}+O(n^{-1})\Bigg]. \tag{2.11}$$

Since $V=\sqrt{n/2}(\exp(\sqrt{2/n}\,Z)-I)$, we have, as in Nagao [10],

$$\frac{\partial(V)}{\partial(Z)}=\left\{\prod_{i<j}\left(\frac{n}{2}\right)^{1/2}\left(\frac{\exp(\sqrt{2/n}\,\lambda_i)-\exp(\sqrt{2/n}\,\lambda_j)}{\lambda_i-\lambda_j}\right)\right\}\exp(\sqrt{2/n}\operatorname{tr}Z), \tag{2.12}$$

where λ_i is an eigenvalue of Z. Then we obtain

$$\frac{\partial(V)}{\partial(Z)}=1+\frac{\sqrt{2}}{2\sqrt{n}}(p+1)\operatorname{tr}Z+O(n^{-1}). \tag{2.13}$$

Since

$$v_{ab}=z_{ab}+\frac{\sqrt{2}}{2\sqrt{n}}\sum_{\alpha=1}^{p}z_{a\alpha}z_{\alpha b}+O(n^{-1}),$$

the asymptotic distribution of $z=(z_{11},\ldots,z_{pp},z_{12},\ldots,z_{p-1,p})$ is given by

$$h_0(z)\Bigg[1+\frac{\sqrt{2}}{\sqrt{n}}\Bigg\{\frac{1}{2}(p+1)\operatorname{tr}Z-\frac{1}{2}\sum_{a\leqslant b}\sum_{c\leqslant d}\sum_{\alpha=1}^{p}g_{ab:cd}z_{ab}z_{c\alpha}z_{\alpha d}$$
$$+\frac{1}{6}\sum_{a\leqslant b}\sum_{c\leqslant d}\sum_{e\leqslant f}d_{ab:cd:ef}\Bigg(\Bigg(\sum_{g\leqslant h}g_{ab:gh}z_{gh}\Bigg)$$
$$\times\Bigg(\sum_{g\leqslant h}g_{cd:gh}z_{gh}\Bigg)\Bigg(\sum_{g\leqslant h}g_{ef:gh}z_{gh}\Bigg)-(g_{ab:ef})$$
$$\times\Bigg(\sum_{g\leqslant h}g_{cd:gh}z_{gh}\Bigg)-(g_{cd:ef})\Bigg(\sum_{g\leqslant h}g_{ab:gh}z_{gh}\Bigg)-(g_{ab:cd})$$
$$\times\Bigg(\sum_{g\leqslant h}g_{ef:gh}z_{gh}\Bigg)\Bigg\}+O(n^{-1})\Bigg], \tag{2.14}$$

where $h_0(z)$ is a $p(p+1)/2$ variate normal distribution with mean 0 and covariance matrix $G^{-1}=(g_{ab:cd})^{-1}=C$.

3. Characteristic Function of Test Criteria T_n

From Section 2, the first term of the characteristic function of T_n, say, $c(t)$ is given by

$$c(t)=\text{const}\cdot|C|^{-1/2}\int\exp\Big[-\frac{1}{2}z'C^{-1}z+(it)\sum_{a\leqslant b}\sum_{c\leqslant d}T_{ab:cd}(I)$$
$$\times\Big(z_{ab}+\frac{1}{\sqrt{2}}\theta_{ab}\Big)\Big(z_{cd}+\frac{1}{\sqrt{2}}\theta_{cd}\Big)\Big]\prod_{a\leqslant b}dz_{ab}$$
$$=|C|^{-1/2}\,|C^{-1}-2itF|^{-1/2}\exp\Big[(it)^2\,\theta'F(C^{-1}-2itF)^{-1}F\theta+\frac{it}{2}\theta'F\theta\Big], \tag{3.1}$$

where $\theta=(\theta_{11},...,\theta_{pp},\theta_{12},...,\theta_{p-1,p})'$. Now there exist orthogonal matrices T_1 and T_2 such that

$$T_1F^{1/2}C^{1/2}T_2=\text{diag}(\sqrt{\lambda_{11}},...,\sqrt{\lambda_{pp}},\sqrt{\lambda_{12}},...,\sqrt{\lambda_{r^*,s^*}},0,...,0)$$
$$=\begin{pmatrix}D_{11}^{1/2} & 0\\ 0 & 0\end{pmatrix}=D^{1/2},$$

where $\lambda_{11}\geqslant\cdots\geqslant\lambda_{pp}\geqslant\lambda_{12}\geqslant\cdots\geqslant\lambda_{r^*,s^*}>0$ and these are eigenvalues of $C^{1/2}FC^{1/2}$. The (r^*,s^*) are some integers such that the number of positive roots is $r(T)$. We note that due to assumption (iii) that F is positive semi-definite and thus $F^{1/2}$ is well defined. We shall now simplify the expression on the exponent of $c(t)$, namely,

$$(it)^2\,\theta'F(C^{-1}-2itF)^{-1}F\theta+\frac{(it)}{2}\theta'F\theta$$
$$=(it)^2\,\theta'F^{1/2}T_1'D^{1/2}(I-2itD)^{-1}D^{1/2}T_1F^{1/2}\theta+\frac{(it)}{2}\theta'F^{1/2}T_1'T_1F^{1/2}\theta$$
$$=(it)^2\,\eta'\,\text{diag}(\lambda_{11}(1-2it\lambda_{11})^{-1},...,\lambda_{r^*,s^*}(1-2it\lambda_{r^*,s^*})^{-1},0,...,0)\,\eta$$
$$+\frac{(it)}{2}\eta'\eta$$
$$=(it)^2\sum_{a\leqslant b}\frac{\lambda_{ab}}{1-2it\lambda_{ab}}\eta_{ab}^2+\frac{(it)}{2}\sum_{a\leqslant b}\eta_{ab}^2$$
$$=\frac{1}{2}\sum_{a\leqslant b}\frac{(it)\,\eta_{ab}^2}{1-2it\lambda_{ab}}, \tag{3.2}$$

where

$$\eta = (\eta_{11}, \ldots, \eta_{pp}, \eta_{12}, \ldots, \eta_{p-1,p})'$$
$$= T_1 F^{1/2}\theta = T_1 F^{1/2} C^{1/2} T_2 T_2' C^{-1/2}\theta$$
$$= \begin{pmatrix} D_{11}^{1/2} & 0 \\ 0 & 0 \end{pmatrix} T_2' C^{-1/2}\theta = (\eta_{11}, \ldots, \eta_{r^*,s^*}, 0, \ldots, 0)'.$$

Also we have

$$|C|^{-1/2}\,|C^{-1} - 2itF|^{-1/2} = |I - 2itC^{1/2}FC^{1/2}|^{-1/2}$$
$$= |I - 2itD|^{-1/2} = \prod_{a \leqslant b} (1 - 2it\lambda_{ab})^{-1/2}. \quad (3.3)$$

Thus the first term of the characteristic function is given by

$$c(t) = \prod_{a \leqslant b} \left\{ (1 - 2it\lambda_{ab})^{-1/2} \exp\left(\frac{(it)\,\lambda_{ab}}{1 - 2it\lambda_{ab}} \lambda_{ab}^{+} \eta_{ab}^2 \right) \right\}, \quad (3.4)$$

where

$$\lambda_{ab}^{+} = \begin{cases} \lambda_{ab}^{-1}, & \lambda_{ab} > 0 \\ 0, & \lambda_{ab} = 0. \end{cases}$$

Since we can regard z as a normal distribution with mean $v = \sqrt{2}(it)(C^{-1} - 2itF)^{-1}\,F\theta$ with $\theta = (\theta_{11}, \ldots, \theta_{pp}, \theta_{12}, \ldots, \theta_{p-1,p})'$ and covariance matrix $(C^{-1} - 2itF)^{-1}$, we shall calculate the expectations of other terms under the following assumption (v):

$$\text{(v)} \quad C^{1/2} T_2 \begin{pmatrix} D_{11}^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} T_1 F^{1/2}\theta = \theta. \quad (3.5)$$

Under assumption (v), the mean v simplifies to

$$v = \frac{\sqrt{2}}{2} \left\{ C^{1/2} T_2 (I - 2itD)^{-1} \begin{pmatrix} D_{11}^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} \eta - \theta \right\}. \quad (3.6)$$

Then we note that $\eta + (\sqrt{2}/2)\,\theta$ contains terms dependent only on $(1 - 2it\lambda_{ab})^{-1}$. We now calculate the expectation:

$$E\left\{1+\sqrt{\frac{2}{n}}\left\{\frac{1}{2}(p+1)\operatorname{tr} Z-\frac{1}{2}\sum_{a\leqslant b}\sum_{c\leqslant d}\sum_{\alpha=1}^{p} g_{ab:cd}z_{ab}z_{c\alpha}z_{\alpha d}\right.\right.$$

$$+\frac{1}{6}\sum_{a\leqslant b}\sum_{c\leqslant d}\sum_{e\leqslant f} d_{ab:cd:ef}\left[\left(\sum_{g\leqslant h} g_{ab:gh}z_{gh}\right)\right.$$

$$\times\left(\sum_{g\leqslant h} g_{cd:gh}z_{gh}\right)\left(\sum_{g\leqslant h} g_{ef:gh}z_{gh}\right)-(g_{ab:ef})$$

$$\times\left(\sum_{g\leqslant h} g_{cd:gh}z_{gh}\right)-(g_{cd:ef})\left(\sum_{g\leqslant h} g_{ab:gh}z_{gh}\right)$$

$$\left.-(g_{ab:cd})\left(\sum_{g\leqslant h} g_{ef:gh}z_{gh}\right)\right]+(it)\left\{\frac{1}{3}\sum_{a\leqslant b}\sum_{c\leqslant d}\sum_{e\leqslant f} T_{ab:cd:ef}(I)\right.$$

$$\times\left(z_{ab}+\frac{\theta_{ab}}{\sqrt{2}}\right)\left(z_{cd}+\frac{\theta_{cd}}{\sqrt{2}}\right)\left(z_{ef}+\frac{\theta_{ef}}{\sqrt{2}}\right)$$

$$+\sum_{a\leqslant b}\sum_{c\leqslant d} T_{ab:cd}(I)\left(z_{ab}+\frac{\theta_{ab}}{\sqrt{2}}\right)\sum_{\alpha=1}^{p}\left[\left(z_{c\alpha}+\frac{\theta_{c\alpha}}{\sqrt{2}}\right)\right.$$

$$\left.\left.\left.\times\left(z_{\alpha d}+\frac{\theta_{\alpha d}}{\sqrt{2}}\right)-\frac{1}{2}\theta_{c\alpha}\theta_{\alpha d}\right]\right\}+O(n^{-1})\right]. \tag{3.7}$$

We obtain some formulas,

$$E\{\sqrt{2}\,z_{ab}\}=\left\{\sum_{c\leqslant d} a_{ab:cd}(1-2it\lambda_{cd})^{-1}\lambda_{cd}^{+/2}-\theta_{ab}\right\}=v_{ab}, \tag{3.8}$$

where $(a_{ab:cd})=C^{1/2}T_2$ and

$$\lambda_{ab}^{+/2}=\begin{cases}\lambda_{ab}^{-1/2}, & \lambda_{ab}>0\\ 0, & \lambda_{ab}=0.\end{cases}$$

Also the covariance matrix $(\sigma_{ab:cd})=(C^{-1}-2itF)^{-1}$ can be expressed as

$$\sigma_{ab:cd}=\sum_{e\leqslant f} a_{ab:ef}a_{cd:ef}(1-2it\lambda_{ef})^{-1}. \tag{3.9}$$

Then

$$E\{\sqrt{2}\,z_{ab}z_{cd}z_{ef}\}=v_{ab}\sigma_{cd:ef}+v_{cd}\sigma_{ab:ef}+v_{ef}\sigma_{ab:cd}+\tfrac{1}{2}v_{ab}v_{cd}v_{ef}. \tag{3.10}$$

Also for the terms containing (it) in the expectations, at first we have

$$\sqrt{2}\,(it)\,E\left(z_{ab}+\frac{1}{\sqrt{2}}\theta_{ab}\right)=(it)\,v'_{ab}, \tag{3.11}$$

where $v'_{ab}=\sum_{c\leqslant d} a_{ab:cd}\lambda_{cd}^{+/2}(1-2it\lambda_{cd})^{-1}$. Put $v^*_{ab}=\frac{1}{2}\sum_{c\leqslant d} a_{ab:cd}(\lambda_{cd}^{+/2})^3 \times\{(1-2it\lambda_{cd})^{-1}-1\}$; then we obtain

$$\sqrt{2}\,(it)\,E\left(z_{ab}+\frac{1}{\sqrt{2}}\theta_{ab}\right)=v^*_{ab} \tag{3.12}$$

and

$$\sqrt{2}\,(it)\,E\left(z_{ab}+\frac{1}{\sqrt{2}}\theta_{ab}\right)\left(z_{cd}+\frac{1}{\sqrt{2}}\theta_{cd}\right)\left(z_{ef}+\frac{1}{\sqrt{2}}\theta_{ef}\right)$$
$$=v^*_{ab}\sigma_{cd:ef}+v^*_{cd}\sigma_{ab:ef}+v^*_{ef}\sigma_{ab:cd}+\frac{1}{2}v^*_{ab}v'_{cd}v'_{ef}. \tag{3.13}$$

4. Asymptotic Expansion of the Distribution of T_n

Let the function $\mathscr{L}(u(t))$ stand for the inverse transformation of a characteristic function $u(t)$. At first we consider the distribution corresponding to $c(t)$. Then we have

$$\mathscr{L}(c(t))=P(x)=P\left(\sum_{a\leqslant b}\lambda_{ab}\chi^2_{[1]}(\lambda^+_{ab}\eta^2_{ab})\leqslant x\right), \tag{4.1}$$

where the $r(T)$ random variables $\chi^2_{[1]}(\lambda^+_{ab}\eta^2_{ab})$ are mutually independent and have χ^2 distribution with one degree of freedom and non-centrality $(\lambda^+_{ab}\eta^2_{ab})/2$. Also, all the χ^2 variates appearing hereafter are mutually independent with some degrees of freedom with some non-centrality.

Next we consider the terms of $1/\sqrt{n}$, which consist of sum/product of v_{ab}, v^*_{ab}, and $\sigma_{ab:cd}$ with some weights. We give some formulas:

$$\mathscr{L}(c(t)\,v_{cd})=\sum_{e\leqslant f} a_{cd:ef}\lambda_{ef}^{+/2}P_{ef}(x)-\theta_{cd}P(x), \tag{4.2}$$

where

$$P_{ef}(x)=P\left(\sum_{a\leqslant b}\lambda_{ab}\chi^2_{[ab|ef]}(\lambda^+_{ab}\eta^2_{ab})\leqslant x\right), \tag{4.3}$$

and

$$[ab\,|\,ef]=\begin{cases}3 & (a,b)=(e,f)\\ 1 & (a,b)\neq(e,f).\end{cases}$$
$$\mathscr{L}(c(t)\,v^*_{cd})=\tfrac{1}{2}\sum_{e\leqslant f} a_{cd:ef}(\lambda_{ef}^{+/2})^3\,(P_{ef}(x)-P(x)). \tag{4.4}$$

Also we have

$$\mathscr{L}(c(t)\, v_{cd}\sigma_{ef:gh})) = \sum_{i\leqslant j}\sum_{k\leqslant l} a_{cd:ij}a_{ef:kl}a_{gh:kl}\lambda_{ij}^{+/2}P_{ij:kl}(x)$$
$$-\theta_{cd}\sum_{i\leqslant j} a_{ef:ij}a_{gh:ij}P_{ij}(x), \tag{4.5}$$

where

$$P_{ij:kl}(x) = P\left(\sum_{a\leqslant b}\lambda_{ab}\chi^2_{[ab\,|\,ij:kl]}(\lambda_{ab}^{+}\eta_{ab}^2)\leqslant x\right) \tag{4.6}$$

and

$$[ab\,|\,ij:kl] = \begin{cases} 5 & (a,b)=(i,j)=(k,l) \\ 3 & (a,b)=(i,j) \text{ or } (a,b)=(k,l) \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have

$$\mathscr{L}(c(t)\, v^*_{cd}\sigma_{ef:gh}) = \tfrac{1}{2}\sum_{i\leqslant j}\sum_{k\leqslant l} a_{cd:ij}(\lambda_{ij}^{+/2})^3$$
$$\times a_{ef:kl}a_{gh:kl}\{P_{ij:kl}(x) - P_{kl}(x)\}. \tag{4.7}$$

Finally,

$$\mathscr{L}(c(t)\, v_{cd}v_{ef}v_{gh})$$
$$= \sum_{i\leqslant j}\sum_{k\leqslant l}\sum_{m\leqslant n} a_{cd:ij}a_{ef:kl}a_{gh:mn}\lambda_{ij}^{+/2}\lambda_{kl}^{+/2}\lambda_{mn}^{+/2}P_{ij:kl:mn}(x)$$
$$-\Bigg[\theta_{cd}\sum_{i\leqslant j}\sum_{k\leqslant l} a_{ef:ij}a_{gh:kl} + \theta_{ef}\sum_{i\leqslant j}\sum_{k\leqslant l} a_{cd:ij}a_{gh:kl}$$
$$+\theta_{gh}\sum_{i\leqslant j}\sum_{k\leqslant l} a_{cd:ij}a_{ef:kl}\Bigg]\lambda_{ij}^{+/2}\lambda_{kl}^{+/2}P_{ij:kl}(x)$$
$$+\left(\theta_{cd}\theta_{ef}\sum_{i\leqslant j} a_{gh:ij} + \theta_{ef}\theta_{gh}\sum_{i\leqslant j} a_{cd:ij} + \theta_{gh}\theta_{cd}\sum_{i\leqslant j} a_{ef:ij}\right)$$
$$\times\lambda_{ij}^{+/2}P_{ij}(x) - \theta_{cd}\theta_{ef}\theta_{gh}P(x),$$

where

$$P_{ij:kl:mn}(x) = P\left(\sum_{a\leqslant b}\lambda_{ab}\chi^2_{[ab\,|\,ij:kl:mn]}(\lambda_{ab}^{+}\eta_{ab}^2)\leqslant x\right) \tag{4.9}$$

and

$$[ab \mid ij : kl : mn] = \begin{cases} 7 & (a, b) = (i, j) = (k, l) = (m, n) \\ 5 & \text{exactly two pairs of three pairs } (i, j), (k, l), \\ & (m, n) \text{ equal to } (a, b) \\ 3 & \text{exactly on pair of three pairs } (i, j), (k, l), \\ & (m, n) \text{ equal to } (a, b) \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathscr{L}(c(t)\, v^*_{cd} v'_{ef} v'_{gh}) = \tfrac{1}{8} \sum_{i \leqslant j} \sum_{k \leqslant l} \sum_{m \leqslant n} a_{cd:ij} a_{ef:kl} a_{gh:mn} \\ \times (\lambda_{ij}^{+/2})^3 \lambda_{kl}^{+/2} \lambda_{mn}^{+/2} \{P_{ij:kl:mn}(x) - P_{kl:mn}(x)\}. \tag{4.10}$$

Thus we have the following theorem:

THEOREM 4.1. *Under local alternatives, the asymptotic distribution of the test statistic* $T_n = nT(S_N)$ *under assumptions* (i)–(v) *can be expressed as a weighted sum of* $r(T)$ *independent non-central* χ^2 *variates, where* $r(T)$ *is the rank of* F.

5. ASYMPTOTIC DISTRIBUTION OF TEST STATISTIC UNDER A t DISTRIBUTION

In this section we consider a t distribution, as distribution. In a multivariate t distribution with k degrees of freedom, we have

$$C = (c_{ab:cd}) = \begin{pmatrix} c_1 I_p + lG_p & 0 \\ 0 & c_1 \frac{1}{2} I_{p(p-1)/2} \end{pmatrix}, \tag{5.1}$$

where $l = \frac{1}{2}(c_1 - 1)$ with $c_1 = (k-2)/(k-4)$. Also I_p and $I_{p(p-1)/2}$ stand for identity matrices of order p and $p(p-1)/2$, respectively. Furthermore, G_p means a matrix with all entries one. For details of the t distribution, we can refer to DeGroot [5]. Also we note that in a case of a normal c_1 reduces to 1 and $C^{1/2}FC^{1/2}$ is idempotent for many statistics $T_n = nf(S_N)$ and equals CF. In a multivariate t distribution, $d_{ab:cd:ef}$ in (2.5) is given by

$$d_{ab:cd:ef} = \tfrac{1}{4} [\delta_{ab} \{(c_2 - 3c_1 + 2)\, \delta_{cd} \delta_{ef} + (c_2 - c_1)(\delta_{ce} \delta_{df} \\ + \delta_{cf} \delta_{de})\} + \delta_{ac} \{(c_2 - c_1)\, \delta_{bd} \delta_{ef} + c_2(\delta_{be} \delta_{df} + \delta_{bf} \delta_{de})\} \\ + \delta_{ad} \{(c_2 - c_1)\, \delta_{bc} \delta_{ef} + c_2(\delta_{be} \delta_{cf} + \delta_{bf} \delta_{ce})\} \\ + \delta_{ae} \{(c_2 - c_1)\, \delta_{bf} \delta_{cd} + c_2(\delta_{bc} \delta_{df} + \delta_{bd} \delta_{cf})\} \\ + \delta_{af} \{(c_2 - c_1)\, \delta_{be} \delta_{cd} + c_2(\delta_{bc} \delta_{ed} + \delta_{bd} \delta_{ce})\}], \tag{5.2}$$

where

$$c_2 = \frac{(k-2)^2}{(k-4)(k-6)}.$$

In this section, we treat the following hypothesis concerning a covariance matrix; the null hypothesis $H_0: \Sigma = \sigma^2 I_p$, where σ^2 is unspecified positive constant, against the alternative hypotheses $H_1: \Sigma \neq \sigma^2 I_p$. For this problem we consider two test criteria, namely the likelihood ratio (=LR) test and Nagao's test. These tests in a normal case are discussed in Anderson [1] and Srivastava and Khatri [13]. The rejection regions of the LR test and Nagao's test are, respectively,

$$\{S_N \mid n\{p \log(p^{-1} \operatorname{tr} S_N) - \log|S_N|\} \geqslant c_\alpha\} \tag{5.3}$$

and

$$\left\{S_N \,\middle|\, \frac{np^2}{2} \operatorname{tr}\{S_N(\operatorname{tr} S_N)^{-1} - p^{-1}I_p\}^2 \geqslant c_\alpha\right\}. \tag{5.4}$$

Then, by Chan and Srivastava [4], the matrices F and $T_{ab:cd:ef}(I)$ of the two tests are given as

$$F = (T_{ab:cd}(I)) = \begin{pmatrix} I_p - \frac{1}{p} G_p & 0 \\ 0 & 2I_{p(p-1)/2} \end{pmatrix}. \tag{5.5}$$

Also for the LR test we have

$$T_{ab:cd:ef}(I) = \begin{cases} 2(p^{-2}-1), & a=b=c=d=e=f \\ 2p^{-2}, & a=b=c=d\neq e=f \\ -2, & a=b=c=e\neq d=f \\ 0, & \text{otherwise}; \end{cases} \tag{5.6}$$

and for Nagao's test,

$$T_{ab:cd:ef}(I) = \begin{cases} -6p^{-1}(1-p^{-1}), & a=b=c=d=e=f \\ -2p^{-1}(1-3p^{-1}), & a=b=c=d\neq e=f \\ -4p^{-1}, & a=b=c=e\neq d=f \\ 0, & \text{otherwise}. \end{cases} \tag{5.7}$$

Then the eigenvalues of CF are c_1 with multiplicity $f=p(p+1)/2-1$ and zero. Thus (3.4) reduces to

$$c(t)=(1-2itc_1)^{-f/2}\exp\left(\frac{itc_1}{1-2itc_1}\frac{1}{c_1}\sum_{a\leqslant b}\eta_{ab}^2\right). \tag{5.8}$$

$\sum_{a\leqslant b}\eta_{ab}^2=\eta'\eta=\theta' F\theta=\mathrm{tr}\,\Theta^2$ since $\mathrm{tr}\,\Theta=0$. Therefore the limiting distributions of the LR test and Nagao's test are the same and they are c_1 times non-central χ^2 distributions with $f=p(p+1)/2-1$ degrees of freedom with non-centrality $\delta^2=\mathrm{tr}\,\Theta^2/(2c_1)$. By this result we find these two test criteria are robust test statistics. Next we give the $1/\sqrt{n}$ terms. Since $Ez=\sqrt{2}(it)(C^{-1}-2itF)^{-1}F\theta$ and

$$\begin{aligned}(\sigma_{ab\,:\,cd})&=(C^{-1}-2itF)^{-1}\\ &=c_1(1-2itc_1)^{-1}\begin{pmatrix} I_p-(h+2itc_1)(1+h)^{-1}(G_p/p) & \vdots & 0\\ \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\\ 0 & \vdots & \frac{1}{2}I_{p(p-1)/2}\end{pmatrix},\end{aligned} \tag{5.9}$$

where $h=c_1\{2[c_1(p+2)-p]^{-1}-c_1^{-1}\}$, we have

$$v_{ab}=\theta_{ab}\{(1-2itc_1)^{-1}-1\} \tag{5.10}$$

and

$$\begin{aligned}\sigma_{ab\,:\,cd}=c_1\Bigg[&\frac{1}{2}(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})(1-2itc_1)^{-1}\\ &+\frac{1}{p}\{(1+h)^{-1}-(1-2itc_1)^{-1}\}\,\delta_{ab}\delta_{cd}\Bigg].\end{aligned} \tag{5.11}$$

Also the v'_{ab} in (3.11) and v^*_{ab} in (3.12) are $v'_{ab}=\theta_{ab}(1-2itc_1)^{-1}$ and $v^*_{ab}=\theta_{ab}\{(1-2itc_1)^{-1}-1\}/(2c_1)$. At first we have

$$\mathscr{L}(c(t))=\Pr(c_1\chi_f^2(\delta^2)\leqslant x). \tag{5.12}$$

We put $\Pr(c_1\chi_f^2(\delta^2)\leqslant x)=P_f(\delta^2)$. Then

$$\mathscr{L}(c(t)\,v_{cd})=\theta_{cd}\{P_{f+2}(\delta^2)-P_f(\delta^2)\},$$

$$\mathscr{L}(c(t)\,v^*_{cd})=\frac{1}{2c_1}\mathscr{L}(c(t)\,v_{cd}),$$

$$\mathscr{L}(c(t)\, v_{cd}\sigma_{ef:gh}) = c_1\theta_{cd}\Big[\frac{1}{2}(\delta_{eg}\,\delta_{fh} + \delta_{eh}\,\delta_{fg})(P_{f+4}(\delta^2) - P_{f+2}(\delta^2))$$

$$-\frac{1}{p}\,\delta_{ef}\,\delta_{gh}\{P_{f+4}(\delta^2) - [(1+h)^{-1}+1]$$

$$\times P_{f+2}(\delta^2) + (1+h)^{-1}\,P_f(\delta^2)\}\Big],$$

$$\mathscr{L}(c(t)\, v^*_{cd}\sigma_{ef:gh}) = \frac{1}{2c_1}\,\mathscr{L}(c(t)\, v_{cd}\sigma_{ef:gh}),$$

$$\mathscr{L}(c(t)\, v_{cd}v_{ef}v_{gh}) = \theta_{cd}\theta_{ef}\theta_{gh}(P_{f+6}(\delta^2) - 3P_{f+4}(\delta^2)$$

$$+ 3P_{f+2}(\delta^2) - P_f(\delta^2)),$$

$$\mathscr{L}(c(t)\, v^*_{cd}v'_{ef}v'_{gh}) = \frac{1}{2c_1}\,\mathscr{L}(c(t)\, v_{cd}v_{ef}v_{gh}). \tag{5.13}$$

Therefore we have

THEOREM 5.1. *Let $p \times p$ matrix Σ be a covariance matrix of a p-variate t-distribution with k degrees of freedom. For testing the hypothesis $H_0: \Sigma = \sigma^2 I_p$ against the alternatives $H_1: \Sigma \neq \sigma^2 I_p$, where σ^2 is unspecified positive number, the distribution of the LR test λ given by* (5, 3) *is expanded as*

$$\Pr(\lambda \leqslant x) = P_f(\delta^2) + \frac{1}{\sqrt{n}}\frac{c_1^{-1}}{6}\operatorname{tr}(\Theta/\sigma^2)^3\Big\{\Big(\frac{c_2}{c_1^2} - 1\Big)P_{f+6}(\delta^2)$$

$$-\Big(\frac{3c_2}{c_1^2} - 4\Big)P_{f+4}(\delta^2) + \Big(\frac{3c_2}{c_1^2} - 6\Big)P_{f+2}(\delta^2)$$

$$-\Big(\frac{c_2}{c_1^2} - 3\Big)P_f(\delta^2)\Big\} + O(n^{-1}), \tag{5.14}$$

where $P_f(\delta^2)$ is c_1 times noncentral χ^2 distribution with f degrees of freedom with noncentrality δ^2 and

$$c_1 = \frac{k-2}{k-4}, \qquad c_2 = \frac{(k-2)^2}{(k-4)(k-6)}, \qquad \delta^2 = \frac{1}{2c_1}\operatorname{tr}(\Theta/\sigma^2)^2. \tag{5.15}$$

In a normal case, c_1 and c_2 reduce to 1. The above result coincides with Nagao's on [9] in case of a normal for the sphericity test. Similarly, THEOREM 5.2. *Under the same assumption and notation as in Theorem* 5.1, *the distribution of Nagao's test statistics is given by*

$$\Pr(T \leqslant x) = P_f(\delta^2) + \frac{1}{\sqrt{n}} \frac{c_1^{-1}}{6} \operatorname{tr}(\Theta/\sigma^2)^3 \left\{ \frac{c_2}{c_1^2} P_{f+6}(\delta^2) \right.$$

$$- 3\left(\frac{c_2}{c_1^2} - 1\right) P_{f+4}(\delta^2) + 3\left(\frac{c_2}{c_1^2} - 2\right) P_{f+2}(\sigma^2)$$

$$\left. - \left(\frac{c_2}{c_1^2} - 3\right) P_f(\delta^2) \right\} + O(n^{-1}). \tag{5.16}$$

The result in a normal case given by Nagao [11] is included in the above result. Also Hayakawa [6] derived the asymptotic distribution of these test criteria under local alternatives for an elliptical distribution. However, his results seem to be very complicated.

6. NUMERICAL EXAMPLES

In this section, we give some numerical powers for the sphericity test. For a t-distribution with k degrees of freedom, we modify the LR test and Nagao's test T as

$$\lambda' = \frac{k-4}{k-2} \lambda \geqslant c_\alpha, \qquad T' = \frac{k-4}{k-2} T \geqslant c_\alpha. \tag{6.1}$$

Numerical powers are calculated by two methods due to asymptotic expansions and the bootstrap method used by Beran [3]. At first we explain the bootstrap powers.

We shall mention how to calculate them under covariance matrix $\Sigma = \Sigma_A$ in a sphericity test. We construct vectors x_i $(i = 1, ..., N)$ consisting of having a multivariate t distribution with k degrees of freedom, mean 0, and covariance Σ_A. We note that when $k = \infty$, this distribution reduces to a multivariate normal distribution with mean 0 and covariance Σ_A. Let, as in Beran and Srivastava [2],

$$y_i = \Sigma_A^{1/2} \, \Sigma_{\hat{F}_N}^{-1/2} x_i \qquad (i = 1, ..., N),$$

where $\Sigma_{\hat{F}_N} = (1/N) \sum_{\alpha=1}^N (x_\alpha - \bar{x})(x_\alpha - \bar{x})'$ and $\hat{F}_N$ means an empirical distribution.

From $\{y_1, ..., y_N\}$, we choose N random vectors $y_1^*, ..., y_N^*$ with replacement. Then they can be regarded as a random sample with mean $\sum_A^{1/2} \sum_{\hat{F}_N}^{-1/2} \bar{x}$ and covariance matrix Σ_A. For each sample $x_1, ..., x_N$, we draw samples from $\hat{F}_N$ a great many of times and calculate the test statistics. An approximation of the power is given by dividing the number of rejection cases with the number B of bootstrap replications. In this place, we set $B = 500$ and we repeat this procedure 100 times and we calculate an average and the standard deviation. We call the averages the bootstrap powers. The value in parenthesis in the tables are the standard deviations of 100 trials. To find the power of the test, we have to obtain the critical value c_α. In order to obtain c_α, we have a few methods such as those based on exact distribution, asymptotic distribution, or empirical distribution. The third method can be found in Beran [3]. In this paper, for a normal case, we use critical values due to asymptotic distributions of test statistics in Anderson [1] and Nagao [9]. Also for a t distribution, critical values are calculated by a limiting distribution which is a χ^2 distribution with $f = p(p+1)/2 - 1$ degrees of freedom as given by Muirhead and Waternaux [8] in case of elliptical distribution.

Case 1. For testing the hypothesis $H: \Sigma = \sigma^2 I_3$ against alternative $\Sigma = (1-\rho) I + \rho 11'$ with $1 = (1, 1, 1)'$, we give the powers under a three-dimensional normal and t distribution with 10 degrees of freedom.

$N = 50$ (normal distribution)

ρ	LRT	B-LRT	Nagao	B-Nagao
0.01	0.051	0.079(0.030)	0.055	0.066(0.025)
0.03	0.056	0.072(0.022)	0.061	0.062(0.020)
0.05	0.067	0.083(0.026)	0.072	0.071(0.023)
0.08	0.096	0.109(0.027)	0.104	0.100(0.025)
0.10	0.126	0.140(0.036)	0.136	0.129(0.034)

$N = 50$ (t distribution)

ρ	LRT	B-LRT	Nagao	B-Nagao
0.01	0.050	0.048(0.035)	0.050	0.035(0.030)
0.03	0.054	0.057(0.052)	0.054	0.042(0.041)
0.05	0.062	0.063(0.051)	0.063	0.048(0.038)
0.08	0.084	0.082(0.065)	0.084	0.064(0.052)
0.10	0.105	0.099(0.048)	0.106	0.080(0.039)

$N = 80$ (normal distribution)

ρ	LRT	B-LRT	Nagao	B-Nagao
0.01	0.051	0.063(0.020)	0.054	0.056(0.019)
0.03	0.060	0.072(0.023)	0.063	0.066(0.023)
0.05	0.078	0.089(0.023)	0.082	0.084(0.022)
0.08	0.129	0.130(0.030)	0.134	0.127(0.028)
0.10	0.186	0.181(0.030)	0.189	0.182(0.029)

$N = 80$ (t distribution)

ρ	LRT	B-LRT	Nagao	B-Nagao
0.01	0.051	0.041(0.028)	0.051	0.034(0.025)
0.03	0.057	0.049(0.030)	0.057	0.043(0.026)
0.05	0.070	0.067(0.042)	0.071	0.059(0.037)
0.08	0.107	0.092(0.051)	0.107	0.085(0.046)
0.10	0.144	0.116(0.046)	0.145	0.111(0.042)

$N = 100$ (normal distribution)

ρ	LRT	B-LRT	Nagao	B-Nagao
0.01	0.051	0.061(0.018)	0.054	0.055(0.017)
0.03	0.062	0.067(0.018)	0.065	0.063(0.017)
0.05	0.085	0.091(0.022)	0.089	0.087(0.021)
0.08	0.152	0.152(0.029)	0.157	0.150(0.028)
0.10	0.221	0.215(0.030)	0.228	0.220(0.028)

$N = 100$ (t distribution)

ρ	LRT	B-LRT	Nagao	B-Nagao
0.01	0.051	0.054(0.042)	0.051	0.048(0.038)
0.03	0.059	0.059(0.034)	0.059	0.053(0.032)
0.05	0.076	0.076(0.059)	0.076	0.071(0.055)
0.08	0.123	0.111(0.040)	0.124	0.107(0.036)
0.10	0.172	0.148(0.050)	0.173	0.148(0.047)

From these tables, we find that the bootstrap powers are good for a normal distribution but not for a t distribution.

Case 2. For testing the hypothesis $H: \Sigma = \sigma^2 I_3$ against alternative $\Sigma = \text{diag}(a_1, 1, a_2)$, we give the powers under normal and t distribution with 10 degrees of freedom.

$N = 50$ (normal distribution)

a_1,	a_2	LRT	B-LRT	Nagao	B-Nagao
0.9	0.9	0.058	0.076(0.036)	0.063	0.064(0.031)
0.8	0.9	0.079	0.095(0.028)	0.085	0.081(0.025)
0.8	0.8	0.092	0.108(0.036)	0.100	0.097(0.032)
0.9	1.1	0.073	0.092(0.027)	0.079	0.079(0.024)
0.8	1.1	0.113	0.122(0.034)	0.120	0.104(0.030)
0.8	1.2	0.157	0.165(0.035)	0.166	0.148(0.034)

$N = 50$ (t distribution)

a_1,	a_2	LRT	B-LRT	Nagao	B-Nagao
0.9	0.9	0.056	0.061(0.054)	0.056	0.044(0.042)
0.8	0.9	0.071	0.063(0.038)	0.071	0.047(0.030)
0.8	0.8	0.081	0.082(0.041)	0.081	0.054(0.031)
0.9	1.1	0.067	0.068(0.043)	0.067	0.049(0.032)
0.8	1.1	0.095	0.089(0.043)	0.095	0.064(0.034)
0.8	1.2	0.127	0.106(0.043)	0.127	0.080(0.033)

$N = 80$ (normal distribution)

a_1,	a_2	LRT	B-LRT	Nagao	B-Nagao
0.9	0.9	0.064	0.074(0.022)	0.067	0.068(0.020)
0.8	0.9	0.098	0.103(0.025)	0.102	0.095(0.023)
0.8	0.8	0.122	0.124(0.029)	0.127	0.119(0.028)
0.9	1.1	0.088	0.098(0.024)	0.092	0.092(0.022)
0.8	1.1	0.157	0.168(0.028)	0.163	0.149(0.028)
0.8	1.2	0.236	0.242(0.030)	0.243	0.224(0.030)

$N=80$ (t distribution)

a_1,	a_2	LRT	B-LRT	Nagao	B-Nagao
0.9	0.9	0.060	0.054(0.037)	0.060	0.045(0.032)
0.8	0.9	0.085	0.086(0.062)	0.085	0.072(0.056)
0.8	0.8	0.102	0.097(0.054)	0.102	0.085(0.046)
0.9	1.1	0.078	0.077(0.054)	0.078	0.065(0.050)
0.8	1.1	0.127	0.114(0.051)	0.126	0.095(0.049)
0.8	1.2	0.182	0.160(0.069)	0.182	0.140(0.065)

$N=100$ (normal distribution)

a_1,	a_2	LRT	B-LRT	Nagao	B-Nagao
0.9	0.9	0.067	0.074(0.020)	0.070	0.069(0.020)
0.8	0.9	0.112	0.116(0.026)	0.115	0.108(0.025)
0.8	0.8	0.143	0.149(0.029)	0.148	0.145(0.028)
0.9	1.1	0.099	0.104(0.021)	0.102	0.095(0.019)
0.8	1.1	0.189	0.200(0.028)	0.194	0.180(0.028)
0.8	1.2	0.292	0.291(0.031)	0.300	0.277(0.031)

$N=100$ (t distriution)

a_1,	a_2	LRT	B-LRT	Nagao	B-Nagao
0.9	0.9	0.063	0.056(0.029)	0.063	0.049(0.026)
0.8	0.9	0.095	0.085(0.039)	0.095	0.074(0.035)
0.8	0.8	0.117	0.098(0.043)	0.117	0.091(0.040)
0.9	1.1	0.085	0.073(0.036)	0.085	0.064(0.032)
0.8	1.1	0.149	0.129(0.048)	0.149	0.111(0.046)
0.8	1.2	0.222	0.196(0.060)	0.222	0.174(0.057)

In a normal case, the values in asymptotic expansions show that Nagao's test is better than LRT. This fact indicates that Nagao's test is locally best invariant test as shown by John [7] and Sugiura [14]. Also Nagao [12] in the normal case has considered another property. However, from Cases 1 and 2, we may say that the bootstrap approximations are not good for a t distribution. Whereas the asymptotic expansions seem to give reasonable values for even a t distribution.

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