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Multivariate Stability and Strong Limiting Behaviour of Intermediate Order Statistics

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We investigate the almost sure stability of intermediate order statistics of a sample from a multivariate distribution. It is shown that the ordering of the multivariate sample—based on an increasing family of conditional quantile surfaces—enables us to use a univariate approach. In the univariate case, we characterise the strong limiting behaviour of weighted uniform quantile processes along an intermediate sequence satisfying suitable conditions of growth and regularity. From this characterisation a necessary and sufficient condition for the almost sure stability is deduced. © 2001 Academic Press

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1. INTRODUCTION

The ordering of a multivariate sample is not natural and several definitions have been already proposed in the literature (see, e.g., Barnett, 1976, Galambos, 1987, Maller, 1990, and Masse and Theodorescu, 1994). A new definition of the order statistics of a multivariate sample has been introduced in Delcroix and Jacob (1991), making use of the intrinsic properties of the distributions or, more precisely, of the isobar surfaces. As it has been noticed, this definition is particularly convenient for some specific problems. The main motivation was to find the asymptotic location of the sample, without using the convex hull of the sample as done classically in Geffroy (1961), Davis *et al.* (1987), and Delcroix (1994), for example.

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Before presenting our results, it is convenient to recall some definitions. We also introduce assumptions and notation which will be used throughout this paper.

Let X be an $\mathbf{R}^k$-valued random variable defined on a probability space $(\Omega, \mathscr{A}, \mathscr{P})$. Denote by $\|\cdot\|$ the Euclidian norm of $\mathbf{R}^k$ and by $\mathbf{S}^{k-1}$ the unit sphere of $\mathbf{R}^k$ which is endowed with the induced topology of $\mathbf{R}^k$.

Suppose that the distribution of X has a continuous density function. If $\|X\| \neq 0$, define the pair (R, Θ) in $\mathbf{R}_+^* \times \mathbf{S}^{k-1}$ by $R = \|X\|$ and $\Theta = \frac{X}{\|X\|}$. We assume that the distribution of R given $\Theta = \theta$, is defined by the continuous conditional distribution function, $F_\theta(r) = P\{R \leqslant r \mid \Theta = \theta\}$, for all θ. We also suppose that F_θ is one-to-one, for all θ.

DEFINITION 1. For a given u, $0 < u < 1$, the mapping

$$\mathbf{S}^{k-1} \to \mathbf{R}_+^*$$

$$\theta \rightsquigarrow F_\theta^{-1}(u),$$

is called **u-level isobar** from the distribution of (R, Θ). The corresponding surface $\rho = F_\theta^{-1}(u)$ is also called isobar.

We assume that for fixed u, this mapping is continuous and strictly positive.

Let $X_1, X_2, ...$ be a sequence of independent and identically distributed (i.i.d.) random variables with the same distribution as X, and with polar representation $(R_1, \Theta_1), (R_2, \Theta_2), ...$. Denote by E_n the n-sample $(X_1, ..., X_n)$, $n \geqslant 1$. Clearly, for each i, $1 \leqslant i \leqslant n$, there is almost surely (a.s.) a unique u_i-level isobar containing X_i.

DEFINITION 2. The **maximum value** in E_n is defined as the element $X_{n,n}^* = (R_n^*, \Theta_n^*)$ which belongs to the upper level surface, i.e., the surface which has a level equal to $\max_{1 \leqslant i \leqslant n} u_i$. In this way, the sample is ordered according to the increasing levels of the corresponding isobar surfaces and, the order statistics of the n-sample are denoted by

$$X_{1,n}^* = (R_1^*, \Theta_1^*),\ X_{2,n}^* = (R_2^*, \Theta_2^*),\ ...,\ X_{n,n}^* = (R_n^*, \Theta_n^*).$$

The weak limiting behaviour of these new extreme values has been discussed by Barme and Gather (1996).

Delcroix and Jacob (1991) investigated the stability of the sequence of extreme values $(X_{n,n}^*)$, according to the following proposed definition:

DEFINITION 3. The sequence $(X^*_{n,n})$ is a.s. stable if and only if there is a sequence (g_n) of isobars such that

$$R^*_n - g_n(\Theta^*_n) \to 0 \qquad \text{a.s.}$$

In particular, the above mentioned authors have shown that the stability of $(X^*_{n,n})$ may be characterised in the same way as in the univariate case.

Now, let (k_n), with $1 \leqslant k_n \leqslant n$, be an increasing sequence of integers, such that $k_n \to \infty$ and $k_n/n \to 0$, as $n \to \infty$, that is, k_n is an **intermediate sequence**. In this paper we investigate the almost sure stability of $(X^*_{n-k_n+1,n})$, that is the sequence of upper intermediate order statistics of the n-sample E_n. We will use a univariate approach as it has been done for the maximum value $X^*_{n,n}$.

We recall that, in the univariate case, a sequence of random variables (W_n) is called **a.s.** (resp., **weakly**) **absolutely stable** if there exists a non-random sequence (a_n) such that $W_n - a_n \to 0$ a.s. (resp., in probability). If $W_n/b_n \to 1$ a.s. (resp., in probability) for a non-random sequence (b_n), $b_n \neq 0$, (W_n) is called **a.s.** (resp., **weakly**) **relatively stable**. Clearly, the study of the almost sure stability of a sequence is closely related to the problem of establishing strong limiting bounds.

The study of the weak absolute stability of $(W_{k,n})$ for fixed k, $1 \leqslant k \leqslant n$, has received considerable attention. In particular, necessary and sufficient conditions were obtained by Gnedenko (1943), Smirnov (1952), and Geffroy (1958–1959). Analogous conditions may be easily deduced for the relative stability, as it has been noticed by Geffroy (1958–1959). These results were extended in several directions (see, e.g., the works of Chevalier, 1976, Gather and Tomkins, 1995, on intermediate order statistics, Gather and Rauhut, 1990, and the references therein).

We will consider here the almost sure (absolute) stability of $(W_{k_n,n})$, giving particular attention to the characterisation of the corresponding strong limiting bounds. For $(W_{k,n})$, this problem has been studied in full generality in works of Geffroy (1958–1959), Barndorff–Nielsen (1963), Kiefer (1972), and Shorack and Wellner (1978) (see also Resnick and Tomkins, 1973, and Tomkins, 1996).

When (k_n) is an intermediate sequence, the strong limiting behaviour of $(W_{k_n,n})$ depends on the order of growth of k_n and it has not been exhaustively investigated. As shown in Kiefer (1972), we may distinguish essentially three cases, according to $k_n/\log_2 n \to 0$, $k_n/\log_2 n \to c \in (0, \infty)$ or $k_n/\log_2 n \to \infty$ as $n \to \infty$, where $\log_2 n = \log(\log(n \vee 3))$. In particular, Kiefer established strong limiting bounds of first order for the three cases. Refinements and extensions of these results have been given by several authors, including Brito (1986, 1990), Chevalier (1976), and Deheuvels (1986, 1987).

These results, which concern mainly the characterisation of strong upper and lower bounds, show that the strong limiting behaviour of the k_nth order statistic is quite different in each of the three ranges. We will concentrate our interest on the case $\log_2 n = o(k_n)$. We will show that it is possible to give a simple necessary and sufficient condition for the almost sure stability of a big class of k_nth order statistics. Furthermore, by the characterisation of the corresponding strong limiting bounds, the connection with the strong limiting behaviour of central order statistics is made explicit. These results extend and complete the work of Brito (1990).

The extension of Definition 3 (Delcroix and Jacob, 1991) to multivariate intermediate order statistics is given in Section 2, where it is also shown that the multivariate stability properties may be treated in a univariate way. The univariate case is considered in Section 3. Finally, in Section 4 the criterion of stability of $(X^*_{n-k_n+1,n})$ is given.

2. STABILITY OF THE MULTIVARIATE INTERMEDIATE ORDER STATISTICS

In this section we consider the sample $(X_1, ..., X_n)$. We begin by stating the results needed to extend Definition 3 to intermediate order statistics.

THEOREM 1. *For all* $1 \leqslant k \leqslant n$,

(i) *The variables* Θ^*_k *and* Θ *have the same distribution.*

(ii) *Any u-level isobar from the distribution of* (R, Θ) *is also isobar from the distribution of* (R^*_k, Θ^*_k) *with a level equal to* $\sum_{i=k}^{n} \binom{n}{i} u^i (1-u)^{n-i}$.

Proof. This result can be proved in the same way as the corresponding result for the maximum value $X^*_{n,n} = (R^*_n, \Theta^*_n)$ (cf. Property 1 of Delcroix and Jacob, 1991). ∎

As a consequence it is possible to give

COROLLARY 1. *Let* $F^*_{k,\theta}$ *be the conditional distribution function of* R^*_k *given* $\Theta^*_k = \theta$. *Then* $F^*_{k,\theta} = \sum_{i=k}^{n} \binom{n}{i} F^i_\theta (1 - F_\theta)^{n-i}$, *for all* θ.

Proof. The idea of the proof is the same as for the maximum value (see Delcroix and Jacob, 1991). ∎

By the results above we may give the following definition, since the distributions of (R, Θ) and of (R^*_k, Θ^*_k) define the same set of isobars.

DEFINITION 4. $(X^*_{k_n, n})$ is a.s. stable if and only if there is a sequence (g_n) of isobars, such that

$$R^*_{k_n} - g_n(\Theta^*_{k_n}) \to 0 \qquad \text{a.s.} \tag{1}$$

It is convenient to fix a point θ_1 in $\mathbf{S}^{k-1}$ and to choose on the axis $(0\theta_1)$ a unit vector $\overrightarrow{0\theta_1}$. For every point r on the positive half axis $0\theta_1^+$, there is a unique surface containing r, which has a level denoted by $u(r)$ and defined by

$$g(\theta, r) = \rho_{u(r)}(\theta).$$

Note that $g(\theta_1, r) = r$. Moreover the mapping $r \rightsquigarrow u(r)$ from $\mathbf{R}^*_+$ into $\mathbf{R}^*_+$ is increasing and one-to-one.

Let W_i be the intersection of $0\theta_1^+$ with the surface containing X_i, for $1 \leqslant i \leqslant n$. In fact, $(W_1, ..., W_n)$ is a sample from the distribution F_{θ_1}, as shown in Delcroix and Jacob (1991). As usual $W_{1,n} \leqslant \cdots \leqslant W_{n,n}$ denote the corresponding order statistics.

We will make use of the following conditions:

(H) There exist $0 < \alpha \leqslant \beta < +\infty$ such that, for all θ in $\mathbf{S}^{k-1}$ and for all $r > 0$,

$$\alpha \leqslant \frac{\partial g}{\partial r}(\theta, r) \leqslant \beta.$$

(K) For all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $r > 0$,

$$\sup_\theta \{ g(\theta, r+\eta) - g(\theta, r-\eta) \} < \varepsilon.$$

Clearly, (H) implies (K).

THEOREM 2. *For all increasing sequences (k_n), $1 \leqslant k_n \leqslant n$, we have*

(i) *Under condition* (K) *the sequence $(X^*_{k_n, n})$ is a.s. stable if $(W_{k_n, n})$ is a.s. stable.*

(ii) *Under condition* (H) *the sequence $(W_{k_n, n})$ is a.s. stable if and only if $(X^*_{k_n, n})$ is a.s. stable.*

Proof. (i) If $(W_{k_n, n})$ is a.s. stable then, there exists a sequence (w_n) such that $W_{k_n, n} - w_n \to 0$ a.s. According to (K), for $\varepsilon > 0$ there exists $\eta > 0$ such that $\sup_\theta \{ g(\theta, w+\eta) - g(\theta, w-\eta) \} < \varepsilon$, for all $w > 0$.

Let $h_n^{\eta}(\theta) = g(\theta, w_n + \eta)$ and $h_n^{-\eta}(\theta) = g(\theta, w_n - \eta)$. Then if $g(\theta, w_n)$ is denoted by $h_n(\theta)$ we have

$$\begin{aligned}\{|W_{k_n, n} - w_n| \leqslant \eta\} &= \{h_n^{-\eta}(\theta_1) \leqslant W_{k_n, n} \leqslant h_n^{\eta}(\theta_1)\}\\ &\subset \{h_n^{-\eta}(\Theta_{k_n}^*) \leqslant R_{k_n}^* \leqslant h_n^{\eta}(\Theta_{k_n}^*)\}\\ &\subset \{|R_{k_n}^* - h_n(\Theta_{k_n}^*)| \leqslant \varepsilon\}.\end{aligned}$$

(ii) Conversely, if there exists a sequence of surfaces (g_n) such that $R_{k_n}^* - g_n(\Theta_{k_n}^*) \to 0$ a.s., denote by w_n the intersection of $0\theta_1^+$ with g_n. According to (H), there exist α and β such that

$$g(\theta, w_n) + \lambda\alpha \leqslant g(\theta, w_n + \lambda) \leqslant g(\theta, w_n) + \lambda\beta$$

and

$$g(\theta, w_n) - \lambda\beta \leqslant g(\theta, w_n - \lambda) \leqslant g(\theta, w_n) - \lambda\alpha$$

for all $\lambda > 0$ and all θ. Given $\varepsilon > 0$, it is possible to choose $\lambda = \varepsilon/\beta$ and $\eta = \varepsilon\alpha/\beta$ and to take

$$\begin{aligned}h_n(\theta) &= g(\theta, w_n + \lambda)\\ \tilde{h}_n(\theta) &= g(\theta, w_n - \lambda).\end{aligned}$$

It follows that

$$\{|R_{k_n}^* - g_n(\Theta_{k_n}^*)| \leqslant \eta\} \subset \{\tilde{h}_n(\Theta_{k_n}^*) \leqslant R_{k_n}^* \leqslant h_n(\Theta_{k_n}^*)\} \subset \{|W_{k_n, n} - w_n| \leqslant \varepsilon\},$$

which concludes the proof of the theorem. ∎

Thus we may investigate the stability of $(X_{k_n, n}^*)$ through the stability of $(W_{k_n, n})$.

3. STRONG LAWS AND STABILITY OF UNIVARIATE INTERMEDIATE ORDER STATISTICS

Let $U_1, U_2, \ldots$ be a sequence of i.i.d. uniform $(0, 1)$ random variables and denote by $U_{1, n} \leqslant U_{2, n} \leqslant \cdots \leqslant U_{n, n}$ the order statistics of $U_1, U_2, \ldots, U_n$. Denote by G_n the right continuous empirical distribution function based on these n random variables and define the uniform empirical quantile function $U_n(\cdot)$, $n = 1, 2, \ldots$, as

$$U_n(s) = U_{k, n}, \qquad (k-1)/n < s \leqslant k/n \qquad (k = 1, 2, \ldots, n),$$

with $U_n(0)=0$. The uniform empirical process α_n is defined by

$$\alpha_n(s)=n^{1/2}(G_n(s)-s), \qquad 0\leqslant s\leqslant 1,$$

and the uniform quantile process u_n by

$$u_n(s)=n^{1/2}(U_n(s)-s), \qquad 0<s<1.$$

In the investigation of the almost sure stability of a sequence of intermediate order statistics, the case where $\log_2 n=o(k_n)$ is of particular interest, since, under general conditions of growth and regularity on k_n, we can derive a simple necessary and sufficient condition. The criterion of stability for $(W_{k_n,n})$ will be established in Corollary 3 of this section via characterising the strong limiting behaviour of $u_n(y_n)$ properly normalised, for a suitable intermediate sequence (y_n). Moreover, this characterisation leads to the result of Corollary 2, which shows that the strong limiting bounds of the corresponding intermediate order statistics may be characterised in the same way as for the central order statistics.

For convenient reference later on, we will now present some facts. The first one concerns the strong limiting behaviour of the oscillation modulus $w_n(a)$ of the empirical process α_n, which is defined by

$$w_n(a)=\sup_{|t-s|\leqslant a}|\alpha_n(t)-\alpha_n(s)|.$$

Fact 1 (Stute, 1982). Let (a_n) be any sequence in $(0,1)$ satisfying

(S1) $a_n\downarrow 0$ and $na_n\uparrow\infty$;

(S2) $\log(1/a_n)=o(na_n)$; and

(S3) $\log(1/a_n)/\log_2 n\to\infty$.

Then

$$\lim_{n\to\infty}\frac{w_n(a_n)}{(2a_n\log(1/a_n))^{1/2}}=1 \qquad \text{a.s.}$$

Following the terminology of Stute, any sequence satisfying the conditions (S1)–(S3) will be called in the sequel a **bandsequence**. The strong limiting behaviour of the oscillation modulus of α_n and of the closely related Bahadur process were further investigated by several authors (see, e.g., Deheuvels and Mason, 1992, and the references therein), but the above result will be sufficient for our needs.

Fact 2 (Theorem 4.5.3 of Csörgő and Révész, 1981). There exists a probability space on which sit a sequence $U_1, U_2, \ldots$ of independent uniform (0, 1) random variables and a Kiefer process $\{K(y, t);\ 0 \leqslant y \leqslant 1, t \geqslant 0\}$ such that

$$\sup_{0<p<1} |u_n(p) - n^{-1/2} K(p, n)| = O(n^{-1/4} (\log n)^{1/2} (\log_2 n)^{1/4}) \qquad \text{a.s.}$$

Fact 3 (Chevalier, 1976). Let k_n be an intermediate sequence.

(i) There exists a sequence (a_n) such that $U_{k_n, n} < a_n$ in probability, if and only if

$$a_n = \frac{k_n + \psi_1(n)(k_n)^{1/2}}{n},$$

with $\lim_{n \to \infty} \psi_1(n) = \infty$.

(ii) There exists a sequence (b_n) such that $b_n < U_{k_n, n}$ in probability, if and only if

$$b_n = \frac{k_n - \psi_2(n)(k_n)^{1/2}}{n},$$

with $\lim_{n \to \infty} \psi_2(n) = \infty$.

The last fact concerns the strong limiting behaviour of central order statistics. Let p be a fixed constant such that $0 < p < 1$.

Fact 4 (Kiefer, 1967). For (c_n) positive and nondecreasing,

$$P\left\{ \pm \frac{n^{1/2}(U_n(p) - p)}{(p(1-p))^{1/2}} > c_n \text{ i.o.} \right\} = 0 \qquad \text{or} \qquad 1,$$

according to whether

$$\sum_n n^{-1} c_n \exp(-c_n^2/2) < \infty \qquad \text{or} \qquad = \infty.$$

Consider now the sequence of real numbers (y_n) such that

$$0 < y_n < 1, \qquad y_n \downarrow 0, \qquad \text{and} \qquad n y_n \uparrow. \tag{2}$$

The strong limiting behaviour of $(y_n(1-y_n))^{-1/2} u_n(y_n)$ for certain sequences (y_n), with $\lim_{n \to \infty} y_n / \log_2 n = \infty$, was investigated in Brito (1990). The main results are summarised in the following two theorems.

THEOREM 3 (Brito, 1990). *Assume that* (y_n) *is a real-valued sequence satisfying* (2) *jointly with*

$$n y_n / (\log n)^4 (\log_2 n) \to \infty. \tag{3}$$

Then one can construct a probability space with a sequence of independent uniform-$(0, 1)$ *random variables* $U_1, U_2, \ldots$ *and a Kiefer process* $\{K(y, t); 0 \leqslant y \leqslant 1, t \geqslant 0\}$ *such that*

$$(y_n(1 - y_n))^{-1/2} \, |u_n(y_n) - n^{-1/2} K(y_n, n)| = o((\log_2 n)^{-1/2}) \qquad a.s. \tag{4}$$

This approximation is the essential tool for characterising the strong limiting behaviour of the weighted quantile process along the sequence (y_n). The proof is based upon the strong approximation of Komlós *et al.* (1975) and the continuity modulus of a Kiefer process (cf. Theorem 1.15.2 of Csörgő and Révész, 1981).

Let Φ_0 denote the class of positive increasing functions ϕ defined on $[t_0, \infty)$, $t_0 \geqslant 1$. For each $\phi \in \Phi_0$ we define

$$I(\phi) = \int_{t_0}^{\infty} t^{-1} \phi(t) \exp(-(\phi(t))^2/2) \, dt.$$

THEOREM 4 (Brito, 1990). *Assume that* $\phi \in \Phi_0$, *with* $t_0 \geqslant \exp(e)$. *If* y_n *satisfies* (2) *and* (3), *then*

$$P\left\{ \pm \frac{u_n(y_n)}{(y_n(1 - y_n))^{1/2}} > \phi(n) \; i.o. \right\} = 0 \qquad or \qquad 1,$$

according to whether

$$I(\phi) < \infty \qquad or \qquad = \infty.$$

The proof of this result is based on the corresponding theorem for the sequence of Gaussian random variables $(Z_n = (y_n(1 - y_n))^{-1/2} K(y_n, n))$, which follows from the results of Csörgő and Hall (1982) concerning the upper and lower classes for triangular arrays.

In Brito (1990) it was remarked that the above characterisation might be derived via the empirical process α_n. We show here that, if we follow this approach, the previous theorem is an easy consequence of the results of Csörgő and Hall (1982) when used in conjunction with Fact 1. We start with the following lemma, which provides a characterisation of the upper and lower classes of weighted empirical processes along the sequence (y_n). Much of this result is given in Example 2 of Csörgő and Hall (1982), thus we omit the proof.

LEMMA 1. *If y_n satisfies* (2) *and* (3), *then*

$$P\left\{\pm\frac{\alpha_n(y_n)}{(y_n(1-y_n))^{1/2}}>\phi(n)\ i.o.\right\}=0 \qquad or \qquad 1,$$

according to whether

$$I(\phi)<\infty \qquad or \qquad =\infty.$$

Now, we turn to the deviation between empirical and quantile processes along the sequence (y_n).

LEMMA 2. *If y_n satisfies* (2) *and* (3), *then*

$$(y_n(1-y_n))^{-1/2}\,|u_n(y_n)+\alpha_n(y_n)|=o((\log_2 n)^{-1/2}) \qquad a.s. \tag{5}$$

Proof. Observe that

$$u_n(s)=-\alpha_n(U_n(s))+n^{1/2}(G_n(U_n(s))-s), \qquad 0<s<1 \qquad \text{a.s.}$$

Using this representation for u_n and the obvious inequality

$$n^{1/2}(G_n(U_n(s))-s)<n^{-1/2}, \qquad 0<s<1 \qquad \text{a.s.}$$

we obtain

$$|u_n(y_n)+\alpha_n(y_n)|\leqslant|\alpha_n(y_n)-\alpha_n(U_n(y_n))|+n^{-1/2} \qquad \text{a.s.}$$

Set $b_n=|U_n(y_n)-y_n|$. By the law of the iterated logarithm for $u_n(y_n)$ (cf. Eicker, 1970, and Kiefer, 1972), we may assume that $b_n\leqslant 2n^{-1/2}(y_n\log_2 n)^{1/2}\equiv h_n$ a.s. for all large n. Consequently,

$$\begin{aligned}|u_n(y_n)+\alpha_n(y_n)|&\leqslant w_n(b_n)+n^{-1/2} \qquad \text{a.s.}\\ &\leqslant w_n(h_n)+n^{-1/2} \qquad \text{a.s.}\end{aligned}$$

It is easily verified that if y_n satisfies (2) and $\lim_{n\to\infty} ny_n\log_2 n/(\log n)^2=\infty$, then (h_n) is a bandsequence and $\log(h_n^{-1})=O(\log n)$ a.s. Using now Fact 1, a straightforward computation taking into account (2) and (3) yields the result. ∎

Proof of Theorem 4. In view of (5), the upper and lower class tests stated in Lemma 1 may be transferred from α_n to u_n by using similar arguments as in Csörgő and Hall (1982, cf. Corollary in Section 3). ∎

Remark 1. From Fact 2, we obtain immediately that, on the Csörgő and Révész probability space, the strong approximation (4) holds for sequences $y_n \downarrow 0$ such that $ny_n \uparrow$ and

$$ny_n/(n^{1/2} \log n(\log_2 n)^{3/2}) \to \infty. \tag{6}$$

Hence, a direct application of the strong approximation of Csörgő and Révész would also yield the result of Theorem 4, but the assumption (3) should be replaced by the much more restrictive condition (6).

Let $k_n =]ny_n[$, where $]x[$ denotes the smallest integer greater than or equal to x. It follows from the definition of u_n that

$$u_n(k_n/n) = n^{1/2}(U_{k_n, n} - k_n/n). \tag{7}$$

Assume now that y_n satisfies the conditions (2) and (3). Then it is easy to check that

$$\frac{u_n(y_n)}{(y_n(1-y_n))^{1/2}} - \frac{u_n(k_n/n)}{(k_n/n(1-k_n/n))^{1/2}} = o((\log_2 n)^{-1/2}) \qquad \text{a.s.}$$

Using this fact jointly with (7), we translate in a similar way the result of Theorem 4 in terms of $u_n(k_n/n)$,

$$P\{(k_n(1-k_n/n))^{-1/2} \, |nU_{k_n, n} - k_n| > \phi(n) \text{ i.o.}\} = 0 \qquad \text{or} \qquad 1, \tag{8}$$

according to whether $I(\phi) < \infty$ or $= \infty$. By applying now an integral test for convergence, we obtain

Corollary 2. *Assume that $\phi \in \Phi_0$, with $t_0 \geqslant \exp(e)$. If y_n satisfies* (2) *and* (3), *then*

$$P\{(k_n(1-k_n/n))^{-1/2} \, |nU_{k_n, n} - k_n| > \phi(n) \ i.o.\} = 0 \qquad or \qquad 1,$$

according to whether

$$\sum_n n^{-1}\phi(n) \exp(-\phi(n)^2/2) < \infty \qquad or \qquad = \infty.$$

Remark 2. In view of this corollary, the upper-lower class test of Kiefer (1967) for central order statistics also holds for "large" intermediate order statistics, under the above conditions.

Remark 3. From the above characterisation of $u_n(k_n/n)$ we obtain the following strong asymptotic bounds (for further details see Brito, 1987, Corollary 3.3.3)

$$P\{(k_n(1-k_n/n))^{-1/2}\,|nU_{k_n,n}-k_n|>(2(\log_2 n+(3/2)\log_3 n+\log_4 n + \cdots +(1+\varepsilon)\log_p n))^{1/2} \text{ i.o.}\}=0 \quad \text{or} \quad 1,$$

according to whether $\varepsilon>0$ or $\varepsilon\leqslant 0$, where $\log_j$ denotes the jth iterated logarithm.

Now, we are in the position of giving a necessary and sufficient condition for the almost sure stability of univariate intermediate order statistics.

COROLLARY 3. *Let $W_1, W_2, \ldots$ be i.i.d. random variables with distribution function F such that $F(x)>0$ for all x, and let $Q(u)=\inf\{x : F(x)\geqslant u\}$ for $0<u<1$. Assume that y_n satisfies (2) and (3). Then, $(W_{k_n,n})$ is a.s stable if and only if there exist two increasing functions ψ_1 and ψ_2, with $\lim_{n\to\infty}\psi_1(n)=\lim_{n\to\infty}\psi_2(n)=\infty$ and such that*

$$I(\psi_1)<\infty, \qquad I(\psi_2)<\infty$$

and

$$\lim_{n\to\infty}(Q(a_n)-Q(b_n))=0,$$

where $a_n=(k_n+\psi_1(n)(k_n)^{1/2})/n$ and $b_n=(k_n-\psi_2(n)(k_n)^{1/2})/n$.

Proof. First note that we may assume without loss of generality that $W_{k,n}=Q(U_{k,n})$, $1\leqslant k\leqslant n$. Thus the sufficiency part is an immediate consequence of Theorem 4, written in terms of $u_n(k_n/n)$ (see (8)). On the other hand, the necessary part follows from Theorem 4 jointly with Fact 3. ▮

Finally, we recall that the characterisation of the upper and lower almost sure classes of $U_{k_n,n}$, where $k_n=\mathrm{O}(\log_2 n)$ has been studied in detail by Deheuvels (1986, 1987) and will not be considered here. However, we note that the situation is more complex for these cases. In particular, the assumed conditions of regularity concern, not only the sequence (k_n), but also the bound sequences (a_n) and (b_n).

4. CONCLUSION

We give now a criterion of stability for $(X^*_{n-k_n+1,n})$. From Theorem 2 and Corollary 3 we obtain

THEOREM 5. *Let* $k_n =]ny_n[$ *and assume that* y_n *satisfies* (2) *and* (3). *Then, under condition* (H) *the sequence* $(X^*_{n-k_n+1,n})$ *is a.s. stable if and only if there exist* θ_1 *and two increasing functions* ψ_1 *and* ψ_2, *with* $\lim_{n\to\infty} \psi_1(n) = \lim_{n\to\infty} \psi_2(n) = \infty$ *and such that*

$$I(\psi_1) < \infty, \qquad I(\psi_2) < \infty \tag{9}$$

and

$$\lim_{n\to\infty} (F^{-1}_{\theta_1}(1-b_n) - F^{-1}_{\theta_1}(1-a_n)) = 0, \tag{10}$$

where $a_n = (k_n + \psi_1(n)(k_n)^{1/2})/n$ *and* $b_n = (k_n - \psi_2(n)(k_n)^{1/2})/n$.

Remark 4. In the same way a sufficient condition is given by the assertion (i) of Theorem 2.

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REFERENCES

1. M. F. Barme-Delcroix, Localisation asymptotique des échantillons multidimensionnels, *Bull. Soc. Math. Belg. Sér. B* **45** (1993).
2. M. F. Barme and U. Gather, Limit laws for multivariate extremes, *in* "XXVIII Journées de Statistique," Quebec, 1996.
3. O. Barndorff-Nielsen, On the limit behaviour of extreme order statistics, *Ann. Math. Statist.* **34** (1963), 992–1002.
4. V. Barnett, The ordering of multivariate data, *J. Roy. Statist. Soc. Ser. A* **139** (1976), 318–355.
5. M. Brito, Sur l'encadrement optimal presque sûr dans un échantillon ordonné, *C. R. Acad. Sci. Paris Sér. I* **303** (1986), 821–824.
6. M. Brito, "Encadrement presque sûr des statistiques d'ordre," Ph.D. thesis, University of Paris VI, 1987.
7. M. Brito, Sobre o comportamento limite forte das estatísticas de ordem, *in* "Actas da 1a Conferência em Estatística e Optimização" (M. A. Turkman and M. L. Salema, Eds.), pp. 113–126, Troia, 1990.
8. J. Chevalier, Majoration e minoration asymptotique dans un échantillon ordonné, *C. R. Acad. Sci. Paris Sér. A* **282** (1976), 235–238.
9. S. Csörgő and P. Hall, Upper and lower classes for triangular arrays, *Z. Wahrsch. Verw. Gebiete* **61** (1982), 207–222.
10. M. Csörgő and P. Révész, "Strong Approximations in Probability and Statistics," Academic Press, New York, 1981.
11. R. Davis, E. Mulrow, and S. Resnick, The convex hull of a random sample in $\mathbf{R}^2$, *Comm. Statist. Stochastic Models* **3** (1987), 1–27.

12. P. Deheuvels, Strong laws for the kth order statistic when $k \leqslant c \log_2 n$, *Probab. Theory Related Fields* **72** (1986), 133–154.
13. P. Deheuvels, Strong laws for the kth order statistic when $k \leqslant c \log_2 n$, II, *in* "Extreme Value Theory" (J. Hüsler and R. D. Reiss, Eds.), Lecture Notes in Statistics, Vol. 51, pp. 21–35, Springer-Verlag, New York, 1987.
14. P. Deheuvels and D. Mason, Functional laws of the iterated logarithm for the increments of empirical and quantile processes, *Ann. Probab.* **20** (1992), 1248–1287.
15. M. F. Delcroix, Un exemple de localisation asymptotique du polyèdre d'appui d'un échantillon de $\mathbf{R}^2$, *Pub. Inst. Statist. Univ. Paris* **38** (1994), 25–36.
16. M. F. Delcroix and P. Jacob, Stability of extreme value for a multidimensional sample, *Statist. Anal. Données* **16** (1991), 1–21.
17. J. Eicker, A Log Log law for double sequences of random variables, I, *Z. Wahrsch. Verw. Gebiete* **16** (1970), 107–133.
18. J. Galambos, "The Asymptotic Theory of Extreme Order Statistics," Krieger, Melbourne, FL, 1987.
19. U. Gather and B. Rauhut, The outlier behaviour of probability distributions, *J. Statist. Plann. Inference* **26** (1990), 237–252.
20. U. Gather and J. Tomkins, On stability of intermediate order statistics, *J. Statist. Plann. Inference* **45** (1995), 175–183.
21. J. Geffroy, Contribution à la théorie des valeurs extrêmes, *Publ. Inst. Statist. Univ. Paris* **7–8** (1958–1959), 37–185.
22. J. Geffroy, Localisation asymptotique du polyèdre d'appui d'un échantillon laplacien à k dimension, *Publ. Inst. Statist. Univ. Paris* **59** (1961).
23. J. Geffroy, Sur la distribution limite du terme maximum d'une série aléatoire, *Ann. Math.* **44** (1961), 423–453.
24. J. Kiefer, On Bahadur's representation of sample quantiles, *Ann. Math. Statist.* **38** (1967), 1323–1342.
25. J. Kiefer, Iterated logarithm analogous for sample quantiles when $p_n \downarrow 0$, *in* "Proc. Sixth Berkeley Symp. on Math. Statist. and Probab.," Vol. 1, pp. 227–244, Univ. of California Press, Berkeley, 1972.
26. J. Komlós, P. Major, and G. Tusnády, An approximation of partial sums of independent R.V.'s and the sample D.F., I, *Z. Wahrsch. Verw. Gebiete* **32** (1975), 111–131.
27. R. A. Maller, Defining extremes and trimming by minimum covering sets, *Stochastic Process. Appl.* **35** (1990), 169–180.
28. J. C. Masse and R. Theodorescu, Halfplane trimming for bivariate distribution, *J. Multivariate Anal.* **48** (1994), 188–202.
29. S. I. Resnick and R. J. Tomkins, Almost sure stability of maxima, *J. Appl. Probab.* **10** (1973), 387–401.
30. G. R. Shorack and J. A. Wellner, Linear bounds on the empirical distribution function, *Ann. Probab.* **6** (1978), 349–353.
31. N. V. Smirnov, Limit distributions for the terms of a variational series, *in* "Amer. Math. Soc. Transl.," Vol. 67, pp. 82–143, Amer. Math. Soc., Providence, 1952.
32. W. Stute, The oscillation behavior of empirical processes, *Ann. Probab.* **10** (1982), 86–107.
33. R. J. Tomkins, Rates of convergence for the stability of large order statistics, *J. Theoret. Probab.* **9** (1996), 841–851.