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Stochastic comparisons of order statistics and their concomitants

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ABSTRACT

Let $X_{1:n} \leq X_{2:n} \cdots \leq X_{n:n}$ be the order statistics from some sample, and let $Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$ be the corresponding concomitants. One purpose of this paper is to obtain results that stochastically compare, in various senses, the random vector $(X_{r:n}, Y_{[r:n]})$ to the random vector $(X_{r+1:n}, Y_{[r+1:n]})$, $r = 1, 2, \ldots, n-1$. Such comparisons are called *one-sample comparisons*. Next, let $S_{1:n} \leq S_{2:n} \cdots \leq S_{n:n}$ be the order statistics constructed from another sample, and let $T_{[1:n]}, T_{[2:n]}, \ldots, T_{[n:n]}$ be the corresponding concomitants. Another purpose of this paper is to obtain results that stochastically compare, in various senses, the random vector $(X_{r:n}, Y_{[r:n]})$ with the random vector $(S_{r:n}, T_{[r:n]})$, $r = 1, 2, \ldots, n$. Such comparisons are called *two-sample comparisons*. It is shown that some of the results in this paper strengthen previous results in the literature. Some applications in reliability theory are described.

1. Introduction

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent copies of a random vector (X, Y). Let $X_{1:n} \leq X_{2:n} \cdots \leq X_{n:n}$ be the order statistics constructed from the sample of the first coordinates $X_1, X_2, \ldots, X_n$. Denote the Y-variate associated with $X_{i:n}$ by $Y_{[i:n]}$, $i = 1, 2, \ldots, n$; that is, $Y_{[i:n]} = Y_k$ if, and only if, $X_{i:n} = X_k$. The random variables $Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}$ are called the concomitants of the order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. One purpose of this paper is to obtain results that stochastically compare, in various senses, the random vector $(X_{r:n}, Y_{[r:n]})$ to the random vector $(X_{r+1:n}, Y_{[r+1:n]})$, $r = 1, 2, \ldots, n-1$. Such comparisons will be called below *one-sample comparisons*.

Next, let $(S_1, T_1), (S_2, T_2), \ldots, (S_n, T_n)$ be another sample of independent copies of a random vector, but this time that random vector is (S, T), which generally has a different distribution function than (X, Y). Let $S_{1:n} \leq S_{2:n} \cdots \leq S_{n:n}$ be the order statistics constructed from the sample of the first coordinates $S_1, S_2, \ldots, S_n$, and let $T_{[1:n]}, T_{[2:n]}, \ldots, T_{[n:n]}$ be the associated concomitants. A purpose of this paper is to obtain results that stochastically compare, in various senses, the random vector $(X_{r:n}, Y_{[r:n]})$ with the random vector $(S_{r:n}, T_{[r:n]})$, $r = 1, 2, \ldots, n$. Such comparisons will be called below *two-sample comparisons*.

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Applications of concomitants of order statistics can be found in many areas of probability and statistics. For example, concomitants are of interest in a variety of estimation problems (see [8]), in selection and prediction problems (see [16]), in insurance (see [5]), and in reliability theory (see [2]).

Some papers that studied positive dependence and/or stochastic orders involving concomitants of order statistics are the following. Khaledi and Kochar [16] identified conditions on the distribution function of (X, Y) under which the concomitants are ordered with respect to several univariate stochastic orders, whereas Khaledi and Kochar [16] and Blessinger [9] identified conditions on the distribution function of (X, Y) under which the concomitants are positively dependent in various senses. Eryilmaz [10] obtained results that stochastically compare concomitants without assuming that $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are identically distributed. Izadi and Khaledi [11] considered stochastic orderings of concomitants of progressive type II censored order statistics, and recently Amini et al. [1] studied properties and orderings of concomitants of record values. A study of concomitants of order statistics for dependent samples was developed by [22].

In this paper "increasing" and "decreasing" stand for "nondecreasing" and "nonincreasing", respectively. For any random variable (or vector) Z and an event A, we will denote by $[Z \big| A]$ any random variable that is distributed according to the conditional distribution function of Z given A. We will use the notation $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

In Sections 2 and 3 below we describe, respectively, our one-sample and two-sample comparison results. In Section 4 we mention examples and applications of the technical results, and we make various remarks on their relations to other results in the literature, and on strictness of the assumptions in different results. But before proceeding to the technical results in Sections 2 and 3 we list below and make some comments on various stochastic orders that will be used in the sequel.

All the stochastic orders that are mentioned in this paper can be found in [19] or in [21]. Specifically, $\le_{\text{st}}$ stands for the univariate *ordinary stochastic order* when two univariate random variables are compared (see, for example, [21, page 3]), and it stands for the multivariate ordinary stochastic order when two random vectors are compared (see its formal definition in [21, page 266]). The notation $\le_{\text{hr}}$ stands for the univariate *hazard rate order* when two univariate random variables are compared (see, for example, [21, page 16]), and it stands for the multivariate hazard rate order when two random vectors are compared (see its formal definition in (6.D.1) in [21]). Another order that will be used in this paper is the *likelihood ratio order* $\le_{\text{lr}}$. The univariate definition of it can be found in [21, page 42], whereas the multivariate definition is given in that reference in page 298. Recall also that for two random vectors (X, Y) and (S, T) we say that (X, Y) is smaller than (S, T) in the *upper orthant order*, denoted as $(X, Y) \le_{\text{uo}} (S, T)$ (respectively, (X, Y) is smaller than (S, T) in the *lower orthant order*, denoted as $(X, Y) \le_{\text{lo}} (S, T)$) if $P\{X > x, Y > y\} \le P\{S > x, T > y\}$ (respectively, $P\{X \le x, Y \le y\} \le P\{S \le x, T \le y\}$) for all $(x, y) \in \mathbb{R}^2$; see Section 6.G.1 in [21].

We further recall the definition of the *strong stochastic order* (see [18] or [21, page 268]). Let (X, Y) and (S, T) be two random vectors. Suppose that

$$X \le_{\text{st}} S \tag{1.1}$$

and that

$$[Y \big| X = x] \le_{\text{st}} [T \big| S = s] \quad \text{whenever } x \le s. \tag{1.2}$$

Then (X, Y) is said to be smaller than (S, T) in the strong stochastic order, and it is denoted as $(X, Y) \le_{\text{sst}} (S, T)$. The multivariate order $\le_{\text{sst}}$ is stronger than the multivariate order $\le_{\text{st}}$.

2. One-sample comparisons

For the proof of the first result we will need the following lemma. To begin with, we recall the definition of the positive dependence concept of *stochastic increasingness*: let (V, W) be a random vector. We say that W is stochastically increasing in V if the conditional random variable $[W \big| V = v]$ is stochastically increasing in $v \in \text{support}\{V\}$ with respect to the univariate ordinary stochastic order $\le_{\text{st}}$.

Lemma 2.1. *Let $(V_1, W_1), (V_2, W_2), \ldots, (V_n, W_n)$ be random vectors. If*

$$V_r \le_{st} V_{r+1}, \quad r = 1, 2, \ldots, n-1, \tag{2.1}$$

and if

$$W_r \text{ is stochastically increasing in } V_r \text{ with respect to } \le_{st}, \quad r = 1, 2, \ldots, n, \tag{2.2}$$

then

$$(V_r, W_r) \le_{st} (V_{r+1}, W_{r+1}), \quad r = 1, 2, \ldots, n-1.$$

Proof. From (2.1) it follows that we can construct, on some probability space, random variables $\widehat{V}_1, \widehat{V}_2, \ldots, \widehat{V}_n$ such that $\widehat{V}_r =_{\text{st}} V_r$, $r = 1, 2, \ldots, n$, and such that

$$\widehat{V}_1 \le \widehat{V}_2 \le \cdots \le \widehat{V}_n \quad \text{almost surely.} \tag{2.3}$$

Furthermore, let U be a uniform(0,1) random variable that is independent of the $\widehat{V}_r$'s. For $r = 1, 2, \dots, n$, denote the conditional distribution function of $[W_r \big| V_r = v]$ by $F_{[W_r|V_r=v]}(w)$, and its inverse by $F^{-1}_{[W_r|V_r=v]}(u)$. For convenience, consider the function g_r defined by $g_r(v, u) = F^{-1}_{[W_r|V_r=v]}(u)$. Now define (on the same probability space as the $\widehat{V}_r$'s) the random variables

$$\widehat{W}_r \equiv g_r(\widehat{V}_r, U), \quad r = 1, 2, \dots, n.$$

We note that

$$(\widehat{V}_r, \widehat{W}_r) =_{\mathrm{st}} (V_r, W_r), \quad r = 1, 2, \dots, n.$$

From assumption (2.2) it follows that $g_r(v, u)$ is increasing in v. This monotonicity, together with (2.3), gives

$$(\widehat{V}_r, \widehat{W}_r) \le_{\mathrm{a.s.}} (\widehat{V}_{r+1}, \widehat{W}_{r+1}), \quad r = 1, 2, \dots, n-1.$$

It follows, from Theorem 6.B.1 in [21], that $(V_r, W_r) \le_{\mathrm{st}} (V_{r+1}, W_{r+1})$, $r = 1, 2, \dots, n-1$. □

Using Lemma 2.1 we get the first comparison result:

Theorem 2.2. *Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be independent copies of random vector (X, Y). Suppose that*

$$Y \text{ is stochastically increasing in } X \text{ with respect to } \le_{st}. \tag{2.4}$$

Then

$$(X_{r:n}, Y_{[r:n]}) \le_{st} (X_{r+1:n}, Y_{[r+1:n]}), \quad r = 1, 2, \dots, n-1. \tag{2.5}$$

Proof. We apply Lemma 2.1 with $(X_{r:n}, Y_{[r:n]})$ playing the role of (V_r, W_r), $r = 1, 2, \dots, n$. In order to do that we need to verify (2.1) and (2.2). Obviously, $X_{r:n} \le_{\mathrm{st}} X_{r+1:n}$, and hence (2.1) holds. Next, the assumption that Y is stochastically increasing in X is the same as the statement that $Y_{[r:n]}$ is stochastically increasing in $X_{r:n}$, and hence (2.2) holds. Thus, the stated result follows from Lemma 2.1. □

We proceed now to a result that assumes a stronger condition than (2.4) on the strength of the dependence between X and Y, and attains a conclusion that is different than (but similar to) (2.5). Recall that the univariate order $\le_{\mathrm{hr}}$ implies the univariate order $\le_{\mathrm{st}}$.

In the proof of the next result, and later in the paper, we denote by $F_{X_{r:n}}$ and $\overline{F}_{X_{r:n}} \equiv 1 - F_{X_{r:n}}$ the distribution function and the survival function of $X_{r:n}$, and we denote by $\overline{F}_{Y|X}(y \big| x) \doteq P\{Y > y \big| X = x\}$ the conditional survival function of Y given $X = x$.

Recall from [13] that a nonnegative function g of two variables x and y is said to be totally positive of order 2 (TP_2) if $g(x, y')g(x', y) \le g(x, y)g(x', y')$ whenever $x \le x'$ and $y \le y'$.

Theorem 2.3. *Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be independent copies of random vector (X, Y). Suppose that*

$$Y \text{ is stochastically increasing in } X \text{ with respect to } \le_{hr}. \tag{2.6}$$

Then

$$(X_{r:n}, Y_{[r:n]}) \le_{hr} (X_{r+1:n}, Y_{[r+1:n]}), \quad r = 1, 2, \dots, n-1. \tag{2.7}$$

Proof. For $r = 1, 2, \dots, n$, the survival function of $(X_{r:n}, Y_{[r:n]})$ is given by

$$\begin{aligned} \overline{F}_{(X_{r:n}, Y_{[r:n]})}(x, y) &= \int_{\tilde{x}=x}^{\infty} \overline{F}_{Y|X}(y \big| \tilde{x})\, dF_{X_{r:n}}(\tilde{x}) \\ &= \int_{\tilde{x}=-\infty}^{\infty} I(x, \tilde{x}) \cdot \overline{F}_{Y|X}(y \big| \tilde{x})\, dF_{X_{r:n}}(\tilde{x}), \end{aligned}$$

where $I(x, \tilde{x}) = 0$ if $\tilde{x} \le x$ and $I(x, \tilde{x}) = 1$ if $\tilde{x} > x$. From Theorem 1.C.37 in [21] we have that $X_{r:n} \le_{\mathrm{lr}} X_{r+1:n}$. Hence $X_{r:n} \le_{\mathrm{hr}} X_{r+1:n}$. That is, the survival function

$$\overline{F}_{X_{r:n}}(\tilde{x}) \quad \text{is } TP_2 \text{ in } r \text{ and } \tilde{x}. \tag{2.8}$$

The assumption (2.6) that Y is stochastically increasing in X with respect to the order $\le_{\mathrm{hr}}$ implies that

$$\overline{F}_{Y|X}(y \big| \tilde{x}) \quad \text{is } TP_2 \text{ in } y \text{ and } \tilde{x} \quad \text{and} \quad \text{is increasing in } \tilde{x}. \tag{2.9}$$

Furthermore, the Heavyside function I satisfies

$$I(x, \tilde{x}) \text{ is } TP_2 \text{ in } x \text{ and } \tilde{x} \quad \text{and} \quad \text{is increasing in } \tilde{x}. \tag{2.10}$$

Using (2.8) and (2.9), we get from Theorem 2.1 of [12] that $\overline{F}_{(X_{r:n},Y_{[r:n]})}(x,y)$ is TP_2 in r and y. Using (2.8) and (2.10), we get, again from Theorem 2.1 of [12], that $\overline{F}_{(X_{r:n},Y_{[r:n]})}(x,y)$ is TP_2 in r and x. Finally, the product $I(x,\tilde{x})\cdot\overline{F}_{Y|X}(y\big|\tilde{x})$ is TP_2 in x and y, and hence $\overline{F}_{(X_{r:n},Y_{[r:n]})}(x,y)$ is TP_2 in x and y. In other words, the function $\overline{F}_{(X_{r:n},Y_{[r:n]})}(x,y)$, of the three variables (r,x,y), is TP_2 in pairs. Since its support is a lattice, we have, from [15], that

$$\overline{F}_{(X_{r:n},Y_{[r:n]})}(x_1,y_1)\overline{F}_{(X_{r+1:n},Y_{[r+1:n]})}(x_2,y_2)\le\overline{F}_{(X_{r:n},Y_{[r:n]})}(x_1\wedge x_2,y_1\wedge y_2)\overline{F}_{(X_{r+1:n},Y_{[r+1:n]})}(x_1\vee x_2,y_1\vee y_2),$$

which yields (2.7). □

Comparing Theorems 2.2 and 2.3 we see, as mentioned earlier, that the assumption (2.6) is stronger than the assumption (2.4), but the conclusions (2.7) and (2.5) are not comparable in the sense that none of them implies the other one.

In the next result we assume a condition that is even stronger than (2.6), and we obtain a conclusion that is stronger than both (2.5) and (2.7).

For the next result we assume that (X,Y) is absolutely continuous, and denote its density function by $f(x,y)$. We also denote the marginal density of X by $f_X(x)$. The conditional density of $[Y\big|X=x]$ is given by

$$f_{Y|X}(y\big|x)=\frac{f(x,y)}{f_X(x)}.$$

In the next result it is assumed that $[Y\big|X=x]$ is stochastically increasing in $x\in\text{support}\{X\}$ with respect to the likelihood ratio order $\le_{\mathrm{lr}}$. That is, for $x\le x'$ we assume that

$$f_{Y|X}(y\big|x')/f_{Y|X}(y\big|x)\quad\text{is increasing in } y;$$

that is,

$$\left(\frac{f(x',y)}{f_X(x')}\right)\Bigg/\left(\frac{f(x,y)}{f_X(x)}\right)\quad\text{is increasing in } y;$$

that is,

$$f(x',y)/f(x,y)\ \text{is increasing in } y;$$

that is, $f(x,y)$ is TP_2.

Theorem 2.4. *Let $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$ be independent copies of an absolutely continuous random vector (X,Y). If*

$$Y\text{ is stochastically increasing in } X \text{ with respect to } \le_{lr}. \tag{2.11}$$

then

$$(X_{r:n},Y_{[r:n]})\le_{lr}(X_{r+1:n},Y_{[r+1:n]}),\quad r=1,2,\ldots,n-1. \tag{2.12}$$

Proof. The density functions of $(X_{r:n},Y_{[r:n]})$ and $(X_{r+1:n},Y_{[r+1:n]})$ are given by

$$f_{(X_{r:n},Y_{[r:n]})}(x,y)=f_{X_{r:n}}(x)\cdot\frac{f(x,y)}{f_X(x)} \tag{2.13}$$

and

$$f_{(X_{r+1:n},Y_{[r+1:n]})}(x,y)=f_{X_{r+1:n}}(x)\cdot\frac{f(x,y)}{f_X(x)}.$$

From Theorem 1.C.37 in [21] we have that $X_{r:n}\le_{\mathrm{lr}}X_{r+1:n}$. Hence

$$f_{X_{r:n}}(x_1)f_{X_{r+1:n}}(x_2)\le f_{X_{r:n}}(x_1\wedge x_2)f_{X_{r+1:n}}(x_1\vee x_2).$$

From the stochastic monotonicity assumption (2.11) (the TP_2-ness) we have

$$f(x_1,y_1)f(x_2,y_2)\le f(x_1\wedge x_2,y_1\wedge y_2)f(x_1\vee x_2,y_1\vee y_2).$$

It follows that

$$\begin{aligned}&f_{X_{r:n}}(x_1)\cdot\frac{f(x_1,y_1)}{f_X(x_1)}\cdot f_{X_{r+1:n}}(x_2)\cdot\frac{f(x_2,y_2)}{f_X(x_2)}\\&\quad\le f_{X_{r:n}}(x_1\wedge x_2)\cdot\frac{f(x_1\wedge x_2,y_1\wedge y_2)}{f_X(x_1\wedge x_2)}\cdot f_{X_{r+1:n}}(x_1\vee x_2)\cdot\frac{f(x_1\vee x_2,y_1\vee y_2)}{f_X(x_1\vee x_2)};\end{aligned}$$

that is,

$$\begin{aligned}&f_{(X_{r:n},Y_{[r:n]})}(x_1,y_1)f_{(X_{r+1:n},Y_{[r+1:n]})}(x_2,y_2)\\&\quad\le f_{(X_{r:n},Y_{[r:n]})}(x_1\wedge x_2,y_1\wedge y_2)f_{(X_{r+1:n},Y_{[r+1:n]})}(x_1\vee x_2,y_1\vee y_2);\end{aligned}$$

and this yields the stated result (2.12). □

As we said earlier, the assumption (2.11) is stronger than both assumptions (2.4) and (2.6) since the univariate order $\le_{\text{lr}}$ implies both univariate orders $\le_{\text{st}}$ and $\le_{\text{hr}}$. On the other hand, by Theorem 6.E.8 and 6.E.6 in [21] it is seen that the conclusion (2.12) is stronger than conclusions (2.5) and (2.7).

The weakest among the positive dependence conditions (2.4), (2.6), and (2.11) is (2.4). This leads one to wonder whether, under a positive dependence condition that is weaker than (2.4), one can obtain a stochastic comparison of $(X_{r:n}, Y_{[r:n]})$ and $(X_{r+1:n}, Y_{[r+1:n]})$ that is weaker than (2.5).

For example, one candidate positive dependence condition that is weaker than (2.4) is the positive quadrant dependent (PQD) condition. Recall that a random vector (X, Y) is said to be PQD if

$$P\{X \le x, Y \le y\} \ge P\{X \le x\}P\{Y \le y\} \quad \text{for all } x, y \in \mathbb{R},$$

or, equivalently, if

$$P\{X > x, Y > y\} \ge P\{X > x\}P\{Y > y\} \quad \text{for all } x, y \in \mathbb{R}.$$

The PQD condition is a positive dependence condition that is indeed weaker than condition (2.4) (see page 146 of [6]). Next, candidate orders that are weaker than $\le_{\text{st}}$ and $\le_{\text{hr}}$ (and hence also weaker than $\le_{\text{lr}}$) are $\le_{\text{uo}}$ and $\le_{\text{lo}}$. Using this notion of positive dependence, and these stochastic orders, we are led to conjecture that if (X, Y) is PQD then $(X_{r:n}, Y_{[r:n]}) \le_{\text{lo}} (X_{r+1:n}, Y_{[r+1:n]})$ and $(X_{r:n}, Y_{[r:n]}) \le_{\text{uo}} (X_{r+1:n}, Y_{[r+1:n]})$, $r = 1, 2, \ldots, n-1$. We have not been able to prove (or disprove) this conjecture.

Furthermore, an examination of the conditions (2.4), (2.6), and (2.11) leads one also to wonder whether a positive dependence relationship between X and Y leads to the same positive dependence relationship between $X_{r:n}$ and $Y_{[r:n]}$. Since the conditional distribution function of Y given $X = x$ is the same as the conditional distribution function of $Y_{[r:n]}$ given $X_{r:n} = x$ it obviously follows that if (X, Y) satisfies (2.4) (respectively, (2.6), (2.11)) then so does $(X_{r:n}, Y_{[r:n]})$. However, the following counterexample shows that the PQD condition on (X, Y), that is weaker than (2.4), does not necessarily imply that the same positive dependence condition also holds for $(X_{r:n}, Y_{[r:n]})$.

Counterexample 2.5. Let (X, Y) have the following joint probability function:

$X \backslash Y$	0	1	2	
0	.1	.1	.1	.3
1	.1	0	.1	.2
2	.1	.1	.3	.5
	.3	.2	.5	1

It is not too hard to verify that (X, Y) is PQD. A straightforward computation yields the following joint probability function of $(X_{1:2}, Y_{[1:2]})$:

$X_{1:2} \backslash Y_{[1:2]}$	0	1	2	
0	.17	.17	.17	.51
1	.12	0	.12	.24
2	.05	.05	.15	.25
	.34	.22	.44	1

Since $P\{X_{1:2} \ge 1, Y_{[1:2]} \ge 1\} = .3200 < .3234 = (.49)(.66) = P\{X_{1:2} \ge 1\}P\{Y_{[1:2]} \ge 1\}$ we see that $(X_{1:2}, Y_{[1:2]})$ is not PQD. ⋆

3. Two-sample comparisons

As is described in Section 1, let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a sample of independent copies of a random vector (X, Y), and let $(S_1, T_1), (S_2, T_2), \ldots, (S_n, T_n)$ be another sample of independent copies of a random vector (S, T). From Section 1, recall the definition of the random vectors $(X_{r:n}, Y_{[r:n]})$ and $(S_{r:n}, T_{[r:n]})$, $r = 1, 2, \ldots, n$. One may expect that if (X, Y) and (S, T) are ordered with respect to some multivariate stochastic order, then $(X_{r:n}, Y_{[r:n]})$ and $(S_{r:n}, T_{[r:n]})$ may be similarly ordered. The results below give some formal versions of such an idea.

Theorem 3.1. *Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a sample of independent copies of a random vector (X, Y), and let $(S_1, T_1), (S_2, T_2), \ldots, (S_n, T_n)$ be another sample of independent copies of a random vector (S, T). If*

$$(X, Y) \le_{sst} (S, T) \tag{3.1}$$

then

$$(X_{1:n}, X_{2:n}, \ldots, X_{n:n}, Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}) \le_{st} (S_{1:n}, S_{2:n}, \ldots, S_{n:n}, T_{[1:n]}, T_{[2:n]}, \ldots, T_{[n:n]}). \tag{3.2}$$

Proof. The assumption $(X, Y) \leq_{\text{sst}} (S, T)$ means that (1.1) and (1.2) hold. From (1.1) it follows that we can construct, on some probability space, random variables $\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_n$ and $\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n$ that satisfy

$$\begin{aligned}
&\widehat{X}_1 =_{\text{st}} \widehat{X}_2 =_{\text{st}} \cdots =_{\text{st}} \widehat{X}_n =_{\text{st}} X,\\
&\widehat{S}_1 =_{\text{st}} \widehat{S}_2 =_{\text{st}} \cdots =_{\text{st}} \widehat{S}_n =_{\text{st}} S,\\
&\widehat{X}_1 \leq_{\text{a.s.}} \widehat{S}_1, \widehat{X}_2 \leq_{\text{a.s.}} \widehat{S}_2, \ldots, \widehat{X}_n \leq_{\text{a.s.}} \widehat{S}_n,
\end{aligned} \tag{3.3}$$

and

$$(\widehat{X}_1, \widehat{S}_1), (\widehat{X}_2, \widehat{S}_2), \ldots, (\widehat{X}_n, \widehat{S}_n) \quad \text{are independent.}$$

Denote the conditional distribution function of $[Y \big| X = x]$ by $F_{[Y|X=x]}(y)$, and its inverse by $F^{-1}_{[Y|X=x]}(u)$. Similarly define $F^{-1}_{[T|S=s]}(u)$. For convenience, consider the functions g and h defined by $g(x, u) = F^{-1}_{[Y|X=x]}(u)$ and $h(s, u) = F^{-1}_{[T|S=s]}(u)$. Note that (1.2) is equivalent to

$$g(x, u) \leq h(s, u) \quad \text{for all } u \in (0, 1) \text{ whenever } x \leq s. \tag{3.4}$$

Consider now, with an obvious notation, the order statistics

$$\widehat{X}_{1:n} \leq \widehat{X}_{2:n} \leq \cdots \leq \widehat{X}_{n:n} \quad \text{and} \quad \widehat{S}_{1:n} \leq \widehat{S}_{2:n} \leq \cdots \leq \widehat{S}_{n:n}.$$

From (3.3) it follows that

$$\widehat{X}_{1:n} \leq_{\text{a.s.}} \widehat{S}_{1:n}, \widehat{X}_{2:n} \leq_{\text{a.s.}} \widehat{S}_{2:n}, \ldots, \widehat{X}_{n:n} \leq_{\text{a.s.}} \widehat{S}_{n:n}. \tag{3.5}$$

Let $U_1, U_2, \ldots, U_n$ be independent uniform(0, 1) random variable that are independent of $(\widehat{X}_1, \widehat{S}_1), (\widehat{X}_2, \widehat{S}_2), \ldots, (\widehat{X}_n, \widehat{S}_n)$. As a result, they are also independent of $\widehat{X}_{1:n}, \widehat{X}_{2:n}, \ldots, \widehat{X}_{n:n}$ and of $\widehat{S}_{1:n}, \widehat{S}_{2:n}, \ldots, \widehat{S}_{n:n}$. Now define (on the same probability space as the $\widehat{X}_i$'s and the $\widehat{S}_i$'s) the random variables

$$\widehat{Y}_{[i:n]} \equiv g(\widehat{X}_{i:n}, U_i), \quad i = 1, 2, \ldots, n,$$

and

$$\widehat{T}_{[i:n]} \equiv h(\widehat{S}_{i:n}, U_i), \quad i = 1, 2, \ldots, n.$$

From (3.4) and (3.5) it follows that

$$\widehat{Y}_{[1:n]} \leq_{\text{a.s.}} \widehat{T}_{[1:n]}, \widehat{Y}_{[2:n]} \leq_{\text{a.s.}} \widehat{T}_{[2:n]}, \ldots, \widehat{Y}_{[n:n]} \leq_{\text{a.s.}} \widehat{T}_{[n:n]}. \tag{3.6}$$

We note that

$$(\widehat{X}_{1:n}, \widehat{X}_{2:n}, \ldots, \widehat{X}_{n:n}, \widehat{Y}_{[1:n]}, \widehat{Y}_{[2:n]}, \ldots, \widehat{Y}_{[n:n]}) =_{\text{st}} (X_{1:n}, X_{2:n}, \ldots, X_{n:n}, Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}) \tag{3.7}$$

and that

$$(\widehat{S}_{1:n}, \widehat{S}_{2:n}, \ldots, \widehat{S}_{n:n}, \widehat{T}_{[1:n]}, \widehat{T}_{[2:n]}, \ldots, \widehat{T}_{[n:n]}) =_{\text{st}} (S_{1:n}, S_{2:n}, \ldots, S_{n:n}, T_{[1:n]}, T_{[2:n]}, \ldots, T_{[n:n]}). \tag{3.8}$$

From (3.5)–(3.8), and Theorem 6.B.2 in [21], we obtain (3.2). □

As a corollary of Theorem 3.1 we see that (3.1) implies

$$(X_{r:n}, Y_{[r:n]}) \leq_{\text{st}} (S_{r:n}, T_{[r:n]}), \quad r = 1, 2, \ldots, n.$$

A slightly stronger result is stated and proven in the next theorem.

Theorem 3.2. *Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a sample of independent copies of a random vector (X, Y), and let (S_1, T_1), $(S_2, T_2), \ldots, (S_n, T_n)$ be another sample of independent copies of a random vector (S, T). If*

$$(X, Y) \leq_{sst} (S, T)$$

then

$$(X_{r:n}, Y_{[r:n]}) \leq_{sst} (S_{r:n}, T_{[r:n]}), \quad r = 1, 2, \ldots, n. \tag{3.9}$$

Proof. From (3.1), which implies $X \leq_{\text{st}} S$, we have

$$X_{r:n} \leq_{\text{st}} S_{r:n}, \quad r = 1, 2, \ldots, n. \tag{3.10}$$

For $x \le s$ we have

$$\begin{aligned}[Y_{[r:n]}\big|X_{r:n}=x] &=_{\rm st} [Y\big|X=x]\\ &\le_{\rm st} [T\big|S=s] \quad \text{by (1.2)}\\ &=_{\rm st} [T_{[r:n]}\big|S_{r:n}=s];\end{aligned}$$

that is,

$$[Y_{[r:n]}\big|X_{r:n}=x] \le_{\rm st} [T_{[r:n]}\big|S_{r:n}=s] \quad \text{whenever } x \le s. \tag{3.11}$$

Now, from the definition of the order $\le_{\rm sst}$ in (1.1) and (1.2) we see that (3.10) and (3.11) yield (3.9). □

A combination of Theorems 2.2 and 3.2 yields the following corollary.

Corollary 3.3. *Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a sample of independent copies of a random vector (X, Y), and let (S_1, T_1), $(S_2, T_2), \ldots, (S_n, T_n)$ be another sample of independent copies of a random vector (S, T). If*

$$(X, Y) \le_{sst} (S, T),$$

and if

Y is stochastically increasing in X with respect to $\le_{st}$,

and/or if

T is stochastically increasing in S with respect to $\le_{st}$,

then

$$(X_{r:n}, Y_{[r:n]}) \le_{sst} (S_{r+1:n}, T_{[r+1:n]}), \quad r = 1, 2, \ldots, n-1.$$

We next aim at obtaining a likelihood ratio order analog of Theorem 3.2. First we state a lemma that will be used in the proof of the next theorem (Theorem 3.5).

Lemma 3.4. *Let $x_1, x_2, \ldots, x_n$ and $x'_1, x'_2, \ldots, x'_n$ be two sets of ordered real numbers such that*

$$x_1 \le x_2 \le \cdots \le x_n \quad \text{and} \quad x'_1 \le x'_2 \le \cdots \le x'_n. \tag{3.12}$$

Then

$$x_1 \wedge x'_1 \le x_2 \wedge x'_2 \le \cdots \le x_n \wedge x'_n \tag{3.13}$$

and

$$x_1 \vee x'_1 \le x_2 \vee x'_2 \le \cdots \le x_n \vee x'_n. \tag{3.14}$$

The lemma holds since whenever $x_i \le x_j$ and $x'_i \le x'_j$ we have $x_i \wedge x'_i \le x_j \wedge x'_j$ and $x_i \vee x'_i \le x_j \vee x'_j$.

Theorem 3.5. *Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a sample of independent copies of an absolutely continuous random vector (X, Y), and let $(S_1, T_1), (S_2, T_2), \ldots, (S_n, T_n)$ be a sample of independent copies of another absolutely continuous random vector (S, T). If*

$$(X, Y) \le_{lr} (S, T) \tag{3.15}$$

then

$$(X_{1:n}, X_{2:n}, \ldots, X_{n:n}, Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]}) \le_{lr} (S_{1:n}, S_{2:n}, \ldots, S_{n:n}, T_{[1:n]}, T_{[2:n]}, \ldots, T_{[n:n]}).$$

Proof. Denote the bivariate density functions of (X, Y) and (S, T) by f and g, respectively. From (3.15) (or from Theorem 6.E.4(a) in [21]) we see that

$$n!\prod_{i=1}^{n} f(x_i, y_i) \times n!\prod_{i=1}^{n} g(x'_i, y'_i) \le n!\prod_{i=1}^{n} f(x_i \wedge x'_i, y_i \wedge y'_i) \times n!\prod_{i=1}^{n} g(x_i \vee x'_i, y_i \vee y'_i) \tag{3.16}$$

for all $(x_1, x_2, \ldots, x_n)$, $(x'_1, x'_2, \ldots, x'_n)$, $(y_1, y_2, \ldots, y_n)$, and $(y'_1, y'_2, \ldots, y'_n)$ in $\mathbb{R}^n$.

The density function of $(X_{1:n}, X_{2:n}, \ldots, X_{n:n}, Y_{[1:n]}, Y_{[2:n]}, \ldots, Y_{[n:n]})$ is given by

$$f_{\boldsymbol{X}_{()},\boldsymbol{Y}_{[]}}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$
$$= \begin{cases} n! \prod_{i=1}^{n} f(x_i, y_i), & \text{if } x_1 \le x_2 \le \cdots \le x_n \text{ and } y_i \in \mathbb{R}, i = 1, 2, \ldots, n; \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the density function of $(S_{1:n}, S_{2:n}, \ldots, S_{n:n}, T_{[1:n]}, T_{[2:n]}, \ldots, T_{[n:n]})$ is given by

$$f_{\boldsymbol{S}_{()},\boldsymbol{T}_{[]}}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$
$$= \begin{cases} n! \prod_{i=1}^{n} g(x_i, y_i), & \text{if } x_1 \le x_2 \le \cdots \le x_n \text{ and } y_i \in \mathbb{R}, i = 1, 2, \ldots, n; \\ 0, & \text{otherwise.} \end{cases}$$

Now, suppose that $x_1, x_2, \ldots, x_n$ and $x'_1, x'_2, \ldots, x'_n$ satisfy (3.12). Then (below LHS and RHS stand for 'left hand side' and 'right hand side')

$$\text{LHS}(3.16) = f_{\boldsymbol{X}_{()},\boldsymbol{Y}_{[]}}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) f_{\boldsymbol{S}_{()},\boldsymbol{T}_{[]}}(x'_1, x'_2, \ldots, x'_n, y'_1, y'_2, \ldots, y'_n).$$

By Lemma 3.4, we see that (3.13) and (3.14) hold, and therefore

$$\text{RHS}(3.16) = f_{\boldsymbol{X}_{()},\boldsymbol{Y}_{[]}}(x_1 \wedge x', x_2 \wedge x', \ldots, x_n \wedge x', y_1 \wedge y', y_2 \wedge y', \ldots, y_n \wedge y')$$
$$\times f_{\boldsymbol{S}_{()},\boldsymbol{T}_{[]}}(x_1 \vee x', x_2 \vee x', \ldots, x_n \vee x', y_1 \vee y', y_2 \vee y', \ldots, y_n \vee y').$$

From the fact that LHS(3.16) $\le$ RHS(3.16) we obtain the stated result. □

From the closure under marginalization property of the order $\le_{\text{lr}}$ (see Theorem 6.E.4(b) in [21]) we obtain the following likelihood ratio order analog of Theorem 3.2 as a corollary of Theorem 3.5.

Theorem 3.6. *Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a sample of independent copies of an absolutely continuous random vector (X, Y), and let $(S_1, T_1), (S_2, T_2), \ldots, (S_n, T_n)$ be another sample of independent copies of an absolutely continuous random vector (S, T). If*

$$(X, Y) \le_{lr} (S, T) \tag{3.17}$$

then

$$(X_{r:n}, Y_{[r:n]}) \le_{lr} (S_{r:n}, T_{[r:n]}), \quad r = 1, 2, \ldots, n. \tag{3.18}$$

A combination of Theorems 2.4 and 3.6 yields the following corollary.

Corollary 3.7. *Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a sample of independent copies of an absolutely continuous random vector (X, Y), and let $(S_1, T_1), (S_2, T_2), \ldots, (S_n, T_n)$ be another sample of independent copies of an absolutely continuous random vector (S, T). If*

$$(X, Y) \le_{lr} (S, T),$$

and if

Y is stochastically increasing in X with respect to $\le_{lr}$,

and/or if

T is stochastically increasing in S with respect to $\le_{lr}$,

then

$$(X_{r:n}, Y_{[r:n]}) \le_{lr} (S_{r+1:n}, T_{[r+1:n]}), \quad r = 1, 2, \ldots, n-1.$$

We are not aware of any analog of Theorems 3.2 and 3.6 in which the conclusion is the ordering of $(X_{r:n}, Y_{[r:n]})$ and $(S_{r:n}, T_{[r:n]})$ with respect to $\le_{\text{hr}}$.

Remark 3.8. A related question that may be asked by looking at Theorems 3.2 and 3.6 is whether a stochastic ordering condition that is weaker than (3.1) and (3.17) can be found, such that it implies a stochastic ordering conclusion that is weaker than (3.9) and (3.18). Two common multivariate stochastic orders that are weaker than $\le_{\text{sst}}$ and $\le_{\text{lr}}$ are the upper and the lower orthant orders, $\le_{\rm uo}$ and $\le_{\rm lo}$. Both orders $\le_{\rm sst}$ and $\le_{\rm lr}$ imply each of the orders $\le_{\rm uo}$ and $\le_{\rm lo}$. The question above becomes: In the setup of Theorems 3.2 and 3.6, is it true that

$$(X, Y) \le_{\rm uo} (S, T) \Longrightarrow (X_{r:n}, Y_{[r:n]}) \le_{\rm uo} (S_{r:n}, T_{[r:n]}),$$

and is it true that

$$(X, Y) \le_{\rm lo} (S, T) \Longrightarrow (X_{r:n}, Y_{[r:n]}) \le_{\rm lo} (S_{r:n}, T_{[r:n]})?$$

The answer to the above question is negative. Let (S, T) have the distribution function of (X, Y) in Counterexample 2.5, and let X and Y be independent random variables such that $X =_{\rm st} S$ and $Y =_{\rm st} T$. Then (S, T) is PQD and as a result we have $(X, Y) \le_{\rm uo} (S, T)$ and also $(X, Y) \le_{\rm lo} (S, T)$. On the other hand, $(S_{r:n}, T_{[r:n]})$ is not PQD, and as a result we do not have $(X_{r:n}, Y_{[r:n]}) \le_{\rm uo} (S_{r:n}, T_{[r:n]})$ and $(X_{r:n}, Y_{[r:n]}) \le_{\rm lo} (S_{r:n}, T_{[r:n]})$. $\lhd$

4. Discussion

4.1. An application in reliability involving clean-up expenses

A typical application in reliability theory, through which one can illustrate many of our technical results, is described next.

Consider an $(n - r + 1)$-out-of-n reliability system with independent absolutely continuous component lifetimes $X_1, X_2, \ldots, X_n$. This system fails at time $X_{r:n}$. Suppose that as a consequence of the system failure, the user of the system encounters some random cost that depends on the failure time of the system, and that may also depend on the particular identity k of the component that caused the system to fail (by absolute continuity, with probability 1, k is uniquely determined from the equality $X_{r:n} = X_k$). Denote this random cost by Y_k, or, equivalently, by $Y_{[r:n]}$. In this paper we assume that the conditional distribution function of $Y_{[r:n]}$, given $X_{r:n} = x$, is determined by a bivariate distribution function of some random vector (X, Y), in the sense that the distribution function of $[Y_{[r:n]} \big| X_{r:n} = x]$ is the same as the distribution function of $[Y \big| X = x]$ for all $x \in$ support(X). The bivariate distribution function of $(X_{r:n}, Y_{[r:n]})$ is then of interest to the user of the reliability system.

Specifically, consider a reliability system that involves some *clean-up expenses* upon failure. Such clean-up expenses often increase (stochastically) with the realization of the failure time, since, because of the routine wear-and-tear of the system, more clean-up is needed after a longer working time span. That is, it is often the case that X and Y, described above, are positively dependent.

In the setup of this application, we see that Theorem 2.2 can be interpreted as follows. Suppose that we compare an $(n - r + 1)$-out-of-n reliability system with an $(n - r)$-out-of-n system. Obviously, the $(n - r)$-out-of-n system has stochastically longer lifetime than the $(n - r + 1)$-out-of-n system — this is usually a desirable quality. Theorem 2.2 points out, however, that the resulting clean-up cost upon failure, for the user, is stochastically larger in the former (better) system — and, this is usually an undesirable quality.

Similar comments apply also to Theorems 2.3 and 2.4.

In the above setup we also see that Theorem 3.2 can be interpreted as follows. Suppose that we compare two $(n-r+1)$-out-of-n reliability systems with component lifetimes $X_1, X_2, \ldots, X_n$ and $S_1, S_2, \ldots, S_n$, respectively. If the component lifetimes in the second system are stochastically larger than the component lifetimes in the first system, we obviously have that the second system lifetime is stochastically larger than the first system lifetime — this is usually a desirable quality. Theorem 3.2 points out that, however, that the resulting clean-up cost upon failure, for the user, is stochastically larger in the second (better) system — and, this is usually an undesirable quality.

A similar comment applies also to Theorem 3.6.

4.2. Relations to results in the literature

In this subsection we point out some instances where our technical results extend and strengthen previous findings in the literature.

First let us consider Theorem 2.2. Note that from (2.5) it follows that

$$Y_{[r:n]} \le_{\rm st} Y_{[r+1:n]}, \quad r = 1, 2, \ldots, n-1. \tag{4.1}$$

Thus, Theorem 2.2 shows that (2.4) $\Longrightarrow$ (4.1). The latter is Theorem 3.1(a) in [16]. So we see that the present Theorem 2.2 strengthens Theorem 3.1(a) of [16].

Theorem 2.2 also strengthens a result in [3]. That result says that if (2.4) holds then $(X_{r:n}, Y_{[r:n]}) \le_{\rm lo} (X_{r+1:n}, Y_{[r+1:n]})$, $r = 1, 2, \ldots, n-1$. Theorem 2.2 is stronger than that result of [3] because of the implication $\le_{\rm st} \Longrightarrow \le_{\rm lo}$. It is worthwhile to mention here that Bairamov [3] corrects a result in [4], which is of the same flavor as Theorem 2.2 above, but which happened to be incorrect.

Regarding Theorem 2.3, note that from (2.7) it follows that

$$Y_{[r:n]} \le_{\rm hr} Y_{[r+1:n]}, \quad r = 1, 2, \ldots, n-1. \tag{4.2}$$

Thus, Theorem 2.3 shows that (2.6) $\Longrightarrow$ (4.2). The latter is Theorem 3.2(b) in [16]. So we see that the present Theorem 2.3 strengthens Theorem 3.2(b) of [16].

Next, regarding Theorem 2.4, note that from (2.12) it follows that

$$Y_{[r:n]} \leq_{\rm lr} Y_{[r+1:n]}, \quad r = 1, 2, \ldots, n-1. \tag{4.3}$$

Thus, Theorem 2.4 shows that (2.11) $\Longrightarrow$ (4.3). The latter is Theorem 3.3(a) in [16]. So we see that the present Theorem 2.4 strengthens Theorem 3.3(a) of [16].

4.3. Examples and remarks

One may wonder how restrictive the positive dependence conditions (2.4) and (2.6), and (2.11) in Theorems 2.2–2.4 are.

The strongest condition of the above three is (2.11). It turns out that it has been used extensively in the literature. In [20, page 200] it is denoted as PLR(X, Y) (PLR stands for positive likelihood ratio). Lehmann [17] showed that if (X, Y) is *bivariate normal* with nonnegative correlation coefficient, then it is PLR(X, Y). As another example, we note that it is not too hard to verify that the *Farlie–Gumbel–Morgenstern copula* (see [20, page 77]), with a nonnegative shape parameter, corresponds to a random vector that satisfies (2.11), and therefore also (2.4) and (2.6).

Regarding condition (2.4), in addition to the above examples, it is also satisfied by the *Plackett copula* (see [20, page 91]) with a shape parameter greater than unity (this is shown in [20, page 197]). Furthermore, Nelsen [20, Exercise 5.34 in page 205] provides conditions on *Archimedean copulas* that correspond to random vectors that satisfy (2.4).

Conditions (3.1) or (3.15) are assumed throughout all the results in Section 3. So, naturally one may wonder how restrictive these conditions are.

It turns out that examples of random vectors that satisfy (3.1) are easy to construct. For example, let X be an exponential random variable with failure rate $\alpha > 0$, and, for $x > 0$, let $[Y \big| X = x]$ be an exponential random variable with failure rate α/x. Then the joint density function of (X, Y) is given by

$$f_\alpha(x, y) = \frac{\alpha^2}{x} \exp\{-\alpha(x + y/x)\}, \quad (x, y) \in (0, \infty)^2.$$

Let (S, T) be another random vector with density function $f_{\alpha'}$. It is easy to see that if $\alpha \geq \alpha' > 0$ then $(X, Y) \leq_{\rm sst} (S, T)$.

We note that constructions of the type above are useful for modeling. For example, in the reliability application in Section 4.1 the above procedure can be used to model the relationship between a lifetime of a component, and the cost that is incurred upon its failure.

Examples of random vectors that satisfy (3.15) (or, equivalently, (3.17)) abound in the literature. For example, Karlin and Rinott [14, Theorems 2.3 and 5.2] give conditions for ordering vectors of absolute values of dependent normal random variables with respect to $\leq_{\rm lr}$, that is, (3.15). Belzunce et al. [7] describe circumstances under which (3.15) holds. In particular, they note that some mixed multivariate distribution functions that arise from proportional hazard models are ordered as in (3.15).

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References

[1] M. Amini, J. Ahmadi, M. Razmkhah, Fisher information in record values and their concomitants: A comparison of two sampling schemes, Commun. Stat. - Theory Methods 40 (2011) 1298–1314.
[2] I. Bairamov, Reliability and mean residual life of complex systems with two dependent components per element, IEEE Trans. Reliab. 62 (2013) 276–285.
[3] I. Bairamov, Corrigendum to "Majorization bounds for distribution functions", [Statist. Probab. Lett. 82 (2012) 1799–1806], Statist. Probab. Lett. 83 (2013) 7–8.
[4] I. Bairamov, Majorization bounds for distribution functions, Statist. Probab. Lett. 82 (2012) 1799–1806.
[5] I. Bairamov, A. Stepanov, Numbers of near bivariate record-concomitant observations, J. Multivariate Anal. 102 (2011) 908–917.
[6] R.E. Barlow, F. Proschan, Statistical Theory of Reliability and Life Testing Probability Models, Holt, Rinehart, and Winston, New York, 1981.
[7] F. Belzunce, J.-A. Mercader, J.-M. Ruiz, F. Spizzichino, Stochastic comparisons of multivariate mixture models, J. Multivariate Anal. 100 (2009) 1657–1669.
[8] P.K. Bhattacharya, Induced order statistics: Theory and applications, in: P.R. Krishnaiah, P.K. Sen (Eds.), in: Handbook of Statistics, Nonparametric Methods, vol. 4, North Holland, Amsterdam, 1984, pp. 383–403.
[9] T. Blessinger, More on stochastic comparisons and dependence among concomitants of order statistics, J. Multivariate Anal. 82 (2002) 367–378.
[10] S. Eryilmaz, Concomitants in a sequence of independent nonidentically distributed random vectors, Communications in Statistics-Theory and Methods 34 (2005) 1925–1933.
[11] M. Izadi, B.-E. Khaledi, Progressive type II censored order statistics and their concomitants: Some stochastic comparisons results, J. Iran. Stat. Soc. (JIRSS) 6 (2007) 111–124.
[12] K. Joag-dev, S. Kochar, F. Proschan, A general composition theorem and its applications to certain partial orderings of distributions, Statist. Probab. Lett. 22 (1995) 111–119.

[13] S. Karlin, Total Positivity, Stanford University Press, 1968.
[14] S. Karlin, Y. Rinott, Total positivity properties of absolute value multinormal variables with applications to confidence interval estimates and related probabilistic inequalities, Ann. Statist. 9 (1981) 1035–1049.
[15] J.H.B. Kemperman, On the FKG inequality for measures on a partially ordered space, Indag. Math. (N.S.) 39 (1977) 313–331.
[16] B.-E. Khaledi, S. Kochar, Stochastic comparisons and dependence among concomitants of order statistics, J. Multivariate Anal. 73 (2000) 262–281.
[17] E.L. Lehmann, Some concepts of dependence, Ann. Math. Stat. 37 (1966) 1137–1153.
[18] H. Li, M. Scarsini, M. Shaked, Linkages: A tool for the construction of multivariate distributions with given nonoverlapping multivariate marginals, J. Multivariate Anal. 56 (1996) 20–41.
[19] A. Müller, D. Stoyan, Comparison Methods for Stochastic Models and Risks, Wiley, Chichester, 2002.
[20] R.B. Nelsen, An Introduction to Copulas, Second ed., Springer, New York, 2006.
[21] M. Shaked, J.G. Shanthikumar, Stochastic Orders, Springer, New York, 2007.
[22] K. Wang, H.N. Nagaraja, Concomitants of order statistics for dependent samples, Statist. Probab. Lett. 79 (2009) 553–558.