

# Improvement on Chi-Squared Approximation by Monotone Transformation

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It is well known that Bartlett adjustment yields an improvement on the chi-squared approximations to the likelihood ratio test statistics. However, it is not possible to obtain such an improvement for some test statistics. Typical examples are the score test statistic and Hotelling's  $T^2$ -statistic. The purpose of the present paper is to give a general method for improving on the chi-squared approximation. The method suggested is based on a monotone transformation. © 1997 Academic Press

## 1. INTRODUCTION

Suppose that a non-negative random variate  $T$  has an asymptotic expansion

$$P(T \leq x) = G_f(x) + \frac{1}{n} \sum_{j=0}^k \beta_j G_{f+2j}(x) + o(n^{-1}), \quad (1.1)$$

where  $n$  is the sample size,  $G_f(\cdot)$  is the distribution function of the chi-squared variate with  $f$  degrees of freedom, and the coefficients  $\beta_j$ 's satisfy a relation  $\sum_{j=0}^k \beta_j = 0$ . Such the asymptotic expansion is called Edgeworth type expansion. Some examples of the variate  $T$  are as follows; for  $k=1$ , likelihood ratio test statistic (see Hayakawa [8]), for  $k=2$ , Lawley-Hotelling trace criterion and Bartlett–Nanda–Pillai trace criterion, which are test statistics for a multivariate linear hypothesis (see Anderson [1] and Siotani et al. [12]), and for  $k=3$ , score test statistic (see Harris [7]), Hotelling's  $T^2$ -statistic (see Fujikoshi [4] and Kano [9]).

Consider the simple transformation  $\tilde{T} = cT$  with a positive bias correction factor  $c$ . This correction is called Bartlett adjustment. If  $k=1$ , the chi-squared approximation to the variate  $T$  is improved in the sense of

$$P(\tilde{T} \leq x) = G_f(x) + o(n^{-1}). \quad (1.2)$$

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However, if  $k \geq 2$ , Bartlett adjustment does not work in the sense of (1.2). Cordeiro and Ferrari [3] obtained a polynomial transformation  $\tilde{T} = p(T)$  with the property (1.2) for any  $k$ . However, it is not always monotone. Fujikoshi [5] obtained a monotone transformation with the property (1.2) for the special case  $k = 2$ .

The purpose of the present paper is to improve the chi-squared approximation to the variate  $T$  by monotone transformation, definitely, to obtain a monotone increasing transformation  $\tilde{T} = q(T)$  with the asymptotic expansion (1.2) for any  $k$ . In Section 2, such a transformation is derived. In Section 3, it is investigated by simulation and compared with other transformations.

## 2. IMPROVEMENT ON CHI-SQUARED APPROXIMATION

In this section, we construct a monotone increasing transformation  $\tilde{T} = q(T)$  and give some comments.

### 2.1. Preliminaries

A polynomial transformation  $\tilde{T} = p(T)$  with the property (1.2), which is given in Lemma 2.1, was suggested by Cordeiro and Ferrari [3]. It is based on Cornish–Fisher inverse transformation.

LEMMA 2.1. *Suppose that a non-negative random variate  $T$  has an asymptotic expansion (1.1). Let*

$$p(t) = t - \frac{1}{n} \sum_{j=1}^k c_j t^j, \quad (2.1)$$

where  $c_j = 2 \sum_{l=j}^k \beta_l / \prod_{l=0}^{j-1} (f + 2l)$ , and let  $\tilde{T} = p(T)$ . Then, we have the asymptotic expansion (1.2).

In general, the transformation  $\tilde{T} = p(T)$  is not monotone. Our interest is to construct a monotone increasing transformation with the same asymptotic property as  $p(T)$ .

### 2.2. Monotone transformation

Let

$$q_1(t) = t + \frac{1}{n} \sum_{j=1}^k c_j^+ t^j, \quad q_2(t) = t - \frac{1}{n} \sum_{j=1}^k c_j^- t^j,$$

where  $c_j^+ = I(c_j \geq 0) c_j$ ,  $c_j^- = I(c_j < 0) c_j$ ,  $I(\cdot)$  is the indicator function, and let

$$q(t) = q_1^{-1}(q_2(t)), \quad (2.2)$$

where  $q_1^{-1}(\cdot)$  is the inverse function. The functions  $q_1(t)$  and  $q_2(t)$  are monotone increasing on  $t \geq 0$ , and so is  $q(t)$ . The following theorem shows that the function  $q(t)$  satisfies our purpose.

**THEOREM 2.1** *Suppose that a non-negative random variate  $T$  has an asymptotic expansion (1.1). Let  $q(t)$  be the monotone increasing function defined by (2.2), and let  $\tilde{T} = q(T)$ . Then, we have the asymptotic expansion (1.2).*

*Proof.* The transformation  $\tilde{T} = q(T)$  implies that  $q_1(\tilde{T}) = q_2(T)$ , that is,

$$\tilde{T} + \frac{1}{n} \sum_{j=1}^k c_j^+ \tilde{T}^j = T - \frac{1}{n} \sum_{j=1}^k c_j^- T^j.$$

Nothing that

$$\tilde{T} = T - \frac{1}{n} \alpha + o_p(n^{-1}),$$

we have

$$T - \frac{1}{n} \alpha + \frac{1}{n} \sum_{j=1}^k c_j^+ T^j + o_p(n^{-1}) = T - \frac{1}{n} \sum_{j=1}^k c_j^- T^j.$$

So,

$$\begin{aligned} \alpha &= \sum_{j=1}^k c_j^+ T^j + \sum_{j=1}^k c_j^- T^j + o_p(1) \\ &= \sum_{j=1}^k c_j T^j + o_p(1). \end{aligned}$$

Hence,

$$\tilde{T} = p(T) + o_p(n^{-1}).$$

This fact and Lemma 2.1 imply Theorem 2.1. ■

### 2.3. Some Comments

If  $k=1$ , we have

$$\begin{aligned} \tilde{T} = q(T) &= (1 + c_1/n)^{-1} T & \text{if } c_1 \geq 0, \\ &= (1 - c_1/n) T & \text{if } c_1 < 0. \end{aligned}$$

This transformation is a usual Bartlett adjustment.

Note that the approximate rejection region  $\{q(T) > c\}$  is the same as  $\{q_2(T) > q_1(c)\}$ . Consequently, it is not necessary to use the inverse transformation in testing problems.

Consider the case that  $\beta_j$ 's are unknown. In this case, let  $\hat{q}(t)$  be the function which is replaced  $\beta_j$  by a consistent estimator  $\hat{\beta}_j$  and let  $\hat{T} = \hat{q}(T)$ . Then, under appropriate regularity conditions, we have  $\hat{T} = \tilde{T} + o_p(n^{-1})$  and

$$P(\hat{T} \leq x) = G_f(x) + o(n^{-1}).$$

The monotone increasing transformation with the property (1.2) is not unique. In fact, if  $\tilde{T} = h_*(T)$  is useful, so is  $\tilde{T} = h_*(T) + T/n^2$ . For  $k=2$ , log-transformation was suggested by Fujikoshi [5]. Also, while the present paper had been submitted, the another transformation was suggested by Kakizawa [10].

### 3. APPLICATION

#### 3.1. Hotelling's $T^2$ -statistic

Let  $x$  be a  $f \times 1$  random vector with mean  $\mu$  and covariance matrix  $A$ . Let  $x_1, \dots, x_n$  be  $n$  independent observations of  $x$ . Then, Hotelling's  $T^2$ -statistic is defined by

$$T_0 = n(\bar{x} - \mu)' S^{-1}(\bar{x} - \mu),$$

where  $\bar{x} = n^{-1} \sum_{i=1}^n x_i$ ,  $S = (n-1)^{-1} \sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})'$ . Let  $\kappa^{(i_1, \dots, i_j)}$  be the  $j$ th cumulant of  $A^{-1/2}(x - \mu)$  and

$$\kappa_3^{(1)} = \left\{ \sum_{i,j,k} (\kappa^{(i,j,k)})^2 \right\}^{1/2}, \quad \kappa_3^{(2)} = \left\{ \sum_{i,j,k} \kappa^{(i,j,i)} \kappa^{(i,k,k)} \right\}^{1/2}, \quad \kappa_4^{(1)} = \sum_{i,j} \kappa^{(i,i,i,j)}.$$

Under appropriate assumptions, Hotelling's  $T^2$ -statistic has an asymptotic expansion (1.1) with  $k=3$  (see Fujikoshi [4] and Kano [9]), given by

$$P(T_0 \leq x) = G_f(x) + \frac{1}{n} \sum_{j=0}^3 \beta_j G_{f+2j}(x) + o(n^{-1}),$$

where

$$\begin{aligned} \beta_0 &= -\frac{1}{4}p^2 + \frac{1}{6}(\kappa_3^{(1)})^2 - \frac{1}{4}\kappa_4^{(1)}, \\ \beta_1 &= -\frac{1}{2}p - \frac{1}{2}(\kappa_3^{(1)})^2 + \frac{1}{2}\kappa_4^{(1)}, \\ \beta_2 &= \frac{1}{4}p(p+2) - \frac{1}{2}(\kappa_3^{(1)})^2 - \frac{1}{4}\kappa_4^{(1)}, \\ \beta_3 &= \frac{1}{3}(\kappa_3^{(1)})^2 + \frac{1}{2}(\kappa_3^{(2)})^2. \end{aligned}$$

Using the transformation  $\tilde{T}=q(T_0)$ , we get the improvement on the chi-squared approximation in the sense of (1.2).

### 3.2. Simulations

The transformation  $\tilde{T}=q(T_0)$  is investigated at the 95% point of  $G_f(\cdot)$  by simulation and compared with the original statistic, Bartlett adjustment, and polynomial transformation. We assume that random vector  $x$  is distributed as a contaminated normal distribution with density  $f_\varepsilon(x) = (1-\varepsilon)\phi(x) + \varepsilon\phi(x/5)$  ( $\kappa_3^{(1)} = \kappa_3^{(2)} = 0$ ), where  $\phi(\cdot)$  is the standard normal density.

Let  $z_\alpha$  be the  $\alpha\%$  point of  $G_f(\cdot)$ , and let

$$d_0 = P(T_0 \leq z_\alpha) - \alpha,$$

$$d_B = P(cT_0 \leq z_\alpha) - \alpha,$$

$$d_p = P(p(T_0) \leq z_\alpha) - \alpha,$$

$$d_q = P(q(t_0) \leq z_\alpha) - \alpha,$$

TABLE I  
The Goodness of Approximation

(i) $p = 1$						
	$n = 20$			$n = 50$		
	$\varepsilon = 0.1$	$\varepsilon = 0.5$	$\varepsilon = 0.9$	$\varepsilon = 0.1$	$\varepsilon = 0.5$	$\varepsilon = 0.9$
$t_0$	-0.0038	-0.0087	-0.0163	-0.0015	-0.0067	-0.0043
$t_B$	0.0092	0.0045	-0.0043	0.0031	-0.0011	0.0003
$t_p$	-0.0019	0.0048	-0.0003	-0.0002	-0.0010	0.0014
$t_q$	-0.0023	0.0038	-0.0036	-0.0005	-0.0012	0.0011
(ii) $p = 2$						
	$n = 20$			$n = 50$		
	$\varepsilon = 0.1$	$\varepsilon = 0.5$	$\varepsilon = 0.9$	$\varepsilon = 0.1$	$\varepsilon = 0.5$	$\varepsilon = 0.9$
$t_0$	-0.0227	-0.0335	-0.0387	-0.0007	-0.0119	-0.0177
$t_B$	0.0032	-0.0033	-0.0094	0.0097	-0.0005	-0.0071
$t_p$	-0.0180	0.0021	0.0050	-0.0009	0.0010	-0.0034
$t_q$	-0.0187	0.0040	-0.0072	-0.0007	-0.0004	-0.0046
(iii) $p = 3$						
	$n = 20$			$n = 50$		
	$\varepsilon = 0.1$	$\varepsilon = 0.5$	$\varepsilon = 0.9$	$\varepsilon = 0.1$	$\varepsilon = 0.5$	$\varepsilon = 0.9$
$t_0$	-0.0479	-0.0533	-0.0687	-0.0050	-0.0169	-0.0185
$t_b$	-0.0039	-0.0035	-0.0168	0.0087	-0.0006	-0.0025
$t_p$	-0.0329	0.0136	0.0187	0.0002	0.0030	0.0032
$t_q$	-0.0375	-0.0057	-0.0158	-0.0007	0.0008	0.0003

where  $c = 1 - (2\beta_0 + A\beta_2)/n$  is a usual Bartlett factor. Each of these values was estimated by 10,000 random samples for some cases;  $p = 1, 2, 3$ ,  $n = 20, 50$ ,  $\varepsilon = 0.1, 0.5, 0.9$ ,  $\alpha = 0.95$ . The results are given in Table I. It is interesting that Bartlett adjustment seems to be as useful as other transformations.

We note that the polynomial transformation  $\tilde{T} = p(T)$  sometimes gives rise to strange phenomena. For example, if  $c_2 > 0$  and  $z_a \geq n(1 - c_1/n)^2 / (4c_2)$ , we have  $P(p(T_0) \leq z_a) = 1$ . This situation, though it is not treated on Table I, is realized when  $p = 1$ ,  $n = 10$ ,  $\varepsilon = 0.9$ ,  $\alpha = 0.975$ , and when  $p = 3$ ,  $n = 20$ ,  $\varepsilon = 0.9$ ,  $\alpha = 0.975$ , and so on.

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