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High-dimensional rank tests for sphericity

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ABSTRACT

In recent years, procedures for testing distributional sphericity have attracted increased attention, especially in high-dimensional settings. A prominent problem in this context is the development of robust and efficient test statistics. In this paper, we propose two novel rank tests inspired by Spearman's rho and Kendall's tau for high-dimensional problems. Due to the "blessing of dimension", estimation of masses of nuisance parameters is avoided, which allows our procedures to work in arbitrary large dimension. The asymptotic normality of the proposed tests is established for elliptical distributions and their performance is investigated over a wide range of simulation set-ups.

1. Introduction

Assessing sphericity is an important issue in a large number of problems arising, e.g., in biostatistics and geostatistics. Mathematically speaking, the hypotheses to be tested are

$$\mathcal{H}_0 : \boldsymbol{\Sigma}_p = \sigma \mathbf{I}_p \quad \text{vs.} \quad \mathcal{H}_1 : \boldsymbol{\Sigma}_p \neq \sigma \mathbf{I}_p, \tag{1}$$

where $\boldsymbol{\Sigma}_p$ denotes the covariance matrix of a p-variate elliptical random vectors. There is a considerable body of work on this issue, most of which cast in a fixed dimension framework. Under a multivariate normal assumption, a commonly used method for testing sphericity is Mauchly's likelihood ratio statistic [11]. John [6,7] also proposed the statistic

$$Q_J = \frac{np^2}{2} \mathrm{tr} \left\{ \frac{\mathbf{S}}{\mathrm{tr}(\mathbf{S})} - \frac{1}{p} \mathbf{I}_p \right\}^2,$$

where $\mathbf{S}$ is the sample covariance matrix. John derived that Q_J is (locally) the most powerful invariant test for sphericity under the multivariate normal assumption. Following [6,7], Muirhead and Waternaux [12] modified Q_J in order to make it applicable for the broader class of elliptical distributions.

With the rapid development of technology, however, high-dimensional datasets are now emerging in many areas of science and industry, e.g., hyperspectral imagery, internet portals, microarray analysis and DNA analysis. For such datasets, the number of variables is often much larger than the sample size; this is the so-called "large p, small n" paradigm. For this reason, an interest in sphericity tests for high-dimensional settings has emerged.

Specifically, Bai et al. [1] proposed corrections to the original likelihood ratio test by using random matrix technology when $p/n \to c \in (0, 1)$. Ledoit and Wolf [9] showed that the existing n-asymptotic theory remains valid when p goes to infinity with n, even when $p > n$. In parallel, Chen et al. [3] could avoid the multivariate normality assumption in their investigation of a high-dimensional test based on Q_J with two accurate estimators of $\mathrm{tr}(\boldsymbol{\Sigma}_p)$ and $\mathrm{tr}(\boldsymbol{\Sigma}_p^2)$. They concluded that

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without explicitly specifying the growth rate of p with respect to n, their test statistic is still asymptotically normal under the diverging factor model. This model, introduced by Bai and Saranadasa [2], states that $\boldsymbol{X}_i = \boldsymbol{\Gamma}\boldsymbol{Z}_i + \boldsymbol{\mu}$, where $\boldsymbol{\Gamma}$ is a $p \times m$ matrix with $m \geq p$ and $\boldsymbol{Z}_i = (Z_{i1}, \ldots, Z_{im})^\top$ is a random vector satisfying the following conditions:

$$\begin{aligned} &\mathrm{E}(\boldsymbol{Z}_i) = 0, \qquad \mathrm{var}(\boldsymbol{Z}_i) = \mathbf{I}_m, \qquad \mathrm{E}(Z_{il}^4) = 3 + \Delta \in (0, \infty), \\ &\qquad\qquad \mathrm{E}(Z_{ik_1}^{\alpha_1} \cdots Z_{ik_q}^{\alpha_q}) = \mathrm{E}(Z_{ik_1}^{\alpha_1}) \cdots \mathrm{E}(Z_{ik_q}^{\alpha_q}). \end{aligned} \tag{2}$$

Here $\sum_{k=1}^q \alpha_k \leq 8$ and $k_1, \ldots, k_q$ are positive integers. Furthermore, Zou et al. [17] studied a sphericity test for a multivariate t-distribution or a mixture of multivariate distributions, which is, however, not a diverging factor model.

Within a host of sphericity tests for fixed dimension cases, multivariate sign- and/or rank-based covariance matrices are often used to construct robust tests for sphericity; see Hallin and Paindaveine [5] and Oja [13] for general overviews. However, when the dimension is larger than the sample size, these methods tend to perform poorly. Zou et al. [17] showed that the type I error of tests based on multivariate signs, used in Marden and Gao [10], Hallin and Paindaveine [5] and Sirkiä et al. [16], are much larger than the nominal level because of poor estimates of the location parameters. On this ground, Paindaveine and Verdebout [15] proposed a high-dimensional sign test assuming known location parameters, and Zou et al. [17] proposed a bias-correction procedure to the existing test statistic. Unfortunately, these tests can only allow the dimension to be, at most, the square of the sample size. In contrast, the dimension of many datasets, such as microarray dataset,can be exponential in terms of the sample size. This motivated us to construct new tests available for ultra-high dimensional settings.

In this paper, we propose two new rank tests for high-dimensional contexts inspired by Spearman's rho and Kendall's tau. As already shown by Sirkiä et al. [16], such rank statistics yield robust and efficient tests for sphericity, though they require the estimation of many nuisance parameters. Such estimates were developed, notably by these authors, but they are not operational in high-dimensional settings because information about the location is unavailable. In addition, the estimators of $\mathrm{tr}(\boldsymbol{\Omega}_p^2)$ or $\mathrm{tr}(\boldsymbol{\Xi}_p^2)$ based on the sample symmetrized sign or rank covariance matrix result in a non-negligible bias term, once the dimension is ultra-high.

The purpose of this paper, therefore, is to propose and investigate improved Spearman's rho and Kendall's tau type rank tests for sphericity in situation of high-dimensional settings. Owing to the "blessing of dimension", the nuisance parameters do not need to be estimated. Based on the leave-one-out method, there are no bias terms in the proposed test statistics. Given that location parameters need not be estimated, the data dimension is unrestricted. Furthermore, the asymptotic normality of these two test statistics is established under the assumption that the underlying distribution is elliptical. Finally, simulation results reported here suggest that the proposed tests outperform the test of Chen et al. [3], valid under a multivariate normality assumption, as well as the test of Zou et al. [17] when p/n^2 is large enough.

2. High-dimensional rank tests

2.1. High-dimensional Spearman's rho-type rank test statistic

We begin by introducing a distributional assumption on $\boldsymbol{X}$:

(A1) $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are generated from a p-variate elliptical distribution with density $\det(\boldsymbol{\Sigma}_p)^{-1/2} g_p\{\|\boldsymbol{\Sigma}_p^{-1/2}(\boldsymbol{X} - \boldsymbol{\theta}_p)\|\}$, where $\|\boldsymbol{X}\| = (\boldsymbol{X}^\top\boldsymbol{X})^{1/2}$ is the Euclidean length of the vector $\boldsymbol{X}$, $\boldsymbol{\theta}_p$ is the center of symmetry and $\boldsymbol{\Sigma}_p$ is a positive definite symmetric $p \times p$ scatter matrix.

Similar to Zou et al. [17], we define $\boldsymbol{\Sigma}_p = \sigma_p \boldsymbol{\Lambda}_p$, where $\mathrm{tr}(\boldsymbol{\Lambda}_p) = p$ and σ_p is a scale parameter. A test of (1) is equivalent to a test of

$$\mathcal{H}_0 : \boldsymbol{\Lambda}_p = \mathbf{I}_p \quad \text{vs.} \quad \mathcal{H}_1 : \boldsymbol{\Lambda}_p \neq \mathbf{I}_p.$$

The spatial-rank function is defined as $R(\boldsymbol{X}) = \mathrm{E}\{U(\boldsymbol{X} - \boldsymbol{Y})|\boldsymbol{X}\}$, where $U(\boldsymbol{X}) = (\boldsymbol{X}/\|\boldsymbol{X}\|)\mathbf{1}(\boldsymbol{X} \neq \mathbf{0})$. The spatial-rank covariance matrix is $\boldsymbol{\Omega}_p = \mathrm{E}\{R(\boldsymbol{X})R(\boldsymbol{X})^\top\}$. Under the null hypothesis, $\boldsymbol{\Omega}_p = \tau_F p^{-1}\mathbf{I}_p$, where τ_F is a constant dependent on g_p. Similar to John's test [6,7], a natural distance measure between $\boldsymbol{\Omega}_p$ and $\tau_F p^{-1}\mathbf{I}_p$ is

$$p\mathrm{tr}\left\{\frac{\boldsymbol{\Omega}_p}{\mathrm{tr}(\boldsymbol{\Omega}_p)} - p^{-1}\mathbf{I}_p\right\}^2 = \frac{p\mathrm{tr}(\boldsymbol{\Omega}_p^2)}{\mathrm{tr}^2(\boldsymbol{\Omega}_p)} - 1.$$

When p is fixed, we use the sample spatial-rank covariance matrix $\boldsymbol{\Omega}_{n,p}$ to estimate $\boldsymbol{\Omega}_p$, i.e.,

$$\boldsymbol{\Omega}_{n,p} = \frac{1}{n}\sum_{i=1}^n \boldsymbol{R}_i\boldsymbol{R}_i^\top = \frac{1}{n^3}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \boldsymbol{U}_{ij}\boldsymbol{U}_{ik}^\top,$$

where $\boldsymbol{R}_i = \sum_{j=1}^n \boldsymbol{U}_{ij}/n$ and $\boldsymbol{U}_{ij} = U(\boldsymbol{X}_i - \boldsymbol{X}_j)$. The Spearman's rho-type rank test statistic is then given by

$$Q_S = p\mathrm{tr}\left\{\frac{\boldsymbol{\Omega}_{n,p}}{\mathrm{tr}(\boldsymbol{\Omega}_{n,p})} - p^{-1}\mathbf{I}_p\right\}^2 = \frac{p\mathrm{tr}(\boldsymbol{\Omega}_{n,p}^2)}{\mathrm{tr}^2(\boldsymbol{\Omega}_{n,p})} - 1.$$

It can be shown that when p is fixed, under the null hypothesis one has

$$\frac{n}{\gamma_S/\tau_F^2} Q_S \rightsquigarrow \chi^2_{(p+2)(p-1)/2},$$

where γ_S and τ_F are two nuisance parameters that depend on g_p and p.

Sirkiä et al. [16] suggested that τ_F could be estimated by $\mathrm{tr}(\boldsymbol{\Omega}_{n,p})/p$. In addition, they proposed two estimators for γ_S. The first estimation is based on the formula of γ_S. However, it is restricted by the assumption that $\boldsymbol{X}_i$ is located at the origin, which is unrealistic in practice. Additionally, if we standardize the samples by the estimated location parameters, as shown in Zou et al. [17], there would be another non-negligible bias term in Q_S when p/n^2 is large enough. The other estimator of γ_S is a complex symmetric U-statistic, which requires $O(n^5p^4)$ computations. Here the total computational complexity of Q_S is of order $O(n^5p^4) + O(p^6)$ because of the inverse of the covariance matrix of $\mathrm{vec}(\boldsymbol{\Omega}_{n,p})$. Obviously, this computational complexity is too high for high-dimensional data.

Fortunately, Lemma 1 in Appendix states that $\mathrm{E}(\boldsymbol{\Omega}_p) = 0.5p^{-1}\mathbf{I}_p\{1 + o(1)\}$ under the null hypothesis as $p \to \infty$. Thus, $\mathrm{tr}(\boldsymbol{\Omega}_p) \to 0.5$, and we only need to propose a better estimator of $\mathrm{tr}(\boldsymbol{\Omega}_p^2)$. However, the estimator $\mathrm{tr}(\boldsymbol{\Omega}_{n,p}^2)$ results in a non-negligible bias term in Q_S when p is ultra-high. Based on the leave-one-out method, we define the following new estimator of $\mathrm{tr}(\boldsymbol{\Omega}_p^2)$,

$$\widehat{\mathrm{tr}(\boldsymbol{\Omega}_p^2)} = \frac{1}{2n(n-1)(n-2)(n-3)} \underbrace{\sum\sum\sum\sum}_{i,j,k,\ell \text{ are not equal}} \boldsymbol{U}_{ij}^\top \boldsymbol{U}_{k\ell} \boldsymbol{U}_{kj}^\top \boldsymbol{U}_{i\ell}.$$

Then we define the following high-dimensional Spearman's rho-type rank test statistic (abbreviated as SR hereafter) as

$$\tilde{Q}_S = 4p\widehat{\mathrm{tr}(\boldsymbol{\Omega}_p^2)} - 1.$$

Obviously, the value of $\tilde{Q}_S$ remains unchanged for $\boldsymbol{Z}_i = a\mathbf{O}\boldsymbol{X}_i + \boldsymbol{c}$, where a is a constant, $\mathbf{O}$ is an orthogonal matrix and $\boldsymbol{c}$ is a vector of constants. Thus, the test statistic $\tilde{Q}_S$ is invariant under rotations. The following result states the asymptotic null distribution of $\tilde{Q}_S$.

Theorem 1. *Under $\mathcal{H}_0$ and Assumption* (A1), *as $n \to \infty$ and $p \to \infty$, $\tilde{Q}_S/\sigma_0 \rightsquigarrow \mathcal{N}(0, 1)$, where $\sigma_0^2 = 4(p-1)/\{n(n-1)(p+2)\}$.*

According to Theorem 1, there are no nuisance parameters in the new proposed test procedure. As n and p go to infinity, $\tilde{Q}_S$ is asymptotically normal and the variance σ_0^2 only depends on p and n. This can be viewed as the phenomenon of the "blessing of dimension". Moreover, the complexity of the entire procedure is only $O(n^4p)$, which is eventually less than the classic Spearman's rho-type rank test procedure.

Theorem 1 also shows that there is no bias term in $\tilde{Q}_S$. As a result, the proposed tests do not need a bias correction procedure as in Zou et al. [17]. Moreover, a relationship between the sample size n and dimension p is not required. On the contrary, the test proposed by Zou et al. [17] (abbreviated as SS hereafter) requires that the dimension is the square of the sample size at most. Also, when $p/n^2 \to \infty$, there will be another bias term in the SS test statistic, which is difficult to compute. Simulation studies will demonstrate these conclusions in Section 3.

Next, we study the asymptotic distribution of $\tilde{Q}_S$ under the alternative $\mathcal{H}_1 : \boldsymbol{\Lambda}_p = \mathbf{I}_p + \mathbf{D}_{n,p}$. In what follows,

$$\sigma_1^2 = \sigma_0^2 + n^{-2}p^{-2}\left\{8p\mathrm{tr}(\mathbf{D}_{n,p}^2) + 4\mathrm{tr}^2(\mathbf{D}_{n,p}^2)\right\} + 8n^{-1}p^{-2}\left\{\mathrm{tr}(\boldsymbol{\Lambda}_p^4) - p^{-1}\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)\right\}.$$

Theorem 2. *Suppose that $n\mathrm{tr}(\mathbf{D}_{n,p}^2)/p = O(1)$. Under $\mathcal{H}_1$ and Assumption* (A1), *as $p, n \to \infty$, $\{\tilde{Q}_S - \mathrm{tr}(\mathbf{D}_{n,p}^2)/p\}/\sigma_1 \rightsquigarrow \mathcal{N}(0, 1)$.*

According to Theorem 2, if $p = O(n^2)$, $\tilde{Q}_S$ has the same power function as the test proposed by Zou et al. [17]. However, when $p/n^2 \to \infty$, the variance of the SS test statistic will be larger than σ_1^2 because of poor estimation of the location parameter $\boldsymbol{\theta}_p$, which will be revisited in Section 3.

As a consequence of Theorem 2, our ST test is consistent. This is formally stated below.

Corollary 1. *If $n\mathrm{tr}(\mathbf{D}_{n,p}^2)/p \to \infty$, the test $\tilde{Q}_S/\sigma_0 > z_\alpha$ is consistent against $\mathcal{H}_1$ as $n \to \infty$ and $p \to \infty$.*

Under the framework of Theorems 1 and 2, our SR test can be theoretically compared with some existing procedures, such as the test of Chen et al. [3]. The following corollary is concerned with the limiting relative efficiency with respect to their test (abbreviated as CZZ hereafter) under the multivariate normality assumption.

Corollary 2. *If $n\mathrm{tr}(\mathbf{D}_{n,p}^2)/p = O(1)$, under multi-normal distributions, the SR test is as asymptotically efficient as the CZZ test.*

It is worth pointing out that from a theoretical perspective, comparing the proposed test with the CZZ test under general multivariate distributions is difficult. This is because the asymptotic validity of the CZZ test relies on the diverging factor model, while Theorems 1 and 2 rely on elliptical distributions. The distinction and connection between elliptical distributions and the diverging factor model is far from clear in the literature.

2.2. High-dimensional Kendall's tau-type rank test statistic

In this subsection, we consider another efficient sphericity test based on Kendall's tau. The classic Kendall's tau covariance matrix is defined as

$$\boldsymbol{\Xi}_{n,p} = \frac{2}{n(n-1)} \sum_{i<j} \boldsymbol{U}_{ij}\boldsymbol{U}_{ij}^{\top}.$$

Under $\mathcal{H}_0$, we have $\mathrm{E}(\boldsymbol{\Xi}_{n,p}) \doteq \boldsymbol{\Xi}_p = p^{-1}\mathbf{I}_p$. Then, the Kendall's tau test statistic is defined as

$$Q_K = p\mathrm{tr}\{\mathrm{tr}^{-1}(\boldsymbol{\Xi}_{n,p})\boldsymbol{\Xi}_{n,p} - p^{-1}\mathbf{I}_p\}^2 = p\mathrm{tr}(\boldsymbol{\Xi}_{n,p}^2) - 1.$$

When p is fixed, under the null hypothesis, one has

$$\frac{n}{\gamma_K} Q_K \rightsquigarrow \chi^2_{(p+2)(p-1)/2}$$

where γ_K is another nuisance parameter which depends on g_p and p. Similarly, the estimator of γ_K in Sirkiä et al. [16] cannot be used in high-dimensional settings, which requires the original location or an $O(n^3p^4)$ computation. Owing to the "blessing of dimension", we do not need this nuisance parameter in high-dimensional data. Moreover, the nature estimator $\mathrm{tr}(\boldsymbol{\Xi}_{n,p}^2)$ also results in a non-negligible bias term in Q_K when p is ultra-high. Thus, based on the leave-one-out method, we propose to estimate $\mathrm{tr}(\boldsymbol{\Xi}_p^2)$ by

$$\widehat{\mathrm{tr}(\boldsymbol{\Xi}_p^2)} = \frac{1}{n(n-1)(n-2)(n-3)} \sum\sum\sum_{i,j,k,\ell \text{ are not equal}}\sum (\boldsymbol{U}_{ij}^{\top}\boldsymbol{U}_{k\ell})^2.$$

Then, we define the following high-dimensional Kendall's tau-type rank test statistic (abbreviated as SK hereafter) as

$$\tilde{Q}_K = p\widehat{\mathrm{tr}(\boldsymbol{\Xi}_p^2)} - 1.$$

Obviously, the test statistic $\tilde{Q}_K$ is also invariant under rotation. The asymptotic properties of $\tilde{Q}_K$ are as follows.

Theorem 3. *As $n \to \infty$ and $p \to \infty$,*

(i) *Under $\mathcal{H}_0$ and Assumption (A1), $\tilde{Q}_K/\sigma_0 \rightsquigarrow \mathcal{N}(0, 1)$.*
(ii) *Under $\mathcal{H}_1$ and Assumption (A1), if $n\mathrm{tr}(\mathbf{D}_{n,p}^2)/p = O(1)$, then*

$$\{\tilde{Q}_K - \mathrm{tr}(\mathbf{D}_{n,p}^2)/p\}/\sigma_1 \rightsquigarrow \mathcal{N}(0, 1).$$

In fact, as shown in the proof of Theorem 3, $\tilde{Q}_K$ is asymptotically equivalent to $\tilde{Q}_S$ under both the null and alternative hypotheses. Thus, similar to Corollary 1, we can also derive the consistency of the SK test. Moreover, the SK test is as asymptotically efficient as the CZZ test under multinormal distributions, via similar arguments as in Corollary 2. We summarize these results in the following corollary.

Corollary 3. *Suppose $n \to \infty$ and $p \to \infty$.*

(i) *If $n\mathrm{tr}(\mathbf{D}_{n,p}^2)/p \to \infty$, the test $\tilde{Q}_K/\sigma_0 > z_\alpha$ is consistent against $\mathcal{H}_1$.*
(ii) *If $n\mathrm{tr}(\mathbf{D}_{n,p}^2)/p = O(1)$, under multinormal distributions, the SK test is asymptotically as efficient as the CZZ test.*

Recently, Onatski et al. [14] obtained the power envelope for the sphericity test in the Gaussian case. They computed the distance between the obtained power envelope and various tests: John's test [6,7], the Ledoit and Wolf [9] procedure, and the corrected LRT of Bai et al. [1] together with the Tracy–Widom type tests studied by Johnstone [8].

According to Theorems 2 and 3, the power functions of the SR and SK tests are

$$\beta_{SR}(\mathbf{D}_{n,p}) = \beta_{SK}(\mathbf{D}_{n,p}) = \Phi\left\{-z_\alpha\sigma_1^{-1}\sigma_0 + \sigma_1^{-1}p^{-1}\mathrm{tr}(\mathbf{D}_{n,p}^2)\right\}$$

where Φ is the standard normal distribution function. Additionally, $\sigma_0/\sigma_1 \to 1$ and $\mathrm{tr}(\mathbf{D}_{n,p}^2) = h^2$ under the spiked alternatives $\mathcal{H}_1 : \boldsymbol{\Sigma}_p = \sigma^2(\mathbf{I}_p + hvv')$ where v is a p-dimensional vector with $\|v\| = 1$ and $p/n \to c \in (0, \infty)$. Thus, the power function of the SR and SK tests under the spiked alternative is

$$\beta_{SR}(h, c) = \beta_{SK}(h, c) = \Phi(-z_\alpha + 0.5c^{-1}h^2).$$

The CZZ test has the same power function as in Corollaries 2 and 3(ii). According to Proposition 10 in Onatski et al. [14], the power function of John's test is

$$\beta_J(h, c) = \beta_{CZZ}(h, c) = \beta_{SR}(h, c) = \beta_{SK}(h, c) = \Phi(-z_\alpha + 0.5c^{-1}h^2).$$

Fig. 1 compares these power functions to the corresponding power envelopes (Proposition 9 in Onatski et al. [14]), where $\theta = \sqrt{-\log(1 - h^2/c)}$.

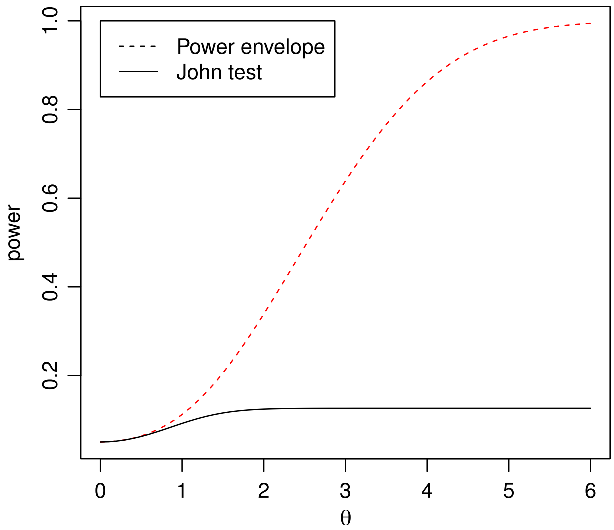

Fig. 1. Asymptotic powers of the tests and the corresponding power envelopes.

3. Simulation

In this section, we consider the following five distribution set-ups for comparison:

(I) The standard multivariate normal.
(II) The standard multivariate t with four degrees of freedom, $t_{p,4}$.
(III) Mixtures of two multivariate normal densities $\kappa f_p(\mu, \mathbf{I}_p) + (1-\kappa) f_p(\mu, 9\mathbf{I}_p)$, where $f_p(\cdot;\cdot)$ is the p-variate multivariate normal density. The value κ was chosen to be 0.8.
(IV) The diverging factor model with the standardized $\mathcal{G}(4, 0.5)$ distribution.
(V) The diverging factor model with the standardized t distribution with four degrees of freedom, t_4.

Here we choose $\mathbf{\Gamma} = \mathbf{I}_p$, and for each $\boldsymbol{Z}_i$, p-independent identically distributed random variables Z_{ij}s were generated from the diverging factor model for Scenarios (IV) and (V). The first three scenarios are well-known multivariate elliptical distributions. However, the last two scenarios are not elliptically distributed. We considered sample sizes $n \in \{20, 30\}$ and dimensions $p \in \{100, 200, 400, 800\}$.

Proceeding as in Chen et al. [3], we set $\boldsymbol{X}_i = \mathbf{A}\boldsymbol{Y}_i$, where $\boldsymbol{Y}_i$ were generated for Scenarios (I)–(V) and $\mathbf{A} = \mathrm{diag}\{2^{1/2}1_{\lfloor vp \rfloor}, 1_{p-\lfloor vp \rfloor}\}$, where $\lfloor x \rfloor$ denotes the integer part of x. Different levels of v were considered for different sample sizes. We compared our SR test and SK test with SS test and CZZ test. Tables 1 and 2 report the empirical sizes and power of these four tests for Scenarios (I)–(III), (IV)–(V), respectively.

First consider the empirical size of these tests. The empirical size of the SR and SS tests was close to the nominal level in all cases, which were not impacted by their dimension. However, SS was not able to control its empirical size very well in many cases. Sometimes it was a little conservative, and other times it was larger than the nominal level.

To evaluate the impact of dimension on the bias term of SS, we also report the mean-standard deviation-ratio $\mathrm{E}(T)/\sqrt{\mathrm{var}(T)}$ and the variance estimator ratio $\mathrm{var}(T)/\widehat{\mathrm{var}(T)}$ of these four tests. Since the explicit form of $\mathrm{E}(T)$ and $\mathrm{var}(T)$ of SS is difficult to calculate for all tests, we estimated them via simulations. Figs. 2 and 3 report the mean-standard deviation-ratio of these four tests. Figs. 4 and 5 report the variance estimator ratio of these tests. We observed that the bias term in SS apparently existed, especially when p/n^2 is large. This could be expected because SS can only allow the dimension to be comparable to the square of the sample size. In contrast, the mean-standard deviation-ratio of our SR and SK test statistics is approximately zero, which shows that, regardless of the dimension, there is no bias term in our test statistics.

For Scenarios (III)–(V), the variance estimator ratio of SS eventually became larger than one when p/n^2 was large. When the dimension became larger, the bias of the spatial-median estimator also increased the variance of the SS test statistic. Thus, the empirical sizes of SS are difficult to maintain in these cases. However, the variance estimator ratio of our SR and SK test statistics were approximately 1. Without estimating the location parameter, the variances of the SR and SK test statistic did not increase with the dimension. In addition, when the sample was generated from the diverging factor model, the empirical sizes of the CZZ test were a little larger than the nominal level in most cases. However, under Scenarios (II) and (III), the mean-standard deviation-ratio of CZZ was less than zero, and the variance estimator ratio eventually became larger than 1. As such, the empirical sizes of the CZZ test were significantly larger than the nominal level. This result is not surprising because neither $t_{p,4}$ nor a mixture of multivariate normal distributions belongs to the diverging factor model.

Next, we compared the power of these tests with both small $n = 20, 30, 40$ and large sample sizes $n = 100, 500$. The SR and SK tests performed similarly, which is consistent with the theoretical results in Section 2. In most cases, the SR and SK

Table 1
Empirical size and power comparison at the 5% level with $n = 20$.

(n, p)	Size				$\upsilon = 0.15$				$\upsilon = 0.30$			
	SR	SK	SS	CZZ	SR	SK	SS	CZZ	SR	SK	SS	CZZ
Scenario (I)												
(20, 100)	5.8	5.8	3.9	5.8	24	24	16	26	33	33	25	34
(20, 200)	6.3	6.3	5.3	6.5	28	28	23	29	36	36	22	36
(20, 400)	6.3	6.3	4.5	7.6	26	26	14	27	34	33	20	35
(20, 800)	6.0	6.0	6.0	7.6	25	25	21	26	36	36	21	37
Scenario (II)												
(20, 100)	5.0	5.3	5.8	9.7	24	26	23	21	30	32	32	25
(20, 200)	4.9	5.8	6.8	10.1	26	28	28	22	32	35	35	27
(20, 400)	5.9	6.7	9.0	11.5	25	27	28	22	32	34	34	27
(20, 800)	5.0	5.7	11.7	10.1	24	26	33	22	34	37	45	28
Scenario (III)												
(20, 100)	6.2	6.2	4.8	11.4	21	23	21	19	29	31	28	23
(20, 200)	5.9	5.8	6.7	12.2	25	27	26	22	32	35	30	25
(20, 400)	5.8	6.3	5.0	12.7	25	27	23	21	34	35	28	24
(20, 800)	5.2	5.9	9.2	11.9	24	27	29	21	34	37	29	26
Scenario (IV)												
(20, 100)	4.8	5.9	4.9	7.1	24	24	18	25	31	31	25	32
(20, 200)	5.0	5.0	5.8	7.8	27	27	23	28	34	34	25	35
(20, 400)	4.5	4.5	3.4	7.0	26	26	15	27	33	33	20	34
(20, 800)	5.0	5.0	6.6	7.4	25	25	22	26	35	35	19	36
Scenario (V)												
(20, 100)	5.5	5.5	5.9	9.8	25	25	20	27	30	30	26	32
(20, 200)	4.9	5.9	5.8	9.7	27	27	18	28	35	35	26	35
(20, 400)	4.6	5.6	5.6	6.8	25	25	21	27	32	32	26	34
(20, 800)	5.7	5.7	4.9	7.6	27	27	19	28	36	36	26	37

Table 2
Empirical size and power comparison at the 5% level with $n = 30$.

(n, p)	Size				$\upsilon = 0.15$				$\upsilon = 0.30$			
	SR	SK	SS	CZZ	SR	SK	SS	CZZ	SR	SK	SS	CZZ
Scenario (I)												
(30, 100)	5.6	5.7	5.2	6.1	39	39	34	41	52	52	48	55
(30, 200)	4.9	4.9	3.6	5.5	42	42	34	43	56	56	51	56
(30, 400)	5.1	5.1	3.0	5.1	40	40	22	41	56	56	43	57
(30, 800)	6.5	6.5	4.2	6.8	41	41	30	42	55	55	47	56
Scenario (II)												
(30, 100)	5.7	4.9	5.3	11.6	37	40	38	28	48	51	50	34
(30, 200)	6.0	5.6	5.5	11.0	40	43	41	30	52	56	55	39
(30, 400)	5.2	5.2	6.4	10.8	38	41	41	30	52	55	57	37
(30, 800)	6.5	6.0	7.9	12.0	38	41	42	31	50	53	57	38
Scenario (III)												
(30, 100)	4.6	6.3	5.3	14.9	36	41	38	31	48	54	50	37
(30, 200)	4.8	4.5	4.6	13.7	38	42	41	29	50	54	54	35
(30, 400)	5.7	5.5	3.6	16.8	37	41	36	31	52	57	54	37
(30, 800)	5.8	5.0	5.9	13.4	37	41	40	28	51	55	55	35
Scenario (IV)												
(30, 100)	4.8	4.8	4.6	6.0	38	38	35	42	51	51	49	53
(30, 200)	5.6	5.8	4.7	6.1	40	40	36	42	55	55	52	56
(30, 400)	5.3	5.3	4.2	5.7	41	41	29	40	55	55	41	56
(30, 800)	5.9	4.9	3.8	7.1	42	42	33	43	57	57	49	57
Scenario (V)												
(30, 100)	4.2	4.2	5.8	8.4	36	36	33	39	50	49	45	51
(30, 200)	5.9	5.9	6.2	8.3	37	37	33	38	50	50	44	49
(30, 400)	4.5	4.5	5.0	7.1	40	40	32	40	54	54	50	55
(30, 800)	4.1	5.1	4.7	7.1	40	40	32	41	55	55	47	55

tests performed a little better than the SS test. The variance of the SS test statistic increased faster than those of the SR and SK test statistics because of the estimation of location parameters. Thus, it is not surprising that the power of SS was smaller than for these two tests. Moreover, the power of SS was larger than those of SR and SK in some cases, such as for Scenario (II) with $(n, p) = (20, 800)$. Note, however, that the empirical sizes of SS were larger than the nominal level in these cases, which mitigates the conclusion. In addition, our SR and SK tests perform similarly to the CZZ test under normal distributions. Even under the non-elliptical distributions (Scenarios (IV) and (V)), the difference between CZZ and SR and SK was marginal.

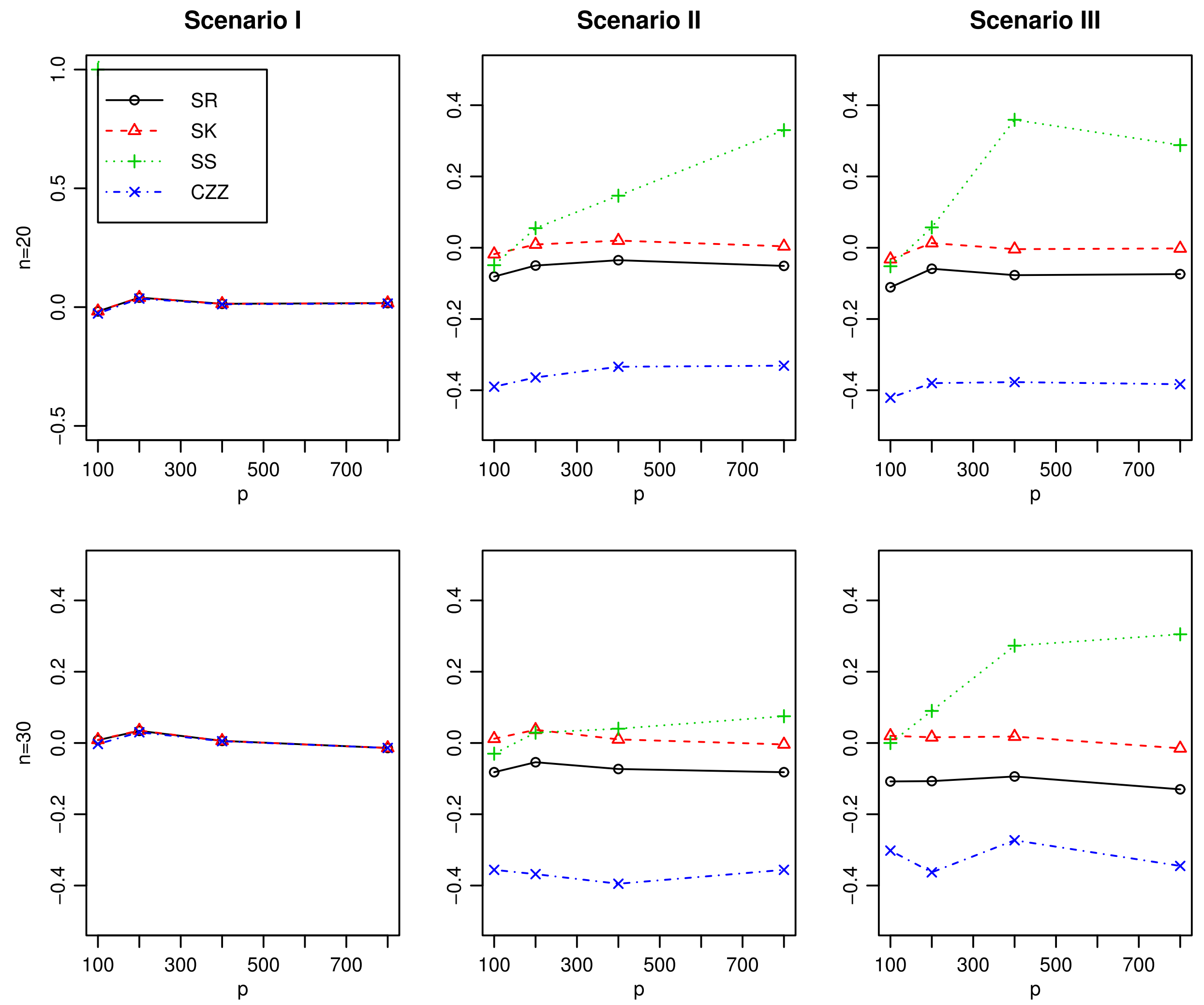

Fig. 2. The mean-standard deviation-ratio of test statistics for Scenarios (I)–(III).

However, under two heavy-tailed elliptical distributions (Scenario (II) and (III)), our SR and SK tests eventually performed better than the CZZ test.

All these results suggest that the two proposed tests are quite robust and efficient in testing sphericity. Without estimating the location parameter, the SR and SK tests can control their empirical sizes suitably, and are more powerful than the SS test when p/n^2 is large. For heavy-tailed elliptical distributions, the SR and SK tests performed much better than the CZZ test both in power and size (see Tables 3–5).

4. Discussion

In this paper, we proposed two new, robust and efficient tests for sphericity based on multivariate ranks, and established their asymptotic normalities. Looking into our future work, this approach can be extended to more problems, such as tests for location parameters. It would also be of considerable interest to relax the multivariate normal assumption into elliptical distributions, and measure the distance between the power envelope and the multivariate sign/rank based tests.

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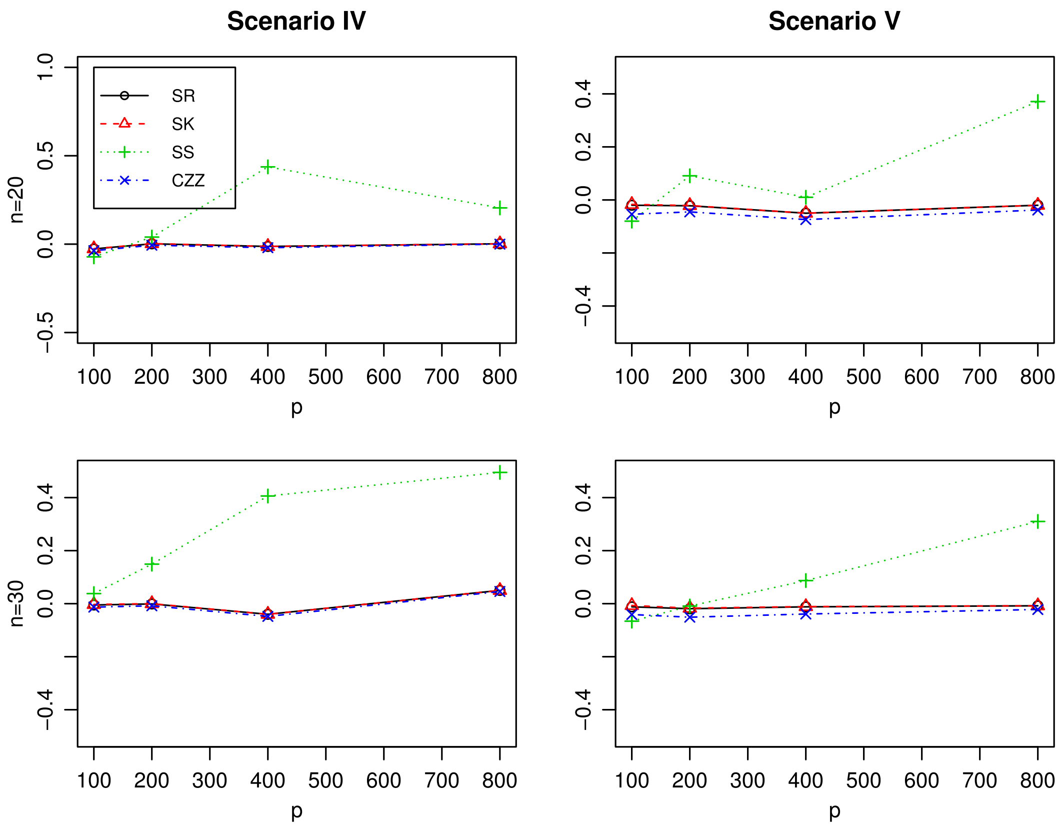

Fig. 3. The mean-standard deviation-ratio of test statistics for Scenarios (IV)–(V).

Appendix A. Some useful lemmas

Denote $\boldsymbol{\varepsilon}_i = \boldsymbol{\Sigma}_p^{-1/2}(\boldsymbol{X}_i - \boldsymbol{\theta}_p)$ and $\boldsymbol{u}_i = \mathrm{E}\{U(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j)|\boldsymbol{\varepsilon}_i\}$. Obviously, $\mathrm{E}(\boldsymbol{u}_i\boldsymbol{u}_i^\top) = \tau_F p^{-1}\mathbf{I}_p$, where τ_F is a constant which depends on the distribution g_p and p.

Lemma 1. $\tau_F \to 0.5$ *as* $p \to \infty$.

Proof.

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\varepsilon}_i^\top\boldsymbol{\varepsilon}_i) &= \mathrm{E}\{(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j)^\top(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_k)\} \\
&= \mathrm{E}\left[\mathrm{E}\left\{(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j)^\top(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_k)\Big|\boldsymbol{\varepsilon}_i\right\}\right] \\
&= \mathrm{E}\left[\mathrm{E}\left\{\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j\|\,\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_k\|U(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j)^\top U(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_k)\Big|\boldsymbol{\varepsilon}_i\right\}\right] \\
&= \mathrm{E}\left[\left\{\mathrm{E}\left(\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j\|\,\Big|\boldsymbol{\varepsilon}_i\right)\right\}^2\right]\mathrm{E}\left\{\mathrm{E}\left(U(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j)^\top U(\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_k)\Big|\boldsymbol{\varepsilon}_i\right)\right\} \\
&= \mathrm{E}\left[\left\{\mathrm{E}\left(\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j\|\,\Big|\boldsymbol{\varepsilon}_i\right)\right\}^2\right]\mathrm{E}(\boldsymbol{u}_i^T\boldsymbol{u}_i) = \tau_F E\left[\left\{\mathrm{E}\left(\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j\|\,\Big|\boldsymbol{\varepsilon}_i\right)\right\}^2\right].
\end{aligned}$$

In addition, $\mathrm{E}(\|\boldsymbol{\varepsilon}_i\|^2) = 0.5\mathrm{E}(\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j\|^2)$. Thus, we only need to show that

$$\frac{\mathrm{E}\left[\left\{\mathrm{E}\left(\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j\|\Big|\boldsymbol{\varepsilon}_i\right)\right\}^2\right]}{\mathrm{E}(\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j\|^2)} \to 1.$$

Because $\boldsymbol{\varepsilon}_i$ has an elliptical distribution, $\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j$ also has an elliptical distribution. Define the density function of $\|\boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j\|$ is $f(t) = c_p t^{p-1}g(t)$ where $c_p = 2\pi^{p/2}/\Gamma(p/2)$. Thus,

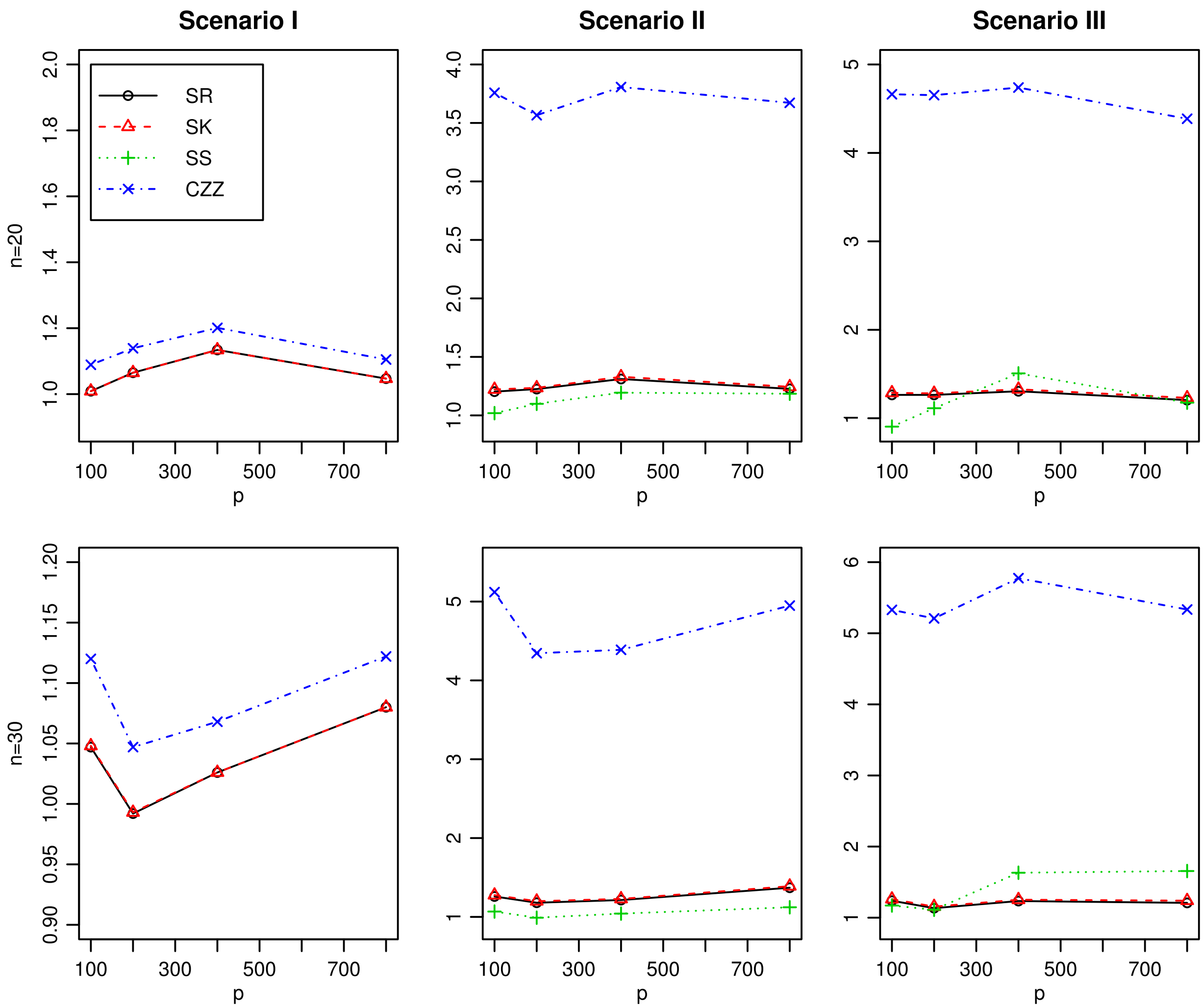

Fig. 4. The variance-ratio of test statistics for Scenarios (I)–(III).

$$
\begin{aligned}
\frac{\mathrm{E}\left[\left\{\mathrm{E}\left(\|\boldsymbol{\varepsilon}_i-\boldsymbol{\varepsilon}_j\|\,|\boldsymbol{\varepsilon}_i\right)\right\}^2\right]}{\mathrm{E}(\|\boldsymbol{\varepsilon}_i-\boldsymbol{\varepsilon}_j\|^2)} &= \frac{\left\{\int c_p t^p g(t)dt\right\}^2}{\int c_p t^{p+1} g(t)dt} \\
&= \frac{c_{p+1}^2}{c_p c_{p+2}} = \frac{\Gamma^2\{(p+1)/2\}}{\Gamma(p/2)\Gamma\{(p+2)/2\}}.
\end{aligned}
$$

From Stirling's formula,

$$
\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x(2\pi x)^{1/2}} = 1,
$$

as $p \to \infty$, we have

$$
\frac{c_{p+1}^2}{c_p c_{p+2}} \to \frac{(p-1)^{p-1}}{p^{p/2}(p-2)^{(p-2)/2}} = (1-p^{-1})^{p/2}\{1+(p-2)^{-1}\}^{(p-2)/2} \to 1.
$$

Hence the proof of Lemma 1 is complete. $\square$

Next, we restate Lemma 4 in Zou et al. [17].

Lemma 2. *Suppose* $\boldsymbol{u}$ *are independently, identically and uniformly distributed on an unit* p *sphere. For any* $p \times p$ *symmetric matrix* $\mathbf{M}$, *we have*

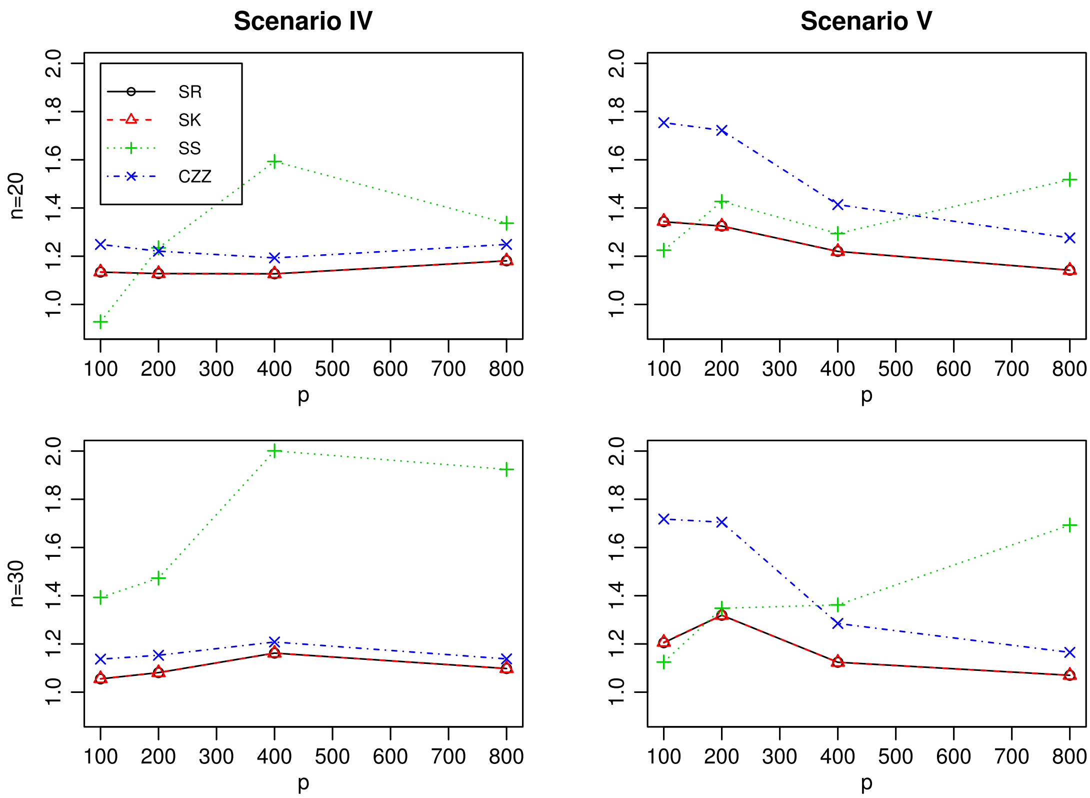

Fig. 5. The variance-ratio of tests for Scenarios (IV)–(V).

$$\mathrm{E}(\boldsymbol{u}^\top \mathbf{M}\boldsymbol{u})^2 = \{\mathrm{tr}^2(\mathbf{M}) + 2\mathrm{tr}(\mathbf{M}^2)\}/(p^2+2p),$$
$$\mathrm{E}(\boldsymbol{u}^\top \mathbf{M}\boldsymbol{u})^4 = \{3\mathrm{tr}^2(\mathbf{M}^2) + 6\mathrm{tr}(\mathbf{M}^4)\}/\{p(p+2)(p+4)(p+6)\}.$$

Lemma 3. *As $n \to \infty$ and $p \to \infty$,*

$$\frac{1}{\sigma_0}\left\{\frac{p}{n(n-1)}\sum\sum_{i\neq j}(\boldsymbol{u}_i^\top \boldsymbol{u}_j)^2/\tau_F^2 - 1\right\} \rightsquigarrow \mathcal{N}(0,1).$$

Proof. Define $\boldsymbol{v}_i = \boldsymbol{u}_i/\sqrt{\tau_F}$. Thus, $\mathrm{E}(\boldsymbol{v}_i\boldsymbol{v}_i^\top) = p^{-1}\mathbf{I}_p$. Define

$$Q_s' = \frac{p}{n(n-1)}\sum\sum_{i\neq j}(\boldsymbol{u}_i^\top \boldsymbol{u}_j)^2/\tau_F^2 - 1 = \frac{p}{n(n-1)}\sum\sum_{i\neq j}(\boldsymbol{v}_i^\top \boldsymbol{v}_j)^2 - 1.$$

The expectation of Q_S' can be easily computed, and hence the details are omitted. We also have

$$\begin{aligned}
\mathrm{var}(Q_S') &= \{n(n-1)\}^{-2}p^2\mathrm{E}\left\{\sum_{i\neq j}(\boldsymbol{v}_i^\top \boldsymbol{v}_j)^2\right\}^2 - 1\\
&= \{n(n-1)\}^{-2}p^2\big[2n(n-1)\mathrm{E}(\boldsymbol{v}_i^\top \boldsymbol{v}_j)^4 + 4n(n-1)(n-2)\mathrm{E}\{(\boldsymbol{v}_i^\top \boldsymbol{v}_j)^2(\boldsymbol{v}_i^\top \boldsymbol{v}_k)^2\}\\
&\quad + n(n-1)(n-2)(n-3)\mathrm{E}\{(\boldsymbol{v}_i^\top \boldsymbol{v}_j)^2(\boldsymbol{v}_k^\top \boldsymbol{v}_l)^2\}\big] - 1\\
&= 4(p-1)/\{n(n-1)(p+2)\}.
\end{aligned}$$

Next, we only need to show the asymptotic normality of Q_S'. Let $\mathcal{F}_0 = \{\varnothing, \boldsymbol{\Omega}\}$, and for each $k \in \{1,\ldots,n\}$, set $\mathcal{F}_k = \sigma\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k\}$. Let $\mathrm{E}_k(\cdot)$ denote the conditional expectation of given $\mathcal{F}_k$ and $\mathrm{E}_0(\cdot) = \mathrm{E}(\cdot)$. We write $Q_S' - \mathrm{E}(Q_S') = \sum_{k=1}^n G_{n,k}$, where $G_{n,k} = (\mathrm{E}_k - \mathrm{E}_{k-1})Q_S'$. Then for every n, $\{G_{n,k}\}_{k=1}^n$ is a martingale difference sequence with respect to the σ-fields $\{\mathcal{F}_k : 1 \le k \le n\}$.

Table 3
Empirical size and power comparison at the 5% level with $n = 40$.

(n, p)	Size				$v = 0.15$				$v = 0.30$			
	SR	SK	SS	CZZ	SR	SK	SS	CZZ	SR	SK	SS	CZZ
Scenario (I)												
(40, 100)	6.3	5.8	6.1	5.3	89	89	88	91	100	100	99	100
(40, 200)	5.7	4.7	4.5	5.6	84	84	79	83	100	100	100	100
(40, 400)	4.6	5.1	5.8	4.9	86	86	50	88	100	100	91	100
(40, 800)	6.2	5.9	5.2	6.2	91	91	79	92	100	100	97	100
Scenario (II)												
(40, 100)	6.4	6.5	7.1	16	87	90	88	66	100	100	100	93
(40, 200)	5.9	6.1	6.7	11	73	76	82	52	100	100	100	88
(40, 400)	4.1	3.9	5.2	7.6	81	82	84	53	100	100	100	89
(40, 800)	4.9	6.4	6.1	10	85	87	88	54	100	100	100	91
Scenario (III)												
(40, 100)	5.7	6.3	7.3	16	82	87	90	60	100	100	100	91
(40, 200)	5.9	5.8	5.8	13	75	81	83	54	100	100	100	92
(40, 400)	5.3	5.7	5.5	16	81	84	84	59	100	100	100	97
(40, 800)	5.1	6.2	6.1	14	81	84	88	52	100	100	99	93
Scenario (IV)												
(40, 100)	5.1	5.8	5.2	6.1	48	48	47	54	65	65	62	68
(40, 200)	5.3	5.4	4.7	4.8	53	53	48	54	77	77	76	77
(40, 400)	5.7	5.2	5.3	5.9	58	58	46	59	72	72	62	73
(40, 800)	4.9	6.1	3.9	6.2	55	55	47	58	71	71	70	71
Scenario (V)												
(40, 100)	4.3	4.9	4.1	6.3	55	56	48	60	70	71	64	72
(40, 200)	5.2	5.6	4.2	5.8	49	49	46	45	64	64	64	67
(40, 400)	3.9	4.3	5.1	4.9	50	50	46	54	72	72	71	69
(40, 800)	5.1	5.2	4.8	5.8	58	58	55	59	76	76	73	77

Table 4
Empirical size and power comparison at the 5% level with $n = 100$.

(n, p)	Size				$v = 0.03$				$v = 0.06$			
	SR	SK	SS	CZZ	SR	SK	SS	CZZ	SR	SK	SS	CZZ
Scenario (I)												
(100, 100)	4.6	3.9	5.1	4.5	33	30	31	31	71	70	73	75
(100, 200)	5.1	5.0	4.3	3.9	32	31	32	35	83	81	81	84
(100, 400)	4.5	4.0	4.9	4.7	36	31	35	38	86	83	86	86
(100, 800)	4.2	6.2	5.3	5.2	39	44	41	42	85	90	87	88
Scenario (II)												
(100, 100)	4.7	5.7	4.4	11	32	30	31	21	71	71	73	46
(100, 200)	5.4	5.2	4.1	14	33	32	33	24	86	79	83	45
(100, 400)	4.9	5.6	5.4	15	37	34	36	31	85	88	86	53
(100, 800)	5.8	5.5	4.9	12	42	37	40	26	89	88	85	52
Scenario (III)												
(100, 100)	4.5	4.3	5.7	16	27	32	31	31	74	71	73	55
(100, 200)	5.3	5.3	4.0	15	29	33	32	31	85	80	82	52
(100, 400)	5.5	6.0	5.1	17	34	36	35	38	88	87	87	61
(100, 800)	5.4	5.3	5.2	15	39	38	41	36	86	84	85	58
Scenario (IV)												
(100, 100)	5.6	6.3	5.6	4.2	19	24	21	26	54	50	51	58
(100, 200)	4.5	5.8	4.3	4.5	32	30	33	34	69	69	69	71
(100, 400)	4.3	4.5	3.8	3.5	31	35	33	34	71	69	70	72
(100, 800)	4.5	4.7	6.1	8.2	36	37	36	37	69	68	71	72
Scenario (V)												
(100, 100)	5.6	5.4	4.6	6.6	18	19	17	19	49	48	46	48
(100, 200)	5.1	4.6	5.7	7.6	19	22	20	28	54	57	58	57
(100, 400)	3.7	4.3	3.7	4.2	28	26	26	30	62	65	65	67
(100, 800)	4.8	5.2	6.3	7.1	36	34	35	39	70	73	72	73

Let $\sigma_{n,k}^2 = \mathrm{E}_{k-1}(G_{n,k}^2)$. According to the Martingale Central Limit Theorem [4], we only need to show that, as $n \to \infty$,

$$\frac{\sum_{k=1}^{n} \sigma_{n,k}^2}{\mathrm{var}(Q_S')} \to 1 \quad \text{in probability and} \quad \frac{\sum_{k=1}^{n} E(G_{n,k}^4)}{\mathrm{var}^2(Q_S')} \to 0. \tag{3}$$

Define $\mathbf{\Gamma}_{k-1} = \sum_{i=1}^{k-1}\left(\boldsymbol{v}_i\boldsymbol{v}_i^\top - p^{-1}\mathbf{I}_p\right)$. We have

$$\sum_{k=1}^{n}\sigma_{n,k}^2 = \sum_{k=1}^{n}E_{k-1}(G_{n,k}^2) = \sum_{k=1}^{n}4\{n(n-1)\}^{-2}p^2\left(\boldsymbol{v}_k^\top\mathbf{\Gamma}_{k-1}\boldsymbol{v}_k\right)^2$$
$$= \frac{8}{\{n(n-1)\}^2}\sum_{k=1}^{n}\mathrm{tr}(\mathbf{\Gamma}_{k-1}^2).$$

By noting that

$$\mathrm{tr}\left(\sum_{k=1}^{n}\mathbf{\Gamma}_{k-1}^2\right) = \sum_{k=1}^{n}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\mathrm{tr}\left\{\left(\boldsymbol{v}_i\boldsymbol{v}_i^\top - p^{-1}\mathbf{I}_p\right)\left(\boldsymbol{v}_j\boldsymbol{v}_j^\top - p^{-1}\mathbf{I}_p\right)\right\}$$
$$= \sum_{i\neq j}2\{n-\max(i,j)\}\,\mathrm{tr}\left\{\left(\boldsymbol{v}_i\boldsymbol{v}_i^\top - p^{-1}\mathbf{I}_p\right)\left(\boldsymbol{v}_j\boldsymbol{v}_j^\top - p^{-1}\mathbf{I}_p\right)\right\} + \frac{n(n-1)(p-1)}{2p},$$

we deduce that

$$\mathrm{E}\left(\sum_{k=1}^{n}\sigma_{n,k}^2\right) = \frac{4(p-1)}{n(n-1)p}, \qquad \mathrm{var}\left(\sum_{k=1}^{n}\sigma_{n,k}^2\right) = \frac{128(n-2)(p-1)}{3\{n(n-1)\}^3p^2(p+2)}.$$

Clearly, $\sum_{k=1}^{n}\sigma_{n,k}^2/\mathrm{var}(Q_S') \to 1$.

Finally, we verify the second part of (3). Note that

$$\sum_{k=1}^{n}\mathrm{E}(G_{n,k}^4) = \frac{16p^4}{\{n(n-1)\}^4}\left[\frac{n(n-1)}{2}\mathrm{E}\left\{\boldsymbol{v}_k^\top\left(\boldsymbol{v}_i\boldsymbol{v}_i^\top - p^{-1}\mathbf{I}_p\right)\boldsymbol{v}_k\right\}^4\right.$$
$$\left.+\,n(n-1)(n-2)\mathrm{E}\left\{\left(\boldsymbol{v}_k^\top\left(\boldsymbol{v}_i\boldsymbol{v}_i^\top - p^{-1}\mathbf{I}_p\right)\boldsymbol{v}_k\right)^2\left(\boldsymbol{v}_k^\top(\boldsymbol{v}_j\boldsymbol{v}_j^\top - p^{-1}\mathbf{I}_p)\boldsymbol{v}_k\right)^2\right\}\right].$$

Because

$$\mathrm{E}\left\{\boldsymbol{v}_k^\top(\boldsymbol{v}_i\boldsymbol{v}_i^\top - p^{-1}\mathbf{I}_p)\boldsymbol{v}_k\right\}^4 = O(p^{-4}),$$
$$\mathrm{E}\left[\left\{\boldsymbol{v}_k^\top(\boldsymbol{v}_i\boldsymbol{v}_i^\top - p^{-1}\mathbf{I}_p)\boldsymbol{v}_k\right\}^2\left\{\boldsymbol{v}_k^\top(\boldsymbol{v}_j\boldsymbol{v}_j^\top - p^{-1}\mathbf{I}_p)\boldsymbol{v}_k\right\}^2\right] = O(p^{-4}),$$

it is straightforward to see that $\sum_{k=1}^{n}E(G_{n,k}^4) = o\{\mathrm{var}^2(Q_S')\}$. Hence the proof of Lemma 3 is complete. □

Appendix B. Proof of the theorems

Proof of Theorem 1. We can decompose $\boldsymbol{U}_{ij}$ as

$$\boldsymbol{U}_{ij} = U(\boldsymbol{X}_i - \boldsymbol{X}_j) = \mathrm{E}\{U(\boldsymbol{X}_i - \boldsymbol{X}_j)|\boldsymbol{X}_i\} - \mathrm{E}\{U(\boldsymbol{X}_i - \boldsymbol{X}_j)|\boldsymbol{X}_j\} + \boldsymbol{\omega}_{ij}.$$

Under $\mathscr{H}_0$, $\mathrm{E}\{U(\boldsymbol{X}_i - \boldsymbol{X}_j)|\boldsymbol{X}_i\} = \boldsymbol{u}_i$. Then, $\boldsymbol{U}_{ij} = \boldsymbol{u}_i - \boldsymbol{u}_j + \boldsymbol{\omega}_{ij}$. Obviously, $\mathrm{E}(\boldsymbol{\omega}_{ij}) = 0$, $\mathrm{E}(\boldsymbol{u}_i^\top\boldsymbol{\omega}_{ij}) = 0$ and $\mathrm{E}(\boldsymbol{\omega}_{ij}^\top\boldsymbol{\omega}_{ik}) = 0$. From Lemma 1, we have $\mathrm{E}(\boldsymbol{\omega}_{ij}^\top\boldsymbol{\omega}_{ij}) = 1 - 2\tau_F = o(1)$. Furthermore,

$$\tilde{Q}_S = \frac{2p}{n(n-1)(n-2)(n-3)}\sum\sum\sum\sum_{i,j,k,\ell\text{ are not equal}}\boldsymbol{U}_{ij}^\top\boldsymbol{U}_{k\ell}\boldsymbol{U}_{kj}^\top\boldsymbol{U}_{i\ell} - 1$$
$$= \frac{4p}{n(n-1)}\sum\sum_{i\neq j}(\boldsymbol{u}_i^\top\boldsymbol{u}_j)^2 - 1 - \frac{2p}{n(n-1)(n-2)}\sum\sum\sum_{i,j,k\text{ are not equal}}\boldsymbol{u}_i^\top\boldsymbol{u}_j\boldsymbol{u}_j^\top\boldsymbol{u}_k$$
$$+\frac{p}{n(n-1)(n-2)(n-3)}\sum\sum\sum\sum_{i,j,k,\ell\text{ are not equal}}\boldsymbol{u}_i^\top\boldsymbol{u}_j\boldsymbol{u}_k^\top\boldsymbol{u}_\ell$$
$$+O(pn^{-4})\sum\sum\sum\sum_{i,j,k,\ell\text{ are not equal}}\left(\boldsymbol{u}_i^\top\boldsymbol{u}_j\boldsymbol{u}_i^\top\boldsymbol{\omega}_{k\ell} + \boldsymbol{u}_i^\top\boldsymbol{u}_j\boldsymbol{u}_k^\top\boldsymbol{\omega}_{i\ell} + \boldsymbol{u}_i^\top\boldsymbol{u}_k\boldsymbol{u}_i^\top\boldsymbol{\omega}_{k\ell}\right.$$
$$\left.+\,\boldsymbol{u}_i^\top\boldsymbol{u}_j\boldsymbol{\omega}_{k\ell}^\top\boldsymbol{\omega}_{i\ell} + \boldsymbol{u}_i^\top\boldsymbol{u}_j\boldsymbol{\omega}_{ij}^\top\boldsymbol{\omega}_{k\ell} + \boldsymbol{u}_i^\top\boldsymbol{\omega}_{k\ell}\boldsymbol{\omega}_{kj}^\top\boldsymbol{\omega}_{i\ell} + \boldsymbol{\omega}_{ij}^\top\boldsymbol{\omega}_{k\ell}\boldsymbol{\omega}_{kj}^\top\boldsymbol{\omega}_{i\ell}\right)$$
$$\doteq J_1 + J_2 + J_3 + J_4.$$

Table 5
Empirical size and power comparison at the 5% level with $n = 500$.

(n, p)	Size				$\upsilon = 0.015$				$\upsilon = 0.02$			
	SR	SK	SS	CZZ	SR	SK	SS	CZZ	SR	SK	SS	CZZ
Scenario (I)												
(500, 100)	4.7	5.0	4.7	5.1	72	71	73	74	81	81	80	82
(500, 200)	4.6	5.1	3.8	3.5	71	73	74	73	98	98	98	99
(500, 400)	4.3	5.8	4.3	4.8	81	81	84	86	99	100	99	100
(500, 800)	5.2	4.7	5.2	4.1	93	95	94	96	100	100	100	100
Scenario (II)												
(500, 100)	5.7	5.6	4.6	23	70	70	71	39	80	82	81	49
(500, 200)	5.4	4.9	3.7	17	73	74	73	41	98	95	97	58
(500, 400)	4.0	4.8	4.4	21	82	82	84	58	100	100	100	73
(500, 800)	5.9	4.1	4.9	12	93	92	94	54	100	100	100	69
Scenario (III)												
(500, 100)	4.8	5.4	5.2	14	68	68	67	42	82	80	81	53
(500, 200)	4.0	5.9	4.0	22	72	73	73	47	97	96	98	68
(500, 400)	4.0	5.9	4.9	16	81	80	82	57	98	99	99	69
(500, 800)	3.9	4.2	5.1	16	94	93	94	63	100	100	100	82
Scenario (IV)												
(500, 100)	3.8	4.3	4.5	3.9	59	58	55	63	68	70	67	73
(500, 200)	4.3	5.5	5.3	6.3	68	69	67	71	90	90	90	92
(500, 400)	5.6	4.8	6.2	6.2	90	90	89	91	99	100	99	99
(500, 800)	4.4	5.1	6.1	6.4	90	89	90	90	100	100	99	99
Scenario (V)												
(500, 100)	3.9	5.0	4.8	6.6	49	48	48	51	60	61	58	62
(500, 200)	6.2	6.2	6.3	6.8	69	68	67	73	86	85	87	85
(500, 400)	3.9	5.1	5.9	7.3	85	84	86	86	94	94	95	93
(500, 800)	5.2	5.3	6.1	6.2	89	90	90	89	96	98	97	96

According to Lemmas 1 and 3, we have $J_1/\sigma_0 \rightsquigarrow \mathcal{N}(0,1)$. Thus, we only need to show the other parts are all $o_p(\sigma_0)$.

$$\begin{aligned}
\mathrm{E}(J_2^2) &= O(p^2n^{-2})\mathrm{E}(\boldsymbol{u}_i^\top \boldsymbol{u}_j\boldsymbol{u}_j^\top \boldsymbol{u}_k\boldsymbol{u}_k^\top \boldsymbol{u}_l\boldsymbol{u}_l^\top \boldsymbol{u}_i) + O(p^2n^{-3})\mathrm{E}(\boldsymbol{u}_i^\top \boldsymbol{u}_j\boldsymbol{u}_j^\top \boldsymbol{u}_k\boldsymbol{u}_k^\top \boldsymbol{u}_j\boldsymbol{u}_j^\top \boldsymbol{u}_i)\\
&= O(p^{-1}n^{-2}) + O(p^{-1}n^{-3}) = o(\sigma_0^2),\\
\mathrm{E}(J_4^2) &= O(p^2n^{-4})\mathrm{E}\{(\boldsymbol{u}_i^\top \boldsymbol{u}_j\boldsymbol{u}_k^\top \boldsymbol{u}_l)^2\} = O(p^{-1}n^{-4}) = o(\sigma_0^2).
\end{aligned}$$

Finally, we consider the first part in J_4, leaving the other parts to the reader, as they can be handled in a similar fashion.

$$\begin{aligned}
&\mathrm{E}\Big\{O(pn^{-4})\underbrace{\sum\sum\sum\sum}_{i,j,k,\ell \text{ are not equal}} \boldsymbol{u}_i^\top \boldsymbol{u}_j\boldsymbol{u}_i^\top \boldsymbol{\omega}_{k\ell}\Big\}^2\\
&= O(p^2n^{-3})\mathrm{E}(\boldsymbol{u}_i^\top \boldsymbol{u}_j\boldsymbol{u}_i^\top \boldsymbol{\omega}_{k\ell}\boldsymbol{u}_s^\top \boldsymbol{u}_j\boldsymbol{u}_s^\top \boldsymbol{\omega}_{k\ell}) + O(p^2n^{-4})\mathrm{E}\{(\boldsymbol{u}_i^\top \boldsymbol{u}_j\boldsymbol{u}_i^\top \boldsymbol{\omega}_{k\ell})^2\}\\
&= O(p^{-1}n^{-3})\mathrm{E}(\boldsymbol{\omega}_{k\ell}^\top \boldsymbol{\omega}_{k\ell}) + O(p^{-1}n^{-4})\mathrm{E}(\boldsymbol{\omega}_{k\ell}^\top \boldsymbol{\omega}_{k\ell})\\
&= o(p^{-1}n^{-3}) + o(p^{-1}n^{-4}) = o(\sigma_0^2).
\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Define $\boldsymbol{V}_i = \mathrm{E}\{U(\boldsymbol{X}_i - \boldsymbol{X}_j)|\boldsymbol{X}_i\}$. Using similar arguments as in the proof of Theorem 1, we can show that

$$\tilde{Q}_S = \frac{4p}{n(n-1)}\sum\sum_{i\neq j}(\boldsymbol{V}_i^\top \boldsymbol{V}_j)^2 - 1 + o_p(\sigma_1).$$

Now, write $\boldsymbol{V}_i = (\boldsymbol{\Lambda}_p^{1/2}\boldsymbol{u}_i)/(1+\boldsymbol{u}_i^\top \mathbf{D}_{n,p}\boldsymbol{u}_i)^{1/2}$, and then

$$\begin{aligned}
\mathrm{E}(\boldsymbol{V}_i^\top \boldsymbol{V}_j)^2 &= \mathrm{tr}\left(\left[\mathrm{E}\left\{\boldsymbol{\Lambda}_p^{1/2}\boldsymbol{u}_i\boldsymbol{u}_i^\top \boldsymbol{\Lambda}_p^{1/2}(1+\boldsymbol{u}_i^\top \mathbf{D}_{n,p}\boldsymbol{u}_i)^{-1}\right\}\right]^2\right)\\
&= \mathrm{tr}\left[\left\{\mathrm{E}\left(\boldsymbol{\Lambda}_p^{1/2}\boldsymbol{u}_i\boldsymbol{u}_i^\top \boldsymbol{\Lambda}_p^{1/2}\right)\right\}^2\right] + \mathrm{tr}\left(\left[\mathrm{E}\left\{C_i\boldsymbol{\Lambda}_p^{1/2}\boldsymbol{u}_i\boldsymbol{u}_i^\top \boldsymbol{\Lambda}_p^{1/2}\left(\boldsymbol{u}_i^\top \mathbf{D}_{n,p}\boldsymbol{u}_i\right)\right\}\right]^2\right),
\end{aligned}$$

where C_i is a bounded random variable between -1 and $-(1+\boldsymbol{u}_i^\top \mathbf{D}_{n,p}\boldsymbol{u}_i)^{-2}$. Obviously,

$$\mathrm{tr}\left[\left\{\mathrm{E}\left(\boldsymbol{\Lambda}_p^{1/2}\boldsymbol{u}_i\boldsymbol{u}_i^\top \boldsymbol{\Lambda}_p^{1/2}\right)\right\}^2\right] = \tau_F^2p^{-2}\mathrm{tr}(\boldsymbol{\Lambda}_p^2) = \tau_F^2p^{-2}\{p + \mathrm{tr}(\mathbf{D}_{n,p}^2)\}.$$

From the Cauchy–Schwarz inequality and Lemma 2,

$$\begin{aligned}\mathrm{tr}([\mathrm{E}\{C_i\boldsymbol{\Lambda}_p^{1/2}\boldsymbol{u}_i\boldsymbol{u}_i^\top\boldsymbol{\Lambda}_p^{1/2}(\boldsymbol{u}_i^\top\mathbf{D}_{n,p}\boldsymbol{u}_i)\}]^2) &\le C\mathrm{tr}[\{\mathrm{E}(\boldsymbol{\Lambda}_p^{1/2}\boldsymbol{u}_i\boldsymbol{u}_i^\top\boldsymbol{\Lambda}_p^{1/2})^2\}]\mathrm{E}\{(\boldsymbol{u}_i^\top\mathbf{D}_{n,p}\boldsymbol{u}_i)^2\}\\ &\le Cp^{-4}\mathrm{tr}(\boldsymbol{\Lambda}_p^2)\mathrm{tr}(\mathbf{D}_{n,p}^2)=Cp^{-4}\{p+\mathrm{tr}(\mathbf{D}_{n,p}^2)\}\mathrm{tr}(\mathbf{D}_{n,p}^2)=o(p^{-1}n^{-1})\end{aligned}$$

by the condition $\mathrm{tr}(\mathbf{D}_{n,p}^2)=O(n^{-1}p)$. Consequently, $\mathrm{E}(Q_S')=p\mathrm{tr}(\boldsymbol{\Lambda}_p^2)-1+o(n^{-1})$. Using the same procedure as $\mathrm{E}\{(\boldsymbol{V}_i^\top\boldsymbol{V}_j)^2\}$, we can show that

$$\begin{aligned}&\mathrm{E}(\boldsymbol{V}_i^\top\boldsymbol{V}_j)^4=\{3\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)+6\mathrm{tr}(\boldsymbol{\Lambda}_p^4)\}/\{p(p+2)(p+4)(p+6)\}[1+O\{p^{-2}\mathrm{tr}(\mathbf{D}_{n,p}^2)\}],\\ &\mathrm{E}\left\{(\boldsymbol{V}_i^\top\boldsymbol{V}_j)^2(\boldsymbol{V}_i^\top\boldsymbol{V}_k)^2\right\}=\{\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)+2\mathrm{tr}(\boldsymbol{\Lambda}_p^4)\}/\{p^3(p+2)\}[1+O\{p^{-2}\mathrm{tr}(\mathbf{D}_{n,p}^2)\}].\end{aligned}$$

Consequently,

$$\mathrm{var}\left\{\frac{1}{n(n-1)}\sum_{i\neq j}(\boldsymbol{V}_i^\top\boldsymbol{V}_j)^2\right\}=\left[\frac{4\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)}{n(n-1)p^4}+\frac{8\{p\mathrm{tr}(\boldsymbol{\Lambda}_p^4)-\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)\}}{(n-1)p^4}\right]\{1+o(1)\}.$$

As a result,

$$\begin{aligned}&\mathrm{E}(\tilde{Q}_S)=\mathrm{tr}(\mathbf{D}_{n,p}^2)/p+o(n^{-1}),\\ &\mathrm{var}(\tilde{Q}_S)=\left[\frac{4\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)}{n(n-1)p^2}+\frac{8\{\mathrm{tr}(\boldsymbol{\Lambda}_p^4)-p^{-1}\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)\}}{(n-1)p^2}\right]\{1+o(1)\}.\end{aligned}$$

Accordingly, it suffices to show that

$$T_n=\{n(n-1)\}^{-1}\sum_{i\neq j}4p(\boldsymbol{V}_i^\top\boldsymbol{V}_j)^2$$

is asymptotically normal. Obviously,

$$\mathrm{var}^2(T_n)\ge K\max\left\{\frac{\{\mathrm{tr}(\boldsymbol{\Lambda}_p^4)-p^{-1}\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)\}\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)}{n(n-1)^2p^4},\frac{\mathrm{tr}^4(\boldsymbol{\Lambda}_p^2)}{\{n(n-1)\}^2p^4}\right\}$$

for sufficiently large n, where K is some constant.

Then we can also use the Martingale Central Limit Theorem [4] to prove asymptotic normality. For this purpose, let $\mathcal{F}_0=\{\varnothing,\boldsymbol{\Omega}\}$ and for each $k\in\{1,\ldots,n\}$, let $\mathcal{F}_k=\sigma\{\boldsymbol{V}_1,\ldots,\boldsymbol{V}_k\}$. Let $\mathrm{E}_k(\cdot)$ denote the conditional expectation of given $\mathcal{F}_k$ and $\mathrm{E}_0(\cdot)=\mathrm{E}(\cdot)$. We can write $T_n-\mathrm{E}(T_n)=\sum_{k=1}^n G_{n,k}$, where $G_{n,k}=(\mathrm{E}_k-\mathrm{E}_{k-1})T_n$. Then for every n, $\{G_{n,k}\}_{k=1}^n$ is a martingale difference sequence with respect to the σ-fields $\{\mathcal{F}_k:1\le k\le n\}$. Let $\sigma_{n,k}^2=\mathrm{E}_{k-1}(G_{n,k}^2)$. It suffices to show that, as $n\to\infty$,

$$\frac{\sum_{k=1}^n\sigma_{n,k}^2}{\mathrm{var}(T_n)}\to 1\quad\text{in probability and}\quad\frac{\sum_{k=1}^n E(G_{n,k}^4)}{\mathrm{var}^2(T_n)}\to 0. \tag{4}$$

As $\mathrm{E}(\sum_{k=1}^n\sigma_{n,k}^2)=\mathrm{var}(T_n)$, to see the first part of (4), we only need to show $\mathrm{var}(\sum_{k=1}^n\sigma_{n,k}^2)=o\{\mathrm{var}^2(T_n)\}$. Define $2\mathrm{E}(\boldsymbol{V}_i\boldsymbol{V}_i^\top)=\boldsymbol{\Gamma}_p$ and $\boldsymbol{\Gamma}_{k-1}=\sum_{i=1}^{k-1}\left(2\boldsymbol{V}_i\boldsymbol{V}_i^\top-\boldsymbol{\Gamma}_p\right)$. By the same procedure as $\mathrm{E}\{(\boldsymbol{V}_i^\top\boldsymbol{V}_j)^2\}$,

$$\begin{aligned}\sigma_{n,k}^2&=\mathrm{E}_{k-1}(G_{n,k}^2)\\ &=\left[\frac{8p^2}{\{n(n-1)\}^2}\frac{\left\{\mathrm{tr}(\boldsymbol{\Gamma}_{k-1}\boldsymbol{\Lambda}_p)^2\mathrm{tr}^2(\boldsymbol{\Lambda}_p)-\mathrm{tr}^2(\boldsymbol{\Gamma}_{k-1}\boldsymbol{\Lambda}_p)\mathrm{tr}(\boldsymbol{\Lambda}_p^2)\right\}}{\mathrm{tr}^4(\boldsymbol{\Lambda}_p)}\right.\\ &\quad+\frac{16p^2}{n^2(n-1)}\frac{\left\{\mathrm{tr}(\boldsymbol{\Gamma}_{k-1}\boldsymbol{\Lambda}_p^3)\mathrm{tr}^2(\boldsymbol{\Lambda}_p)-\mathrm{tr}(\boldsymbol{\Gamma}_{k-1}\boldsymbol{\Lambda}_p)\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)\right\}}{\mathrm{tr}^5(\boldsymbol{\Lambda}_p)}\\ &\quad\left.+\frac{8p^2}{n^2}\frac{\{\mathrm{tr}(\boldsymbol{\Lambda}_p^4)-p^{-1}\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)\}}{\mathrm{tr}^4(\boldsymbol{\Lambda}_p)}\right]\left[1+o\{p^{-2}\mathrm{tr}(\mathbf{D}_{n,p}^2)\}\right].\end{aligned}$$

Then

$$\sum_{k=1}^n\sigma_{n,k}^2=(R_{1,n}+R_{2,n}+R_{3,n}+R_{4,n}+R_{5,n}+C)\{1+o(1)\},$$

where C is a constant, and

$$R_{1,n} = \frac{32p^2}{\{n(n-1)\}^2} \frac{\mathrm{tr}^2(\mathbf{\Lambda}_p^2) \sum_{k=1}^{n} (k-1) \left(\sum_{i=1}^{k-1} \boldsymbol{V}_i^\top \mathbf{\Lambda}_p \boldsymbol{V}_i \right)}{\mathrm{tr}^5(\mathbf{\Lambda}_p)},$$

$$R_{2,n} = -\frac{32p^2}{\{n(n-1)\}^2} \frac{\sum_{k=1}^{n} (k-1) \left(\sum_{i=1}^{k-1} \boldsymbol{V}_i^\top \mathbf{\Lambda}_p^3 \boldsymbol{V}_i \right)}{\mathrm{tr}^3(\mathbf{\Lambda}_p)},$$

$$R_{3,n} = \frac{32p^2}{n^2(n-1)} \frac{\left(\sum_{k=1}^{n} \sum_{i=1}^{k-1} \boldsymbol{V}_i^\top \mathbf{\Lambda}_p^3 \boldsymbol{V}_i \right)}{\mathrm{tr}^3(\mathbf{\Lambda}_p)},$$

$$R_{4,n} = -\frac{32p^2}{n^2(n-1)} \frac{\mathrm{tr}^2(\mathbf{\Lambda}_p^2) \left(\sum_{k=1}^{n} \sum_{i=1}^{k-1} \boldsymbol{V}_i^\top \mathbf{\Lambda}_p \boldsymbol{V}_i \right)}{\mathrm{tr}^5(\mathbf{\Lambda}_p)},$$

$$R_{5,n} = \frac{32p^2}{\{n(n-1)\}^2} \frac{\sum_{k=1}^{n} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (\boldsymbol{V}_i^\top \mathbf{\Lambda}_p \boldsymbol{V}_j)^2}{\mathrm{tr}^2(\mathbf{\Lambda}_p)}.$$

It suffices to show $\mathrm{var}(R_{i,n}) = o\{\mathrm{var}^2(T_n)\}$ for all $i \in \{1, \ldots, 6\}$. Using

$$\begin{aligned} \mathrm{var}\left\{ \sum_{k=1}^{n} (k-1) \left(\sum_{i=1}^{k-1} \boldsymbol{V}_i^\top \mathbf{\Lambda}_p \boldsymbol{V}_i \right) \right\} &= \left\{ \sum_{i=1}^{n} \frac{(n-i)^2(n+i-1)^2}{4} \right\} \left[\mathrm{E}(\boldsymbol{V}_i^\top \mathbf{\Lambda}_p \boldsymbol{V}_i)^2 - \left\{ \mathrm{E}(\boldsymbol{V}_i^\top \mathbf{\Lambda}_p \boldsymbol{V}_i) \right\}^2 \right] \\ &= \left\{ \sum_{i=1}^{n} \frac{(n-i)^2(n+i-1)^2}{2} \right\} \frac{\left\{ \mathrm{tr}(\mathbf{\Lambda}_p^4) - p^{-1}\mathrm{tr}^2(\mathbf{\Lambda}_p^2) \right\}}{4p^2} \{1 + o(1)\}, \end{aligned}$$

we have

$$\frac{\mathrm{var}(R_{1,n})}{\mathrm{var}^2(T_n)} \le K \frac{\mathrm{tr}^2(\mathbf{\Lambda}_p^2)}{\mathrm{tr}^4(\mathbf{\Lambda}_p)} \to 0.$$

By carrying out similar procedures we can show that $\mathrm{var}(R_{i,n}) = o\{\mathrm{var}^2(T_n)\}$ for all $i \in \{1, \ldots, 6\}$, and hence complete the proof for the first part of (4) is complete.

To show the second part of (4), start from the fact that

$$\sum_{k=1}^{n} \mathrm{E}(G_{n,k}^4) \le \frac{128p^4}{n^3} \mathrm{E}\left\{ 2\boldsymbol{V}_k^\top \mathbf{\Gamma}_p \boldsymbol{V}_k - \mathrm{tr}(\mathbf{\Gamma}_p^2) \right\}^4 + \frac{128p^4}{\{n(n-1)\}^4} \sum_{k=1}^{n} \mathrm{E}\left\{ 2\boldsymbol{V}_k^\top \mathbf{\Gamma}_{k-1} \boldsymbol{V}_k - \mathrm{tr}(\mathbf{\Gamma}_{k-1}\mathbf{\Gamma}_p) \right\}^4.$$

By some algebra, we get

$$\mathrm{E}\left\{ 2\boldsymbol{V}_k^\top \mathbf{\Gamma}_p \boldsymbol{V}_k - \mathrm{tr}(\mathbf{\Gamma}_p^2) \right\}^4 \le K \frac{\mathrm{tr}(\mathbf{\Lambda}_p^4) \left\{ \mathrm{tr}(\mathbf{\Lambda}_p^4) - p^{-1}\mathrm{tr}^2(\mathbf{\Lambda}_p^2) \right\}}{\mathrm{tr}^8(\mathbf{\Lambda}_p)},$$

which leads to

$$\frac{1}{\mathrm{var}^2(T_n)} \frac{128p^4}{n^3} \mathrm{E}\{2\boldsymbol{V}_k^\top \mathbf{\Gamma}_p \boldsymbol{V}_k - \mathrm{tr}(\mathbf{\Gamma}_p^2)\}^4 \le K \frac{\mathrm{tr}(\mathbf{\Lambda}_p^4)}{\mathrm{tr}^2(\mathbf{\Lambda}_p^2)}.$$

From the Cauchy–Schwarz inequality, $\mathrm{tr}(\mathbf{D}_{n,p}^4) \le \mathrm{tr}^2(\mathbf{D}_{n,p}^2)$ and $\mathrm{tr}^2(\mathbf{D}_{n,p}^3) \le \mathrm{tr}(\mathbf{D}_{n,p}^4)\mathrm{tr}(\mathbf{D}_{n,p}^2)$, so $\mathrm{tr}(\mathbf{\Lambda}_p^4) = o(p^2) = o\{\mathrm{tr}^2(\mathbf{\Lambda}_p^2)\}$ by the condition $\mathrm{tr}(\mathbf{D}_{n,p}^2) = O(n^{-1}p)$. Thus,

$$\frac{128p^4}{n^3} \mathrm{E}\left\{ 2\boldsymbol{V}_k^\top \mathbf{\Gamma}_p \boldsymbol{V}_k - \mathrm{tr}(\mathbf{\Gamma}_p^2) \right\}^4 = o\{\mathrm{var}^2(T_n)\}.$$

Similarly, we can get

$$\frac{128p^4}{\{n(n-1)\}^4} \sum_{k=1}^{n} \mathrm{E}\left\{ 2\boldsymbol{V}_k^\top \mathbf{\Gamma}_{k-1} \boldsymbol{V}_k - \mathrm{tr}(\mathbf{\Gamma}_{k-1}\mathbf{\Gamma}_p) \right\}^4 = o\{\mathrm{var}^2(T_n)\}.$$

This completes the proof for the second part of (4) and the proof of Theorem 2. □

Proof of Theorem 3. Under $\mathscr{H}_0$, similar to $\tilde{Q}_S$, we can decompose $\tilde{Q}_K$ as follows,

$$\begin{aligned}
\tilde{Q}_K &= \frac{p}{n(n-1)(n-2)(n-3)} \sum\sum\sum\sum_{i,j,k,\ell \text{ are not equal}} (\boldsymbol{U}_{ij}^\top \boldsymbol{U}_{k\ell})^2 - 1 \\
&= \frac{p}{n(n-1)(n-2)(n-3)} \sum\sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \{(\boldsymbol{u}_i - \boldsymbol{u}_j + \boldsymbol{\omega}_{ij})^\top (\boldsymbol{u}_k - \boldsymbol{u}_\ell + \boldsymbol{\omega}_{k\ell})\}^2 - 1 \\
&= \frac{4p}{n(n-1)} \sum\sum_{i \neq j} (\boldsymbol{u}_i^\top \boldsymbol{u}_j)^2 - 4\tau_F^2 - \frac{2p}{n(n-1)(n-2)} \sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \boldsymbol{u}_i^\top \boldsymbol{u}_j \boldsymbol{u}_j^\top \boldsymbol{u}_k \\
&\quad + \frac{p}{n(n-1)(n-2)(n-3)} \sum\sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \boldsymbol{u}_i^\top \boldsymbol{u}_j \boldsymbol{u}_k^\top \boldsymbol{u}_\ell + O(pn^{-3}) \sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \boldsymbol{u}_i^\top \boldsymbol{u}_j \boldsymbol{u}_i^\top \boldsymbol{\omega}_{jk} \\
&\quad + O(pn^{-4}) \sum\sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \boldsymbol{u}_i^\top \boldsymbol{u}_k \boldsymbol{u}_j^\top \boldsymbol{\omega}_{k\ell} + O(pn^{-3}) \sum\sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \boldsymbol{u}_i^\top \boldsymbol{u}_k \boldsymbol{\omega}_{ij}^\top \boldsymbol{\omega}_{k\ell} \\
&\quad + O(pn^{-3}) \sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \{(\boldsymbol{u}_i^\top \boldsymbol{\omega}_{jk})^2 - p^{-1}(1 - 2\tau_F)\} \\
&\quad + O(pn^{-4}) \sum\sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \{(\boldsymbol{\omega}_{ij}^\top \boldsymbol{\omega}_{k\ell})^2 - (1 - 2\tau_F)^2\}.
\end{aligned}$$

According to the proof of Theorem 1, we only need to show the last two parts are $o_p(\sigma_0^2)$. On one hand,

$$\begin{aligned}
&\mathrm{E}\Big[O(pn^{-3}) \sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \left\{(\boldsymbol{u}_i^\top \boldsymbol{\omega}_{jk})^2 - p^{-1}(1 - 2\tau_F)\right\}\Big]^2 \\
&= O(p^2 n^{-3}) \mathrm{E}[\{(\boldsymbol{u}_i^\top \boldsymbol{\omega}_{jk})^2 - p^{-1}(1 - 2\tau_F)\}^2] \\
&\quad + O(p^2 n^{-2}) \mathrm{E}[\{(\boldsymbol{u}_i^\top \boldsymbol{\omega}_{jk})^2 - p^{-1}(1 - 2\tau_F)\}\{(\boldsymbol{u}_\ell^\top \boldsymbol{\omega}_{jk})^2 - p^{-1}(1 - 2\tau_F)\}] \\
&= O(p^2 n^{-3})[\mathrm{E}\{(\boldsymbol{u}_i^\top \boldsymbol{\omega}_{jk})^4\} - p^{-2}(1 - 2\tau_F)^2] O(p^2 n^{-2})[\mathrm{E}\{(\boldsymbol{u}_i^\top \boldsymbol{\omega}_{jk})^2 (\boldsymbol{u}_\ell^\top \boldsymbol{\omega}_{jk})^2\} - p^{-2}(1 - 2\tau_F)^2] \\
&= o(n^{-3}) + o(n^{-2}) = o(\sigma_0^2).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathrm{E}\left[O(pn^{-4}) \sum\sum\sum\sum_{i,j,k,\ell \text{ are not equal}} \left\{(\boldsymbol{\omega}_{ij}^\top \boldsymbol{\omega}_{k\ell})^2 - (1 - 2\tau_F)^2\right\}\right]^2 \\
&= O(p^2 n^{-4}) \mathrm{E}\{(\boldsymbol{\omega}_{ij}^\top \boldsymbol{\omega}_{k\ell})^4 - (1 - 2\tau_F)^2\} + O(p^2 n^{-2}) \mathrm{E}\{(\boldsymbol{\omega}_{ij}^\top \boldsymbol{\omega}_{k\ell})^2 (\boldsymbol{\omega}_{is}^\top \boldsymbol{\omega}_{kt})^2 - (1 - 2\tau_F)^2\} \\
&= o(n^{-2}) = o(\sigma_0^2).
\end{aligned}$$

Thus, part (i) is proved. Similarly, we can also prove result (ii) under $\mathscr{H}_1$. This completes the proof of Theorem 3. □

Appendix C. Proof of the corollaries

Proof of Corollary 1. From Theorems 1–2,

$$\liminf_n \Pr\left(\frac{\tilde{Q}_S - p\delta_{n,p}}{\sigma_0} > z_\alpha\right) \geq 1 - \limsup_n \Phi\left\{\frac{\sigma_0 z_\alpha - p^{-1}\mathrm{tr}(\mathbf{D}_{n,p}^2)}{\sigma_1}\right\}.$$

Obviously, $\sigma_0/\sigma_1 = O(1)$ due to $\mathrm{tr}(\boldsymbol{\Lambda}_p^4) - p^{-1}\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2) \geq 0$. Denote

$$\begin{aligned}
\gamma_{1n} &= \frac{8\left\{\mathrm{tr}(\boldsymbol{\Lambda}_p^4) - p^{-1}\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)\right\}}{p^2}, \\
\gamma_{2n} &= \frac{8\left\{\mathrm{tr}(\boldsymbol{\Lambda}_p^4)\mathrm{tr}^2(\boldsymbol{\Lambda}_p) + \mathrm{tr}^3(\boldsymbol{\Lambda}_p^2) - 2\mathrm{tr}(\boldsymbol{\Lambda}_p)\mathrm{tr}(\boldsymbol{\Lambda}_p^2)\mathrm{tr}(\boldsymbol{\Lambda}_p^3)\right\}}{\mathrm{tr}^2(\boldsymbol{\Lambda}_p^2)p^2}.
\end{aligned}$$

First consider the case $p/\mathrm{tr}(\mathbf{D}_{n,p}^2) = o(1)$. The condition $n\mathrm{tr}(\mathbf{D}_{n,p}^2)/p \to \infty$ leads to

$$\frac{\sigma_1^2}{p^{-2}\mathrm{tr}^2(\mathbf{D}_{n,p}^2)} = O\left\{\frac{p^2}{n^2\mathrm{tr}^2(\mathbf{D}_{n,p}^2)}\right\} + O\left\{\frac{\mathrm{tr}(\mathbf{\Lambda}_p^4)}{n\mathrm{tr}^2(\mathbf{D}_{n,p}^2)}\right\}$$
$$= O\left\{\frac{\mathrm{tr}^2(\mathbf{D}_{n,p}^2)}{n\mathrm{tr}^2(\mathbf{D}_{n,p}^2)}\right\} + o(1) \to 0,$$

which implies the assertion of Corollary 1. For the case $p/\mathrm{tr}(\mathbf{D}_{n,p}^2) = O(1)$, it can be seen that $\gamma_{2n}/\gamma_{1n} = O(1)$. From Theorem 4(i) in Chen et al. [3], we have $\gamma_{2n}/\{np^{-2}\mathrm{tr}^2(\mathbf{D}_{n,p}^2)\} \to 0$ from which the corollary follows immediately. □

Proof of Corollary 2. From Theorem 1 in Chen et al. [3],

$$\frac{C_n - \mathrm{tr}(\mathbf{D}_{n,p}^2)/p}{\sqrt{4n^{-2} + \gamma_{2n}n^{-1}}} \rightsquigarrow \mathcal{N}(0, 1),$$

where C_n is the test statistic proposed by Chen et al. [3]. Thus, the asymptotic power function of C_n is

$$\beta_{C_n} = \Phi\left\{-\frac{2n^{-1}}{\sqrt{4n^{-2} + \gamma_{2n}n^{-1}}}z_\alpha + \frac{\mathrm{tr}(\mathbf{D}_{n,p}^2)/p}{\sqrt{4n^{-2} + \gamma_{2n}n^{-1}}}\right\}.$$

According to Theorems 1 and 2, the asymptotic power function of $\tilde{Q}_S$ is

$$\beta_{\tilde{Q}_S} = \Phi\left\{-\frac{\sigma_0}{\sigma_1}z_\alpha + \frac{\mathrm{tr}(\mathbf{D}_{n,p}^2)/p}{\sigma_1}\right\}.$$

Obviously, $\sigma_0 = 2n^{-1}\{1 + o(1)\}$ as $p \to \infty$. Thus the asymptotic relative efficiency of $\tilde{Q}_S$ with respect to C_n is 1 in this case. □

Proof of Corollary 3. From the proof of Theorem 3(ii), $\tilde{Q}_K = \tilde{Q}_S + o_p(\sigma_1)$. Thus, from Corollaries 1 and 2, we can easily obtain the results. □

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