

On the behavior of the DFA and DCCA in trend-stationary processes

Taiane Schaedler Prass*, Guilherme Pumi

Programa de Pós-Graduação em Estatística - Instituto de Matemática e Estatística - Universidade Federal do Rio Grande do Sul - Av. Bento Gonçalves, 9500, Porto Alegre - RS, Brazil

ARTICLE INFO

Article history:

Received 21 October 2019

Received in revised form 5 November 2020

Accepted 6 November 2020

Available online 17 November 2020

AMS 2010 subject classifications:

primary 62H20

62M10

secondary 62H12

62F12

Keywords:

Cross-correlation

DCCA

Trend-stationary time series

ABSTRACT

In this work, we develop the asymptotic theory of the Detrended Fluctuation Analysis (DFA) and Detrended Cross-Correlation Analysis (DCCA) for trend-stationary stochastic processes without any assumption on the specific form of the underlying distribution. All results are presented and derived under the general framework of potentially overlapping boxes for the polynomial fit. We prove the stationarity of the DFA and DCCA, viewed as stochastic processes, obtain closed forms for moments up to second order, including the covariance structure for DFA and DCCA and a miscellany of law of large number related results. Our results generalize and improve several results presented in the literature. To verify the behavior of our theoretical results in small samples, we present a Monte Carlo simulation study and an empirical application to econometric time series.

© 2020 Elsevier Inc. All rights reserved.

1. Introduction

Calculating basic statistics as variance, correlation, cross-correlation among others from non-stationary data is a challenging problem. In this context, commonly applied statistics such as sample autocorrelation, sample variance and sample cross-correlation lose their traditional meaning. Given the importance of such quantities, circumventing this problem becomes an essential matter.

The Detrended Fluctuation Analysis (DFA), introduced by [7], is often heuristically described as an indirect way to quantifying variation in trend-stationary time series (understood as the sum of a stationary process plus a polynomial trend). A generalization of the DFA for the context where the interest lies in the joint behavior of a pair of time series is the Detrended Cross-Correlation Analysis (DCCA), introduced by [13] based on the detrended covariance of [9] and the DFA. In this sense, the DCCA is an indirect quantifier of cross-correlation.

Applications of the DFA and DCCA are abundant. A list of near 100 applications of DFA is presented in [4,6] and references therein. Applications include areas such as physics, medicine, economics, bioinformatics, meteorology, among others. In applications, the DFA is employed as a tool to detect and quantify long-range dependence in trend-stationary time series. As for the DCCA, it is usually applied as a measure of long-range cross-correlation between time series. For instance, applications of the DFA and DCCA in economics include the study of the correlation and cross-correlation structure between Brazilian stock and commodity markets [10] and between Chinese A-share and B-share markets [11]. In the modeling of traffic flow, the DFA and DCCA are applied to the study of long-range correlation and cross-correlation

* Corresponding author.

E-mail address: taiane.prass@ufrgs.br (T.S. Prass).

between different traffic fluctuations signals [12]. Recently, [14] applies the DFA and DCCA to analyze the relationship between relative air humidity and temperature. Other applications of the DCCA can be found in [8] and reference therein.

In the literature, both DFA and DCCA, are usually defined in a constructively fashion based on a sample from a given stochastic process, that is, as estimators. Interestingly, the literature is remarkably vague about their theoretical counterparts. Instead, the focus usually lies on the relationship between the DFA/DCCA and the underlying time series, especially in the context of long-range dependence non-stationary time series, which are the core of applications of these methodologies. In other words, what does the DCCA and DFA measure is still unknown. In this paper, we make an effort to solve this problem by investigating and giving meaning to the DFA and DCCA theoretical counterparts. Incidentally, the precise definition of the DFA and DCCA's theoretical counterparts will open the possibility of discussion about the properties of the DFA and DCCA as estimators, such as consistency and unbiasedness, absent in the current literature.

Large sample results for the DFA and DCCA are known under restrictions on the underlying process. For the DFA and DCCA, [1] presents large sample results in the context of fractional Gaussian noise and fractional Brownian Motion. For the DCCA, asymptotic theory is available only for long-range dependent trend-stationary time series decomposable as a sum of a polynomial trend plus a fractional Gaussian noise [2,3]. To the best of our knowledge, large sample results under the general scope of stationary processes are not available. In time series, long-range dependence is often regarded as a complicated and delicate subject, especially when compared to classical ARMA processes. So a fair question is: why the asymptotic theory for the DCCA and DFA is established under non-stationary fractional Gaussian noise assumptions? We can think of three good reasons for that. First, the DFA and DCCA were designed with long-range dependent data in mind. So it is only natural that the theory has been developed in this direction. Second, mathematically speaking, the context of fractional Gaussian noise (or fractional Brownian motion) is very convenient not only because it presents a well-developed theory, but also because when working with DFA/DCCA, it entails several simplifications that hold specifically for these processes, but not for general stationary time series. Third, adding a polynomial trend to the base stationary process allows working in the more general context of trend-stationary time series. Such a trend, however, does not affect either the DFA or the DCCA, in such a way that results valid for stationary processes will also be valid for trend-stationary time series, without any modification.

In this work, we are interested in developing the theory of DCCA for jointly trend-stationary processes. As we shall show, it is sufficient to work in the context of jointly stationary processes because deterministic polynomial trends play no role in the asymptotic theory. Some of the established literature consider non-overlapping boxes to calculate the DFA and DCCA. This approach allows for some simplifications, but there is no theoretical reason for doing so and, in practice, applying overlapping boxes can be advantageous, especially in small sample sizes. Hence, we shall consider the more general framework of potentially overlapping boxes. In this work, we shall derive several results regarding DFA and DCCA under stationarity conditions and the existence of the appropriate moments. Among these results, we highlight the stationarity of the DFA and DCCA, derivation of closed forms for the moments up to second-order (including the covariance structure for DFA and DCCA) as well as a miscellany of law of large number related results. We also present a Monte Carlo simulation study and an empirical application.

The paper is organized as follows. In the next section, we define the DFA and DCCA and introduce some notation. Section 3 is concerned with stationarity results for the DCCA and DFA and the study of the DFA and DCCA's theoretical counterparts. The covariance structure of the DCCA and DFA (as stochastic processes) and a miscellany of law of large number results are derived. In Section 4 we discuss some special cases of general interest. The proofs of all results are presented in the Appendix. This paper accompanies a supplementary material, which contains further examples, a Monte Carlo simulation study and also an empirical application of the DCCA to the study of the joint behavior of 4 stock indexes (S&P500, Nasdaq, Dow Jones and Ibovespa) and the Bitcoin cryptocurrency.

2. Detrended cross-correlation coefficient

Throughout this paper, given a sequence $\{Y_{k,t}\}_{t=1}^n$, let $\mathbf{Y}_{k,j}^{(i)}$ be defined by

$$\mathbf{Y}_{k,j}^{(i)} = (Y_{k,i}, \dots, Y_{k,j})^\top, \quad i, j \in \{1, \dots, n\}, \quad i \leq j.$$

For any $\ell \times \ell$ matrix A_ℓ , let $A_\ell^{(i)}$ be the matrix containing the elements of A_ℓ , from row i up to row ℓ . Given a block matrix A , let $[A]^{p,q}$ denote its (p, q) th block. Also, let $\mathbf{0}_n$ and $\mathbf{1}_n$ denote vectors of zeros and ones in $\mathbb{R}^n$, respectively, $\mathbf{0}_{m,n}$ and $\mathbf{1}_{m,n}$ denote $m \times n$ matrices of zeros and ones, respectively, and I_n denote the $n \times n$ identity matrix.

Let $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ be two stochastic processes and let $\{X_{1,t}\}_{t=1}^n$ and $\{X_{2,t}\}_{t=1}^n$ be two samples of size n obtained from these processes, respectively. Define the integrated signals $\{R_{k,t}\}_{t=1}^n$ by

$$R_{k,t} := \sum_{j=1}^t X_{k,j}, \quad k \in \{1, 2\}, \quad t \in \{1, \dots, n\}. \quad (1)$$

Fig. 1 presents the time series plot of a sample $\{X_t\}_{t=1}^{50}$ from a stationary time series and the corresponding integrated time series $\{R_t\}_{t=1}^{50}$.

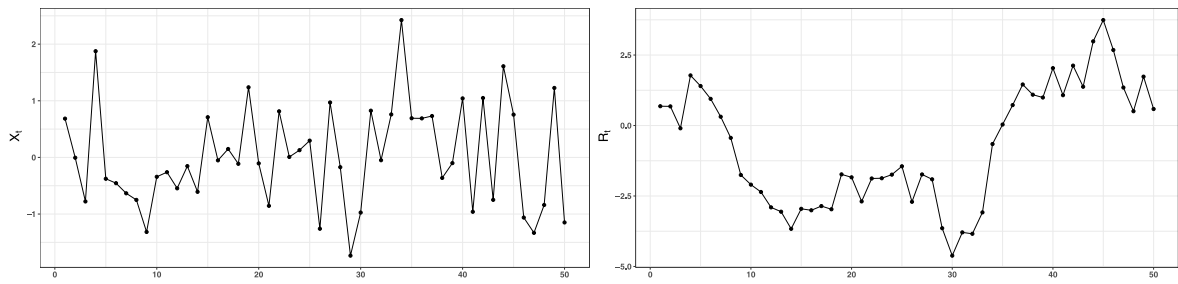


Fig. 1. A simulated sample $\{X_t\}_{t=1}^{50}$ from a stationary time series and the corresponding integrated time series $\{R_t\}_{t=1}^{50}$.

Let J_ℓ be the $\ell \times \ell$ matrix whose (r, s) th element is given by

$$[J_\ell]_{r,s} = \begin{cases} 1, & \text{if } 1 \leq s \leq r \leq \ell, \\ 0, & \text{if } 1 \leq r < s \leq \ell. \end{cases} \quad (2)$$

It follows that, for $0 < m < n$,

$$\mathbf{R}_{k,n}^{(1)} = J_n \mathbf{X}_{k,n}^{(1)}, \quad \mathbf{R}_{k,m+i}^{(i)} = J_{m+i}^{(i)} \mathbf{X}_{k,m+i}^{(1)}, \quad i \in \{1, \dots, n-m\}. \quad (3)$$

The set $\{\mathbf{R}_{k,m+i}^{(i)}\}_{i=1}^{n-m}$ is a sequence of $n-m$ overlapping boxes each containing $m+1$ values from the integrated signal, starting at i and ending at $m+i$.

Remark 2.1. Notice that, upon replacing the right-hand-side equality in (3) by

$$\mathbf{R}_{k,(m+1)i}^{((m+1)(i-1)+1)} = J_{(m+1)i}^{((m+1)(i-1)+1)} \mathbf{X}_{k,(m+1)i}^{(1)}, \quad i \in \{1, \dots, \lfloor n/(m+1) \rfloor\},$$

the corresponding set $\{\mathbf{R}_{k,(m+1)i}^{((m+1)(i-1)+1)}\}_{i=1}^{\lfloor n/(m+1) \rfloor}$ is a sequence of $\lfloor n/(m+1) \rfloor$ non-overlapping boxes each containing $m+1$ values from the integrated signal, starting at $(m+1)(i-1)+1$ and ending at $(m+1)i$. All calculations and results that follow can be obtained analogously for this case.

Now, for each $k \in \{1, 2\}$ and $i \in \{1, \dots, n-m\}$, let $\tilde{\mathbf{R}}_{k,i}$ be the vector with the ordinates $\tilde{R}_{k,t}(i)$, $i \leq t \leq m+i$, of a polynomial least-squares fit associated to the i th box $\mathbf{R}_{k,m+i}^{(i)}$ and $\mathcal{E}_{k,i}$ be the vector with the corresponding error terms $\mathcal{E}_{k,t}(i)$, $i \leq t \leq m+i$, that is,

$$\begin{aligned} \tilde{\mathbf{R}}_{k,i} &= P_{m+1} \mathbf{R}_{k,m+i}^{(i)} = (\tilde{R}_{k,i}(i), \dots, \tilde{R}_{k,m+i}(i))^\top, \\ \mathcal{E}_{k,i} &= \mathbf{R}_{k,m+i}^{(i)} - \tilde{\mathbf{R}}_{k,i} = Q_{m+1} \mathbf{R}_{k,m+i}^{(i)} = (\mathcal{E}_{k,i}(i), \dots, \mathcal{E}_{k,m+i}(i))^\top, \end{aligned} \quad (4)$$

with

$$D_{m+1}^\top := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & m+1 \\ \vdots & \vdots & \ddots & \vdots \\ 1^{v+1} & 2^{v+1} & \dots & (m+1)^{v+1} \end{pmatrix}, \quad \begin{aligned} P_{m+1} &:= D_{m+1} (D_{m+1}^\top D_{m+1})^{-1} D_{m+1}^\top, \\ Q_{m+1} &:= I_{m+1} - P_{m+1} \end{aligned} \quad (5)$$

and $v \in \mathbb{N}$. The dependence on v in the matrices D_{m+1} , P_{m+1} and Q_{m+1} and the dependence on m in the vectors $\tilde{\mathbf{R}}_{k,i}$ and $\mathcal{E}_{k,i}$ was suppressed for simplicity. The error term $\mathcal{E}_{k,i}$ is often called the “detrended walk”. Notice that, since we are considering overlapping boxes, the index i in the notation $\tilde{R}_{k,t}(i)$ and $\mathcal{E}_{k,t}(i)$ is necessary to indicate the boxes to which the values are associated. Fig. 2 illustrates the local polynomial fit and, incidentally, the use of the notation, in the case of overlapping and non-overlapping boxes.

For $0 < m < n$ and $i \in \{1, \dots, n-m\}$, let $f_{k,DFA}^2(m, i)$ be the sample variance of the residuals $\{\mathcal{E}_{k,t}(i)\}_{t=i}^{m+i}$, for $k \in \{1, 2\}$, and $f_{DCCA}(m, i)$ be the sample covariance between the residuals $\{\mathcal{E}_{1,t}(i)\}_{t=i}^{m+i}$ and $\{\mathcal{E}_{2,t}(i)\}_{t=i}^{m+i}$, corresponding to the i th box, that is,

$$f_{k,DFA}^2(m, i) := \frac{1}{m} \sum_{t=i}^{m+i} (R_{k,t} - \tilde{R}_{k,t}(i))^2 = \frac{1}{m} \mathcal{E}_{k,i}^\top \mathcal{E}_{k,i}, \quad k \in \{1, 2\}, \quad (6)$$

and

$$f_{DCCA}(m, i) := \frac{1}{m} \sum_{t=i}^{m+i} (R_{1,t} - \tilde{R}_{1,t}(i))(R_{2,t} - \tilde{R}_{2,t}(i)) = \frac{1}{m} \mathcal{E}_{1,i}^\top \mathcal{E}_{2,i}. \quad (7)$$

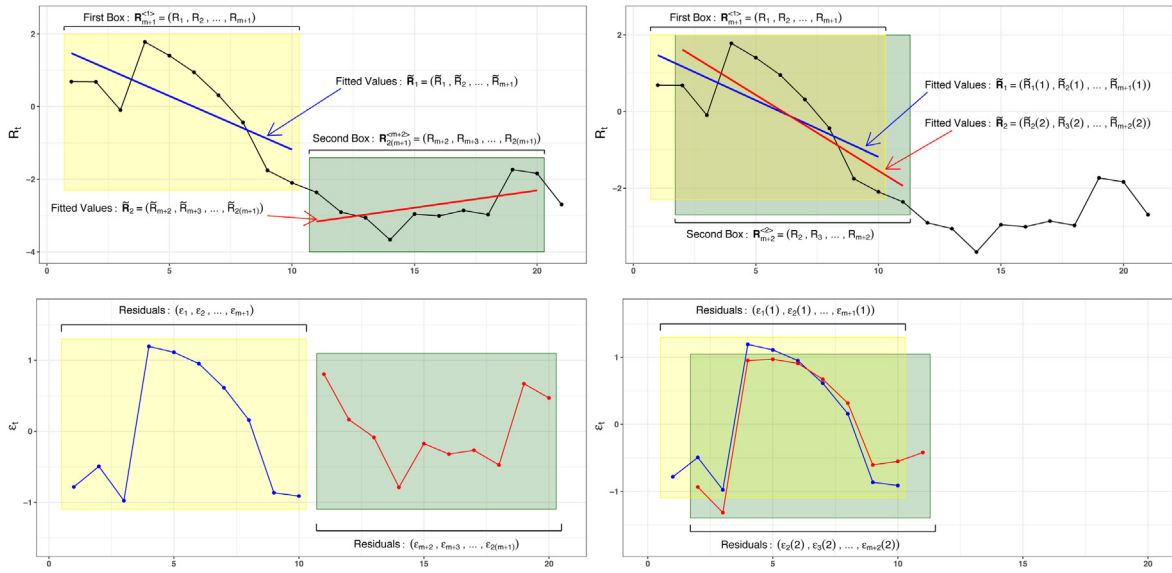


Fig. 2. Visual representation of the integrated signal obtained from a stationary process and the corresponding local linear fit (top panels) and residuals (bottom panels) in two different scenarios. The left panels correspond to non-overlapping boxes of size 10 ($m = 9$) while the right panels consider overlapping boxes of the same size.

The detrended variance $F_{k,DFA}^2$, $k = 1, 2$, the detrended covariance F_{DCCA} and the detrended correlation coefficient ρ_{DCCA} are defined, respectively, by

$$F_{k,DFA}^2(m) := \frac{1}{n-m} \sum_{i=1}^{n-m} f_{k,DFA}^2(m, i), \quad F_{DCCA}(m) := \frac{1}{n-m} \sum_{i=1}^{n-m} f_{DCCA}(m, i), \quad (8)$$

and

$$\rho_{DCCA}(m) := \frac{F_{DCCA}(m)}{\sqrt{F_{1,DFA}^2(m)} \sqrt{F_{2,DFA}^2(m)}}. \quad (9)$$

Notice that, from (6), (7), and upon invoking the Cauchy-Schwarz inequality twice, we obtain

$$\left| \sum_{i=1}^{n-m} f_{DCCA}(m, i) \right| \leq \sum_{i=1}^{n-m} f_{1,DFA}(m, i) f_{2,DFA}(m, i) \leq \sqrt{\sum_{i=1}^{n-m} f_{1,DFA}^2(m, i)} \sqrt{\sum_{i=1}^{n-m} f_{2,DFA}^2(m, i)}.$$

Hence, just as the classical Pearson correlation coefficient, the DCCA also satisfy $|\rho_{DCCA}(m)| \leq 1$.

Remark 2.2. In the literature, the expression in (7) and the detrended covariance are often denoted respectively by $f_{DCCA}^2(m, i)$ and $F_{DCCA}^2(m)$ instead of $f_{DCCA}(m, i)$ and $F_{DCCA}(m)$. Here we consider the second notation as it is more coherent with traditional notation of variance, covariance and correlation of random variables, namely, $\text{Var}(X) = \sigma_X^2$, $\text{Cov}(X, Y) = \sigma_{X,Y}$ and $\text{Corr}(X, Y) = \rho_{X,Y}$. Moreover, the notation $f_{DCCA}^2(m, i)$ and $F_{DCCA}^2(m)$ is somewhat misleading and can induce the reader to draw the wrong conclusion that the detrended covariance is always positive.

As mentioned before, the theory presented in [1] for the DFA and [2] and [3] for the DCCA is developed under a (stationary) fractional Gaussian noise (or fractional Brownian motion) plus a polynomial trend assumption, which results in a trend-stationary process. The next result shows that the DFA and DCCA are always invariant under polynomial trend, regardless the underlying distribution, so that, in a trend-stationary context, it suffices to focus on the underlying stationary process. For ease of presentation, we defer the proof of all results to the [Appendix](#).

Proposition 2.1 (Invariance to Polynomial Trend). *Let $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ be any two stochastic process and let p_1 and p_2 denote two polynomial of degree v_1 and v_2 , respectively. Let $Y_{k,t} := X_{k,t} + p_k(t)$, $k = 1, 2$, and $v = \max\{v_1, v_2\}$. Then the DFA and DCCA of $\{X_{k,t}\}_{t \in \mathbb{Z}}$ and $\{Y_{k,t}\}_{t \in \mathbb{Z}}$ are the same.*

The invariance to polynomial trend property of the DFA and DCCA allows extending the results for a stationary process to non-stationary process that can be written as a polynomial trend plus a stationary signal without loss of generality.

3. Stationarity results and the theoretical counterpart of ρ_{DCCA}

In this section, we derive several results for the DFA and DCCA and also present the theoretical counterpart of ρ_{DCCA} . We shall enunciate the results for (jointly) stationary processes only. Because of Proposition 2.1, the results in this section hold unaltered for trend-stationary processes, except stated otherwise.

We start by noticing that P_{m+1} (the projection matrix) and Q_{m+1} are bisymmetric, hermitian and idempotent matrices. Moreover,

$$E(\mathcal{E}_{k,i}) = Q_{m+1}E(\mathbf{R}_{k,m+i}^{(i)}) = 0_{m+1} \quad \text{if, and only if,} \quad E(\mathbf{R}_{k,m+i}^{(i)}) = D_{m+1}\beta_i, \quad (10)$$

for some $\beta_i \in \mathbb{R}^{m+1}$, or, in other words, if $E(\mathbf{R}_{k,m+i}^{(i)})$ is a polynomial trend of degree at most $\nu + 1$. From (1), $E(\mathbf{R}_{k,t}) = \sum_{j=1}^t E(X_{k,j})$, $k = 1, 2$ and $t \geq 1$, so that a sufficient condition for (10) to hold is that $E(X_{k,j})$ does not depend on j . Hence, if $\{X_{k,t}\}_{t \in \mathbb{Z}}$ is stationary, we have

$$Q_{m+1}J_{m+1+h}^{(1+h)}[X_{k,m+1+h}^{(1)} - E(X_{k,m+1+h}^{(1)})] = Q_{m+1}J_{m+1+h}^{(1+h)}X_{k,m+1+h}^{(1)}, \quad h \geq 0,$$

so that, without loss of generality, we can assume that $E(X_{k,t}) = 0$.

Notice that, for a fixed i , $\{\mathcal{E}_{k,t}(i)\}_{t=i}^{m+i}$ is not an identically distributed sequence. However, Lemma 3.1 provides a sufficient condition for the joint stationarity of $\{\mathcal{E}_{1,i}\}_{i=1}^{n-m}$ and $\{\mathcal{E}_{2,i}\}_{i=1}^{n-m}$.

Lemma 3.1. *If $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ are two jointly strictly stationary processes, then so are $\{\mathcal{E}_{1,i}\}_{i=1}^{n-m}$ and $\{\mathcal{E}_{2,i}\}_{i=1}^{n-m}$.*

As a consequence of Lemma 3.1 we have Corollary 3.1, which shows that, if $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ are jointly strictly stationary, then the distribution of $f_{k,DFA}^2(m, i)$ and $f_{DCCA}(m, i)$ do not depend on i . This result generalizes lemma 2.2 in Bardet and Kammoun [1], where the authors show that $\{f_{k,DFA}^2(m, i)\}_{i=1}^{\lfloor n/m \rfloor}$, obtained by considering non-overlapping boxes of size $\lfloor n/m \rfloor$, and also lemma 1.3 in [3], where the authors consider only the case where the underlying process is long-range dependent.

Corollary 3.1. *If $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ are two jointly strictly stationary processes, then both processes, $\{f_{k,DFA}^2(m, i)\}_{i=1}^{n-m}$ and $\{f_{DCCA}(m, i)\}_{i=1}^{n-m}$, are strictly stationary.*

As mentioned in the introduction, the DCCA and the DFA are usually defined in terms of a sample from a given underlying process, as in (8) and (9). Thus, they can be viewed as an estimator of some quantity. It is interesting to notice, however, that the literature does not mention the DCCA's theoretical counterpart. To fill in this gap, observe that as a consequence of Corollary 3.1 and (8),

$$E(F_{k,DFA}^2(m)) = \frac{1}{n-m} \sum_{i=1}^{n-m} E(f_{k,DFA}^2(m, i)) = E(f_{k,DFA}^2(m, 1)) = \frac{1}{m} E(\mathcal{E}_{k,1}^\top \mathcal{E}_{k,1})$$

and

$$E(F_{DCCA}(m)) = \frac{1}{n-m} \sum_{i=1}^{n-m} E(f_{DCCA}(m, i)) = E(f_{DCCA}(m, 1)) = \frac{1}{m} E(\mathcal{E}_{k,1}^\top \mathcal{E}_{k,2}).$$

Hence, the theoretical counterpart of $\rho_{DCCA}(m)$ is given by

$$\rho_{\mathcal{E}}(m) := \frac{E(F_{DCCA}(m))}{\sqrt{E(F_{1,DFA}^2(m))} \sqrt{E(F_{2,DFA}^2(m))}} = \frac{\sum_{t=i}^{m+i} \text{Cov}(\mathcal{E}_{1,t}(i), \mathcal{E}_{2,t}(i))}{\sqrt{\sum_{t=i}^{m+i} \text{Var}(\mathcal{E}_{1,t}(i))} \sqrt{\sum_{t=i}^{m+i} \text{Var}(\mathcal{E}_{2,t}(i))}}, \quad (11)$$

$0 < m < n$, $1 \leq i \leq n-m$, where $\mathcal{E}_{k,i} = (\mathcal{E}_{k,i}(i), \dots, \mathcal{E}_{k,m+i}(i))^\top$ is defined by (4), $k \in \{1, 2\}$.

A closer look into (11) suggests why the theoretical counterpart of the ρ_{DCCA} is never mentioned in the literature and also why applications only focus on its decay: there is no simple interpretation for (11). Observe that (11) can be rewritten as the average covariance divided by the square root of the average variances corresponding to the processes $\{\mathcal{E}_{1,t}(i)\}_{t=i}^{m+i}$ and $\{\mathcal{E}_{2,t}(i)\}_{t=i}^{m+i}$, which are the residuals of a local polynomial fit applied to the i th window associated to the integrated processes $\{R_{1,t}\}_{t=1}^n$ and $\{R_{2,t}\}_{t=1}^n$. Hence, it is clear that $\rho_{\mathcal{E}}(m)$ is not a direct measure of the cross-correlation between the original processes. It is also obvious that $\rho_{DCCA}(m)$ is a biased estimator for $\rho_{\mathcal{E}}(m)$, for fixed n . However, as it will be shown in the sequel, $\rho_{\mathcal{E}}(m)$ is consistent under some mild conditions.

Another consequence of Corollary 3.1 is that, if $f_{k,DFA}^2(m, i)$ and $f_{DCCA}(m, i)$ have finite variance, then

$$\gamma_{k,DFA}(h) := \text{Cov}(f_{k,DFA}^2(m, 1), f_{k,DFA}^2(m, 1+h)) = \text{Cov}(f_{k,DFA}^2(m, i), f_{k,DFA}^2(m, i+h)), \quad (12)$$

$$\gamma_{DCCA}(h) := \text{Cov}(f_{DCCA}(m, 1), f_{DCCA}(m, 1+h)) = \text{Cov}(f_{DCCA}(m, i), f_{DCCA}(m, i+h)), \quad (13)$$

for all $i \geq 1$ and $h \geq 0$ for which the last terms in (12) and (13) make sense. Closed expressions for (12) and (13) shall be derived in the sequel, but first we need to introduce some notation.

For any $k_1, k_2 = 1, 2$, $0 < m < n$ and $0 \leq h_1, h_2 < n - m$, let

$$\Gamma_{k_1, k_2}^{h_1, h_2} := \text{Cov}\left(\mathbf{X}_{k_1, m+1+h_1}^{(1)}, \mathbf{X}_{k_2, m+1+h_2}^{(1)}\right), \quad \Sigma_{k_1, k_2}^{h_1, h_2} := \text{Cov}\left(\mathbf{R}_{k_1, m+1+h_1}^{(h_1+1)}, \mathbf{R}_{k_2, m+1+h_2}^{(h_2+1)}\right),$$

and observe that, from (3),

$$\Sigma_{k_1, k_2}^{h_1, h_2} = J_{m+1+h_1}^{(h_1)} \Gamma_{k_1, k_2}^{h_1, h_2} \left[J_{m+1+h_2}^{(h_2)} \right]^\top \quad (14)$$

where J_{m+1+h} is defined by (2), for $h \geq 0$. Let $\mathcal{K}_{k_1, k_2}(p, r, q, s)$ denote the joint cumulant of $X_{k_1, p}, X_{k_1, r}, X_{k_2, q}, X_{k_2, s}$ and let $\mathcal{K}_{k_1, k_2}(h)$ be the $[(m+1)(m+1+h)] \times [(m+1)(m+1+h)]$ block matrix, for which the (r, s) th element in the (p, q) th block is given by

$$\left[\mathcal{K}_{k_1, k_2}(h) \right]_{r, s}^{p, q} := \mathcal{K}_{k_1, k_2}(p, r, q, s), \quad 1 \leq p, q \leq m+1, \quad 1 \leq r, s \leq m+1+h. \quad (15)$$

For sake of simplicity, for any $h, h_1, h_2 \geq 0$ and $k, k_1, k_2 \in \{1, 2\}$, define

$$\Gamma_k^{h_1, h_2} := \Gamma_{k, k}^{h_1, h_2}, \quad \Gamma_k := \Gamma_{k, k}^{0, 0}, \quad \Sigma_k^{h_1, h_2} := \Sigma_{k, k}^{h_1, h_2}, \quad \Sigma_k := \Sigma_{k, k}^{0, 0}, \quad (16)$$

$$\Gamma_{1,2} := \Gamma_{1,2}^{0,0}, \quad \Sigma_{1,2} := \Sigma_{1,2}^{0,0}, \quad \mathcal{K}_k(h) := \mathcal{K}_{k,k}(h), \quad \mathcal{K}_k := \mathcal{K}_k(0) \quad \text{and} \quad \mathcal{K}_{k_1, k_2} := \mathcal{K}_{k_1, k_2}(0). \quad (17)$$

Moreover, let $K_{m+1} = K_{m+1}(0) := J_{m+1}^\top Q_{m+1} J_{m+1}$ and observe that, for all $h > 0$,

$$J_{m+1+h}^{(h+1)} = [1_{m+1, h} J_{m+1}] \implies K_{m+1}(h) := \left[J_{m+1+h}^{(h+1)} \right]^\top Q_{m+1} J_{m+1+h}^{(h+1)} = \begin{bmatrix} 0_{h, h} & 0_{h, m+1} \\ 0_{m+1, h} & K_{m+1} \end{bmatrix}. \quad (18)$$

Also, let $K_{m+1}^\otimes = K_{m+1}^\otimes(0) := K_{m+1} \otimes K_{m+1}$ and, for $h > 0$,

$$K_{m+1}^\otimes(h) := \left[J_{m+1} \otimes J_{m+1+h}^{(h+1)} \right]^\top (Q_{m+1} \otimes Q_{m+1}) \left[J_{m+1} \otimes J_{m+1+h}^{(h+1)} \right] = K_{m+1} \otimes K_{m+1}(h). \quad (19)$$

Theorem 3.1 presents closed form expressions for the expectation, variance and covariance function related to the processes $\{f_{k, \text{DFA}}^2(m, i)\}_{i=1}^{n-m}$ and $\{f_{\text{DCCA}}(m, i)\}_{i=1}^{n-m}$, under joint stationarity and finite fourth moment assumptions for $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$. This result is a generalization of the results presented in Bardet and Kammoun [1] for the DFA and in [3] for the DCCA, where the authors consider non-overlapping windows and a fractional Gaussian noise (plus a polynomial trend) as the underlying process. Moreover, while the expressions derived in [3] are presented in terms of the covariance matrices related to the integrated process $\{(R_{1,t}, R_{2,t})\}_{t=1}^n$, the results given in Theorem 3.1 are written in terms of the covariance matrices related to the original process $\{(X_{1,t}, X_{2,t})\}_{t \in \mathbb{Z}}$, which is often more useful.

Theorem 3.1. Let $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ be two jointly strictly stationary stochastic processes with $E(|X_{k,t}|^4) < \infty$, $k = 1, 2$. Then, for all $0 < m < n$, $1 \leq i \leq n - m$, $0 \leq h < n - m$ and $k \in \{1, 2\}$,

$$E(f_{k, \text{DFA}}^2(m, i)) = \frac{1}{m} \text{tr}(K_{m+1} \Gamma_k), \quad (20)$$

$$\gamma_{k, \text{DFA}}(0) = \frac{1}{m^2} \left[\text{tr}(K_{m+1}^\otimes \mathcal{K}_k) + 2 \text{tr}(K_{m+1} \Gamma_k K_{m+1} \Gamma_k) \right], \quad (21)$$

$$\gamma_{k, \text{DFA}}(h) = \frac{1}{m^2} \left[\text{tr}(K_{m+1}^\otimes(h) \mathcal{K}_k(h)) + 2 \text{tr}(K_{m+1} \Gamma_k^{0, h} K_{m+1}(h) \Gamma_k^{h, 0}) \right], \quad (22)$$

and

$$E(f_{\text{DCCA}}(m, i)) = \frac{1}{m} \text{tr}(K_{m+1} \Gamma_{1,2}), \quad (23)$$

$$\gamma_{\text{DCCA}}(0) = \frac{1}{m^2} \left[\text{tr}(K_{m+1}^\otimes \mathcal{K}_{1,2}) + \text{tr}(K_{m+1} \Gamma_1 K_{m+1} \Gamma_2) + \text{tr}(K_{m+1} \Gamma_{1,2} K_{m+1} \Gamma_{1,2}) \right], \quad (24)$$

$$\gamma_{\text{DCCA}}(h) = \frac{1}{m^2} \left[\text{tr}(K_{m+1}^\otimes \mathcal{K}_{1,2}(h)) + \text{tr}(K_{m+1} \Gamma_1^{0, h} K_{m+1}(h) \Gamma_2^{h, 0}) + \text{tr}(K_{m+1} \Gamma_{1,2}^{0, h} K_{m+1}(h) \Gamma_{1,2}^{h, 0}) \right], \quad (25)$$

with $\Gamma_k, \Gamma_k^{h_1, h_2}, \Gamma_{1,2}, \Gamma_{1,2}^{h_1, h_2}, \mathcal{K}_k, \mathcal{K}_k(h), \mathcal{K}_{1,2}, \mathcal{K}_{1,2}(h), K_{m+1}, K_{m+1}^\otimes, K_{m+1}^\otimes(h)$ defined in (15)–(19), $h_1, h_2 \in \{0, h\}$.

Remark 3.1. For non-overlapping windows, we define

$$\Sigma_{k_1, k_2}^{h_1, h_2} := \text{Cov}\left(\mathbf{R}_{k_1, (m+1)(h_1+1)}^{((m+1)h_1+1)}, \mathbf{R}_{k_2, (m+1)(h_2+1)}^{((m+1)h_2+1)}\right), \quad h_1, h_2 \geq 0,$$

so that

$$\Sigma_{k_1, k_2}^{h_1, h_2} = J_{(m+1)(h_1+1)}^{((m+1)h_1+1)} \Gamma_{k_1, k_2}^{h_1, h_2} \left[J_{(m+1)(h_2+1)}^{((m+1)h_2+1)} \right]^\top, \quad \Gamma_{k_1, k_2}^{h_1, h_2} := \text{Cov} \left(\mathbf{X}_{k_1, (m+1)(h_1+1)}^{(1)}, \mathbf{X}_{k_2, (m+1)(h_2+1)}^{(1)} \right).$$

Also, in (15) make $1 \leq r, s \leq (m+1)(h+1)$ and in (18) and (19) replace $J_{m+1+h}^{(h+1)}$ with $J_{(m+1)(h+1)}^{((m+1)h+1)}$. Theorem 3.1 remains unchanged.

As mentioned before, $\rho_{DCCA}(m)$ is a biased estimator for (11). However, using the results derived in Theorem 3.1, Theorem 3.2 provides sufficient conditions for consistence and almost sure convergence of $\rho_{DCCA}(m)$. As a direct consequence of this theorem, it is showed that $\rho_{DCCA}(m)$ is asymptotically unbiased.

Theorem 3.2. Let $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ be two jointly stationary processes. If $\gamma_{k,DFA}(h) \rightarrow 0$ and $\gamma_{DCCA}(h) \rightarrow 0$, as $h \rightarrow \infty$, then

$$F_{k,DFA}^2(m) \xrightarrow{P} E(f_{k,DFA}^2(m, 1)) = \frac{1}{m} \text{tr}(K_{m+1} \Gamma_k), \quad \text{as } n \rightarrow \infty,$$

and

$$F_{DCCA}(m) \xrightarrow{P} E(f_{DCCA}(m, 1)) = \frac{1}{m} \text{tr}(K_{m+1} \Gamma_{1,2}), \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$\rho_{DCCA}(m) \xrightarrow{P} \frac{\text{tr}(K_{m+1} \Gamma_{1,2})}{\sqrt{\text{tr}(K_{m+1} \Gamma_1) \text{tr}(K_{m+1} \Gamma_2)}} = \rho_{\mathcal{E}}(m), \quad \text{as } n \rightarrow \infty,$$

where Γ_k , $\Gamma_{1,2}$ and K_{m+1} are defined, respectively, by (16), (17) and (18). Furthermore, if

$$\sum_{h=1}^{\infty} \frac{|\gamma_{k,DFA}(h)|}{h^{q_k}} < \infty, \quad \sum_{h=1}^{\infty} \frac{|\gamma_{DCCA}(h)|}{h^{q_{12}}} < \infty,$$

for some $0 \leq q_k, q_{12} < 1$, then the convergence holds almost surely.

In order to derive the limit in probability of $F_{k,DFA}^2(m)$, $F_{DCCA}(m)$ and $\rho_{DCCA}(m)$, as $n \rightarrow \infty$, Theorem 3.2 requires that $\gamma_{k,DFA}(h) \rightarrow 0$ and $\gamma_{DCCA}(h) \rightarrow 0$, as $h \rightarrow \infty$. These conditions are very mild, but can still be difficult to verify in some specific contexts. Proposition 3.1 presents sufficient conditions for the hypothesis of Theorem 3.2 to hold. These conditions are related to the behavior of the cross-covariance and joint cumulants of the original time series, often much simpler to verify.

Proposition 3.1. Suppose that, for $k_1, k_2 \in \{1, 2\}$ and any $p, q, \tau > 0$ fixed,

$$\gamma_{k_1, k_2}(h) \rightarrow 0, \text{ as } |h| \rightarrow \infty \quad \text{and} \quad \mathcal{K}_{k_1, k_2}(p, h + \tau, p, h + q) \rightarrow 0, \quad \text{as } h \rightarrow \infty,$$

where $\gamma_{k_1, k_2}(h) := \text{Cov}(X_{k_1, t}, X_{k_2, t+h})$. Then $\gamma_{k,DFA}(h) \rightarrow 0$ and $\gamma_{DCCA}(h) \rightarrow 0$, as $h \rightarrow \infty$.

4. Specializations

The results presented in Theorem 3.2 are general ones, holding for jointly stationary processes with some mild assumptions. In this section we examine the consequences of Theorem 3.2 in some specific contexts, focusing in asymptotic results when both, the sample size n and the window size m go to infinity.

We start by studying the asymptotic behavior of $F_{k,DFA}^2(m)$ and $F_{DCCA}(m)$, as $m \rightarrow \infty$, but first we need to introduce some notation. Recall that the degree of the polynomial fit in (5) is given by $\nu + 1$. Let M_j and M_j^* , for all $0 \leq j \leq m$, be two matrices of size $(m+1) \times (m+1)$ for which the (r, s) th coefficients are given respectively by

$$[M_j]_{r,s} = \begin{cases} 1, & \text{if } |r-s|=j, \\ 0, & \text{otherwise,} \end{cases} \quad [M_j^*]_{r,s} = \begin{cases} 1, & \text{if } s-r=j, \\ 0, & \text{otherwise.} \end{cases} \quad (26)$$

Observe that the matrices Γ_k and $\Gamma_{1,2}$ defined respectively by (16) and (17), can be rewritten as

$$\Gamma_k = \sum_{h=0}^m \gamma_k(h) M_h, \quad \Gamma_{1,2} = \gamma_{1,2}(0) M_0^* + \sum_{h=1}^m \gamma_{1,2}(h) M_h^* + \sum_{h=1}^m \gamma_{1,2}(-h) [M_h^*]^\top.$$

Let

$$\alpha_0^{(m)} := \text{tr}(K_{m+1}), \quad \alpha_j^{(m)} := \text{tr}(K_{m+1} M_j^*), \quad \beta_j^{(m)} := \alpha_j^{(m)} / \alpha_0^{(m)}, \quad 1 \leq j \leq m, \quad (27)$$

for K_{m+1} defined by (18), so that, for any $\nu \geq 0$ one can write

$$\text{tr}(K_{m+1}\Gamma_k) = \alpha_0^{(m)} \left\{ \gamma_k(0) + 2 \sum_{h=1}^{m-1} \beta_h^{(m)} \gamma_k(h) \right\}, \quad (28)$$

$$\text{tr}(K_{m+1}\Gamma_{1,2}) = \alpha_0^{(m)} \left\{ \gamma_{1,2}(0) + \sum_{h=1}^{m-1} \beta_h^{(m)} [\gamma_{1,2}(-h) + \gamma_{1,2}(h)] \right\}. \quad (29)$$

These equations yield Lemma 4.1. In what follows, for two sequences of real numbers $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, we write $a_n \sim b_n$ if $a_n/b_n \rightarrow 1$, as $n \rightarrow \infty$.

Lemma 4.1. Let $\nu + 1$ be the degree of the polynomial fit in (5) and $\alpha_j^{(m)}$, $0 \leq j \leq m$, be defined by (27). Also, let K_{m+1} and M_j be the matrices defined by (18) and (26), respectively. Then

$$\text{tr}(K_{m+1}M_0) = \alpha_0^{(m)}, \quad \text{tr}(K_{m+1}M_m) = 0, \quad \text{tr}(K_{m+1}M_j) = 2\alpha_j^{(m)}, \quad 1 \leq j < m.$$

In particular, if $\nu = 0$, then $\alpha_j^{(m)} \sim \alpha_0^{(m)}$, as $m \rightarrow \infty$.

From the proof of Lemma 4.1 (presented in the Appendix), when $\nu = 0$ one concludes that

$$0 < \beta_j^{(m)} \leq 1, \quad \text{if } 0 \leq j < j_0(m), \quad \text{and} \quad -1 \leq \beta_j^{(m)} \leq 0, \quad \text{if } j_0(m) \leq j < m,$$

where $j_0(m) = [\sqrt{105m^2 + 210m + 9} - 9(m + 1)]/6$. Also, from Lemma 4.1, for all $0 \leq j < m$, $\beta_j^{(m)} \rightarrow 1$, as $m \rightarrow \infty$. In Theorem 4.1, (28), (29) and Lemma 4.1 are combined to obtain the asymptotic behavior of $E(f_{k,DFA}^2(m, 1))$ and $E(f_{DCCA}(m, 1))$, as $m \rightarrow \infty$. Further notice that, under the hypothesis of Theorem 3.2, the expressions derived also provide the asymptotic behavior of $F_{DCCA}(m)$, $F_{k,DFA}^2(m)$ and $\rho_{DCCA}(m)$:

Theorem 4.1. Let $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ be two jointly strictly stationary stochastic processes with $E(|X_{k,t}|^4) < \infty$, autocovariance $\gamma_k(\cdot)$, $k = 1, 2$, and cross-covariance $\gamma_{1,2}(\cdot)$. If $\nu = 0$, $\sum_{h \in \mathbb{Z}} |\gamma_k(h)| < \infty$ and $\sum_{h \in \mathbb{Z}} |\gamma_{1,2}(h)| < \infty$, then

$$E(f_{k,DFA}^2(m, 1)) \sim \frac{m}{15} \sum_{h \in \mathbb{Z}} \gamma_k(h), \quad E(f_{DCCA}(m, 1)) \sim \frac{m}{15} \sum_{h \in \mathbb{Z}} \gamma_{1,2}(h), \quad \text{as } m \rightarrow \infty. \quad (30)$$

From Theorem 4.1 we conclude that, under the conditions of Theorem 3.2 and $\nu = 0$,

$$\rho_{DCCA}(m) \xrightarrow{P} \frac{\sum_{h \in \mathbb{Z}} \gamma_{1,2}(h)}{\sqrt{\sum_{h \in \mathbb{Z}} \gamma_1(h)} \sqrt{\sum_{h \in \mathbb{Z}} \gamma_2(h)}}, \quad \text{as } m, n \rightarrow \infty. \quad (31)$$

Fig. 3 presents the behavior of $F_{DCCA}(m)$, $F_{k,DFA}^2(m)$ and $\rho_{DCCA}(m)$ calculated from a sample of size 50 obtained from a jointly stationary process. The plots display the results for overlapping (red) and non-overlapping (blue) windows, as well as the asymptotic relation presented in Theorem 4.1 (green).

Wold's decomposition theorem states that any zero mean nondeterministic weakly stationary process $\{X_t\}_{t \in \mathbb{Z}}$ can be uniquely decomposed as $X_t = \sum_{j=0}^\infty \psi_j u_{t-j} + d_t$, where $\sum_{j=0}^\infty \psi_j^2 < \infty$, $\{u_t\}_{t \in \mathbb{Z}}$ is a white noise process and $\{d_t\}_{t \in \mathbb{Z}}$ is a deterministic process. Inspired by this result, Corollary 4.1 provides the limit in probability for the ρ_{DCCA} in the context of linear sequences with absolutely summable coefficients, described in the sequel.

Let $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ be two jointly stationary processes satisfying

$$X_{k,t} = \sum_{j \in \mathbb{Z}} \psi_{k,j} \varepsilon_{k,t-j}, \quad t \in \mathbb{Z}, \quad \text{with} \quad \sum_{j \in \mathbb{Z}} |\psi_{k,j}| < \infty, \quad k \in \{1, 2\}. \quad (32)$$

where $\{\varepsilon_{k,t}\}_{t \in \mathbb{Z}}$, $k \in \{1, 2\}$, are white noise processes with zero mean, $\text{Var}(\varepsilon_{k,t}) = \tau_k^2$, $k \in \{1, 2\}$, and $\text{Cov}(\varepsilon_{1,r}, \varepsilon_{2,s}) = \tau_{1,2}$, if $r = s$, and zero otherwise. Structure (32) can be used to describe a wide variety of scenarios. For instance,

- (1) if $\tau_{1,2} = 0$, then $\{X_{k,t}\}_{t \in \mathbb{Z}}$, $k \in \{1, 2\}$, are stationary uncorrelated processes;
- (2) if $\psi_{k,j} = 0$, for all $j \neq 0$, and $\tau_{1,2} \neq 0$, then $\{(X_{1,t}, X_{2,t})\}_{t \in \mathbb{Z}}$ is a bivariate white noise;
- (3) if $\psi_{k,j} = 0$, for all $j < 0$ and $j > j_0$, for some $j_0 \in \mathbb{N}$, $\{X_{k,t}\}_{t \in \mathbb{Z}}$ for $k \in \{1, 2\}$ are moving average processes;
- (4) if $\psi_{k,j} = 0$, for all $j < 0$, then $\{X_{k,t}\}_{t \in \mathbb{Z}}$ for $k \in \{1, 2\}$ are causal processes.

Observe that

$$\gamma_k(h) = \tau_k^2 \sum_{j \in \mathbb{Z}} \psi_{k,j} \psi_{k,j+h}, \quad \gamma_{1,2}(h) = \tau_{1,2} \sum_{j \in \mathbb{Z}} \psi_{1,j} \psi_{2,j+h}, \quad \text{for all } h \in \mathbb{Z}, \quad (33)$$

so that $\gamma_k(h) \rightarrow 0$ and $\gamma_{1,2}(h) \rightarrow 0$, as $|h| \rightarrow \infty$. If, in particular, $\{(\varepsilon_{1,t}, \varepsilon_{2,t})\}_{t \in \mathbb{Z}}$ is such that $\varepsilon_{k_1,t} \perp \varepsilon_{k_2,s}$, $t \neq s$ and $k_1, k_2 \in \{1, 2\}$, (that is, the random variables are independent for $t \neq s$), with finite joint fourth moment, then

$$\text{Cov}(X_{k_1,p} X_{k_2,p}, X_{k_1,\tau+h} X_{k_2,q+h}) = \text{Var}(\varepsilon_{k_1,0} \varepsilon_{k_2,0}) \sum_{j \in \mathbb{Z}} \psi_{k_1,j} \psi_{k_1,j-p+\tau+h} \psi_{k_2,j} \psi_{k_2,j-p+q+h} \rightarrow 0,$$

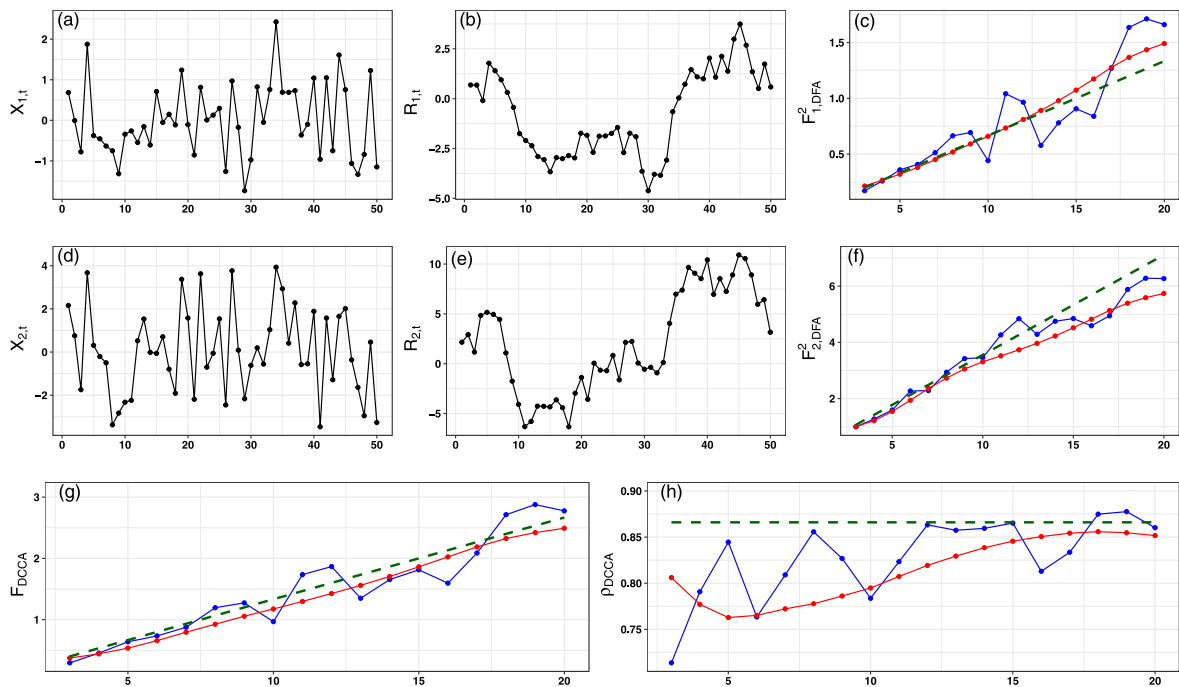


Fig. 3. Consider jointly stationary processes $(X_{1,t}, X_{2,t})$ obtained as follows: $X_{2,t} = 2X_{1,t} + \varepsilon_t$, where $X_{1,t}$ is i.i.d $\mathcal{N}(0, 1)$ and ε_t is i.i.d $\mathcal{U}(-2, 2)$. Plots of $X_{1,t}$ and $X_{2,t}$ are presented in (a) and (d); (b) and (e) present the respective associate integrated signal; (c) and (f) present the associated $F_{k,DFA}^2(m)$ as a function of m . F_{DCCA} and ρ_{DCCA} are presented in (g) and (h). In each plot, the blue line represents the respective quantities calculated using non-overlapping windows, while the same quantities calculated using overlapping windows are presented in red. The green line is the appropriate right hand side of (30) or (31).

as $h \rightarrow \infty$, for $k_1, k_2 \in \{1, 2\}$, which is equivalent to $\mathcal{K}_{k_1, k_2}(p, \tau + h, p, q + h) \rightarrow 0$, as $h \rightarrow \infty$. Moreover, if $\{(\varepsilon_{1,t}, \varepsilon_{2,t})\}_{t \in \mathbb{Z}}$ is a bivariate Gaussian process then

$$\mathcal{K}_{k_1, k_2}(p, \tau + h, p, q + h) = 0, \quad \forall p, h, \tau, q \in \mathbb{Z}.$$

Corollary 4.1. Let $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$ be two jointly stationary processes satisfying (32) and let $\nu = 0$. Then, under the conditions of Theorem 3.2,

$$\rho_{DCCA}(m) \xrightarrow{P} \text{sign}(\Psi_{1,2}) \frac{\tau_{1,2}}{\tau_1 \tau_2} = \text{sign}(\Psi_{1,2}) \rho_{1,2}^\varepsilon, \quad \text{as } n, m \rightarrow \infty,$$

where $\Psi_{1,2} = \sum_{j \in \mathbb{Z}} \psi_{1,j} \sum_{\ell \in \mathbb{Z}} \psi_{2,\ell}$ and $\rho_{1,2}^\varepsilon = \text{Corr}(\varepsilon_{1,t}, \varepsilon_{2,t})$.

4.1. Effect of ν on Theorem 3.2

In view of Proposition 2.1, the results in Lemma 4.1, Theorem 4.1, and Corollary 4.1 also hold for stationary process with linear trends, without any modification. Similar results can, in principle, be obtained for polynomial trends of higher degrees as long as one can carry on the delicate analysis in Lemma 4.1. It entails a careful investigation of the general form of Q_{m+1} in (5), which depends on obtaining $(D_{m+1}^\top D_{m+1})^{-1}$ in closed form. Even for $\nu = 1$, this task is very complicated. Also, the analysis is ν by ν dependent in the sense that no general formula can be obtained.

To visualize the effect of $\nu > 0$ on the asymptotic behavior of the $F_{k,DFA}^2(m)$, $F_{DCCA}(m)$ and $\rho_{DCCA}(m)$, we perform a Monte Carlo Simulation study considering $\nu \in \{0, 2, 5\}$ and $1,000 \leq m \leq 1,100$ with increments of size 10. The data generating process in this example is

$$X_{1,t} = 0.6X_{1,t-1} + \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} \mathcal{N}(0, 1), \quad X_{2,t} = 0.6\eta_{t-1} + \eta_t, \quad \eta_t \stackrel{iid}{\sim} \mathcal{N}(0, 1),$$

with ε_t and η_t independent of each other. We perform 1,000 replications and the sample size was set to 2,000 in each case. The results are shown in Fig. 4. The difference between the $F_{k,DFA}^2(m)$, $F_{DCCA}(m)$ and $\rho_{DCCA}(m)$ for $\nu = 2$ and $\nu = 5$ is shown in the rightmost column. From the plots, the similarity between the plots is remarkable. Just as in the case of $\nu = 0$, the $F_{k,DFA}^2(m)$, $F_{DCCA}(m)$ and $\rho_{DCCA}(m)$ behaves as a linear function of m for $\nu \in \{2, 5\}$.

Observe that the degree of the local polynomial fitted on each window of the integrated noise is, by definition, $\nu + 1$. It is a well-known fact from regression analysis that, increasing the degree of a polynomial fit tends to present diminishing

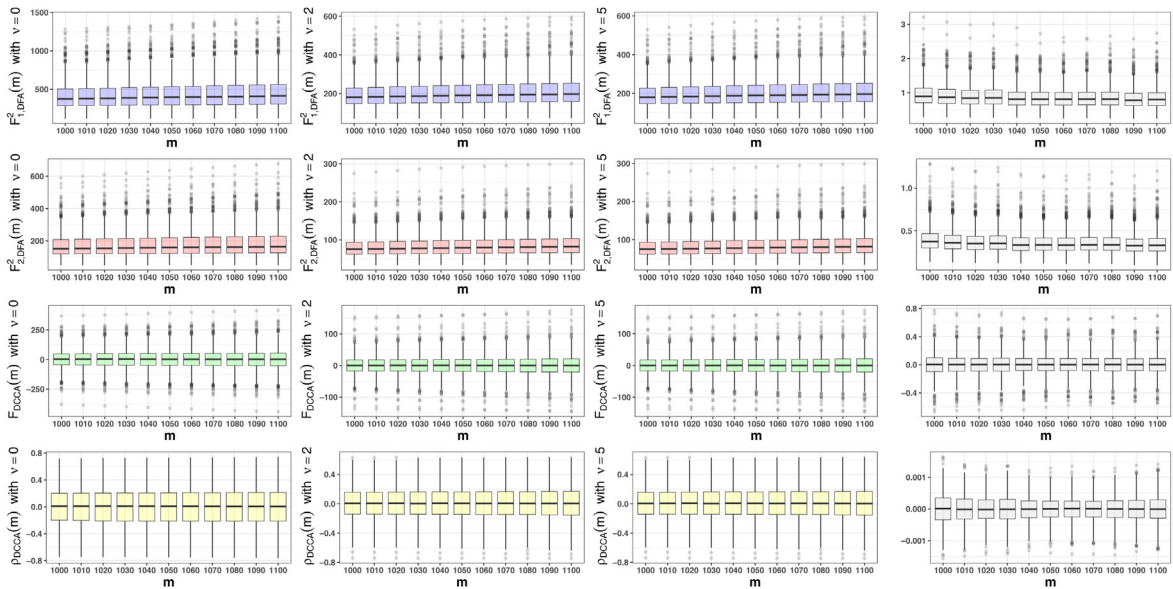


Fig. 4. Boxplots of the $F^2_{k,DFA}(m)$, $F_{DCCA}(m)$ and $\rho_{DCCA}(m)$ for $\nu \in \{0, 2, 5\}$. The rightmost panel shows the boxplots of the differences for $\nu = 2$ and $\nu = 5$ in each case.

returns on the quality of the approximation. So, it is expected that the values of $F^2_{k,DFA}(m)$, $F_{DCCA}(m)$ and $\rho_{DCCA}(m)$ tend to stabilize as ν increases.

CRedit authorship contribution statement

Taiane Schaedler Prass: Conceptualization, Methodology, Formal analysis, Software, Writing - original draft, Writing - review & editing, Visualization, Supervision, Funding acquisition. **Guilherme Pumi:** Methodology, Formal analysis, Software, Writing - original draft, Writing - review & editing, Visualization.

Acknowledgments

We thank the Editor, Associate Editor and two anonymous referees for comments and suggestions. T.S. Prass gratefully acknowledges the support of FAPERGS, Brazil (ARD 01/2017, Processo 17/2551-0000826-0).

Appendix. Proof of theorems

Proof of Proposition 2.1. Let $h_{\ell+1}(t) := \sum_{j=1}^t j^\ell$, $\ell \in \mathbb{N}$. Observe that $h_{\ell+1}(t)$ is the generalized harmonic number and hence, for each $\ell \in \mathbb{N}$, $h_\ell(t)$ is a polynomial with order $\ell + 1$ given by

$$h_{\ell+1}(t) = \frac{b_{\ell+1}(-1)^\ell}{\ell + 1} + \frac{1}{\ell + 1} B_{\ell+1}(t + 1),$$

where b_ℓ are the Bernoulli numbers and $B_\ell(\cdot)$ are the Bernoulli polynomials defined, respectively, through

$$b_\ell = \sum_{\tau=0}^{\ell} \frac{1}{\tau + 1} \sum_{v=0}^{\tau} \binom{\tau}{v} (-1)^v (v + 1)^\ell, \quad B_\ell(t) := \sum_{\tau=0}^{\ell} \binom{\ell}{\tau} b_\tau t^{\ell-\tau}.$$

Observe further that $p_k(j) = \sum_{\ell=0}^{\nu_k} a_{k,\ell} j^\ell$ so that we can write

$$R_{k,t} = \sum_{j=1}^t X_{k,j} + \sum_{j=1}^t p_k(j) = \sum_{j=1}^t X_{k,j} + \sum_{\ell=0}^{\nu_k} a_{k,\ell} h_{\ell+1}(t) = \sum_{j=1}^t X_{k,j} + p_k^*(t).$$

where $p_k^*(t)$ is a polynomial of order $\nu_k + 1$. Upon denoting $T_{k,j} = p_k(j)$ and $T_{k,j}^* = p_k^*(j)$, we have $\mathbf{Y}_{k,n}^{(1)} = \mathbf{X}_{k,n}^{(1)} + \mathbf{T}_{k,n}^{(1)}$,

$$\mathbf{R}_{k,n}^{(1)} = J_n \left[\mathbf{X}_{k,n}^{(1)} + \mathbf{T}_{k,n}^{(1)} \right] = J_n \mathbf{X}_{k,n}^{(1)} + \mathbf{T}_{k,n}^{*(1)}, \quad \mathbf{R}_{k,m+i}^{(i)} = J_{m+i}^{(i)} \mathbf{X}_{k,m+i}^{(1)} + \mathbf{T}_{k,m+i}^{*(i)},$$

$i \in \{1, \dots, n-m\}$. Now, with D_{m+1} , P_{m+1} and Q_{m+1} as in (5), let $\mathcal{E}_{k,i}^X$ and $\mathcal{E}_{k,i}^Y$ denote the detrended walk based on $\{X_{k,t}\}_{t \in \mathbb{Z}}$ and $\{Y_{k,t}\}_{t \in \mathbb{Z}}$, respectively. Since $P_{m+1} \mathbf{T}_{k,m+i}^{*(i)} = \mathbf{T}_{k,m+i}^{*(i)}$, it follows that

$$\mathcal{E}_{k,i}^Y = Q_{m+1} \mathbf{R}_{k,m+i}^{(i)} = Q_{m+1} \left[J_{m+1}^{(i)} \mathbf{X}_{k,m+i}^{(1)} + \mathbf{T}_{k,m+i}^{*(i)} \right] = Q_{m+1} J_{m+1}^{(i)} \mathbf{X}_{k,m+i}^{(1)} = \mathcal{E}_{k,i}^X,$$

and the result follows from (6) and (7). $\square$

Proof of Lemma 3.1. Assume $k \in \{1, 2\}$ and $i \in \{1, \dots, n-m\}$. Let $\mathcal{X}_{k,i}$ be the vector defined as

$$\mathcal{X}_{k,i} := \mathbf{R}_{k,m+i}^{(i)} - R_{k,i} \mathbf{1}_{m+1} = \left(0, X_{k,i+1}, \dots, \sum_{t=i+1}^{m+i} X_{k,t} \right)^\top \quad (34)$$

and $\mathcal{M} := \{1, \dots, n-m\}$. For any $1 \leq \tau \leq n-m$, $h \geq 0$ and $\mathbf{i} = (i_1, \dots, i_\tau) \in \mathcal{M}^\tau$, let

$$\mathbf{i} + h := (i_1 + h, \dots, i_\tau + h), \quad \mathbf{Q}_{k,i} := (\mathcal{X}_{k,i_1}^\top Q_{m+1}, \dots, \mathcal{X}_{k,i_\tau}^\top Q_{m+1}),$$

with Q_{m+1} be given by (5). Hence, from the joint stationarity of $\{X_{1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{2,t}\}_{t \in \mathbb{Z}}$,

$$(\mathbf{Q}_{1,i}, \mathbf{Q}_{2,j}) \stackrel{d}{=} (\mathbf{Q}_{1,i+h}, \mathbf{Q}_{2,j+h}), \quad \tau, \ell \in \mathcal{M}, \quad \mathbf{i} \in \mathcal{M}^\tau, \quad \mathbf{j} \in \mathcal{M}^\ell, \quad (35)$$

and all $h \geq 0$ for which the $(\mathbf{Q}_{1,i+h}, \mathbf{Q}_{2,j+h})$ is well defined. The result then follows by observing that, since $E(X_{k,t})$ does not depend on t , from (34), $Q_{m+1} R_{k,i} \mathbf{1}_{m+1} = 0_{m+1}$, which implies $Q_{m+1} \mathcal{X}_{k,i} = Q_{m+1} \mathbf{R}_{k,m+i}^{(i)} = \mathcal{E}_{k,i}$. $\square$

Proof of Corollary 3.1. The result follows from (6), (7) and Lemma 3.1. $\square$

Proof of Theorem 3.1. Assume $k, k_1, k_2 \in \{1, 2\}$, $0 < m < n$, $0 < i < n-m$ and $0 \leq h, h_1, h_2 < n-m$.

The stationarity assumption for $\{X_{k,t}\}_{t \in \mathbb{Z}}$ and (1) imply that $Q_{m+1} E(\mathbf{R}_{k,m+1}^{(1)}) = 0_{m+1}$ so that

$$Q_{m+1} E(\mathbf{R}_{k_1,m+1}^{(1)} [\mathbf{R}_{k_2,m+1}^{(1)}]^\top) = Q_{m+1} \text{Cov}(\mathbf{R}_{k_1,m+1}^{(1)}, \mathbf{R}_{k_2,m+1}^{(1)}) = Q_{m+1} \Sigma_{k_1,k_2}. \quad (36)$$

From Corollary 3.1, $E(\mathcal{E}_{k_1,i}^\top \mathcal{E}_{k_2,i}) = E(\mathcal{E}_{k_1,i}^\top \mathcal{E}_{k_2,i})$ so that, from (4) and the properties of trace,

$$E(\mathcal{E}_{k_1,i}^\top \mathcal{E}_{k_2,i}) = E\left(\text{tr}([\mathbf{R}_{k_1,m+1}^{(1)}]^\top Q_{m+1} \mathbf{R}_{k_2,m+1}^{(1)})\right) = \text{tr}\left(Q_{m+1} E(\mathbf{R}_{k_1,m+1}^{(1)} [\mathbf{R}_{k_2,m+1}^{(1)}]^\top)\right). \quad (37)$$

From (14), the definition of K_{m+1} and the properties of the trace, (36) and (37) imply (20), when $k_1 = k_2 = k$, and (23), when $k_1 = 1$ and $k_2 = 2$.

In order to derive $\gamma_{k,DFA}$ and γ_{DCCA} , first let

$$\Lambda_{k_1,k_2}(h) := E\left([\mathbf{X}_{k_1,m+1}^{(1)} \otimes \mathbf{X}_{k_1,m+1+h}^{(1)}][\mathbf{X}_{k_2,m+1}^{(1)} \otimes \mathbf{X}_{k_2,m+1+h}^{(1)}]^\top\right), \quad (38)$$

$$\Upsilon_{k_1,k_2}(h) := E\left([\mathbf{R}_{k_1,m+1}^{(1)} \otimes \mathbf{R}_{k_1,m+1+h}^{(h+1)}][\mathbf{R}_{k_2,m+1}^{(1)} \otimes \mathbf{R}_{k_2,m+1+h}^{(h+1)}]^\top\right), \quad (39)$$

and observe that, from (3) and the properties of the Kronecker product,

$$\mathbf{R}_{k,m+1}^{(1)} \otimes \mathbf{R}_{k,m+1+h}^{(h+1)} = [J_{m+1} \otimes J_{m+1+h}^{(h+1)}] [\mathbf{X}_{k,m+1}^{(1)} \otimes \mathbf{X}_{k,m+1+h}^{(1)}], \quad k \in \{1, 2\}$$

so that, from (38) and (39),

$$\Upsilon_{k_1,k_2}(h) = [J_{m+1} \otimes J_{m+1+h}^{(h+1)}] \Lambda_{k_1,k_2}(h) [J_{m+1} \otimes J_{m+1+h}^{(h+1)}]^\top. \quad (40)$$

Notice that $\Lambda_{k_1,k_2}(h)$ is a $[(m+1)(m+1+h)] \times [(m+1)(m+1+h)]$ block matrix, for which the (r, s) th element in the (p, q) th block is $E(X_{k_1,p} X_{k_1,r} X_{k_2,q} X_{k_2,s})$, $1 \leq p, q \leq m+1$, $1 \leq r, s \leq m+1+h$. Moreover, under the assumption $E(X_{k,t}) = 0$,

$$[\Lambda_{k_1,k_2}(h)]_{r,s}^{p,q} = \mathcal{K}_{k_1,k_2}(p, r, q, s) + \gamma_{k_1}(r-p)\gamma_{k_2}(s-q) + \gamma_{k_1,k_2}(q-p)\gamma_{k_1,k_2}(s-r) + \gamma_{k_1,k_2}(s-p)\gamma_{k_1,k_2}(q-r),$$

$1 \leq p, q \leq m+1$, $1 \leq r, s \leq m+1+h$, where γ_{k_1,k_2} is the cross-covariance function associated to $\{X_{k_1,t}\}_{t \in \mathbb{Z}}$ and $\{X_{k_2,t}\}_{t \in \mathbb{Z}}$, $\gamma_k := \gamma_{k,k}$ and $\mathcal{K}_{k_1,k_2}(p, r, q, s)$ is the joint cumulant of $X_{k_1,p}, X_{k_1,r}, X_{k_2,q}, X_{k_2,s}$. Furthermore, by letting $\Gamma_{k_1,k_2}^{h_1,h_2}$, $\Gamma_k^{h_1,h_2}$ and $\mathcal{K}_{k_1,k_2}(h)$ be the matrices defined in (14)–(17) and observing that $[\Gamma_k^{0,h}]^\top = \Gamma_k^{h,0}$, one can write

$$\Lambda_{k_1,k_2}(h) = \mathcal{K}_{k_1,k_2}(h) + \text{vec}(\Gamma_{k_1}^{h,0}) \text{vec}(\Gamma_{k_2}^{h,0})^\top + \Gamma_{k_1,k_2}^{0,0} \otimes \Gamma_{k_1,k_2}^{h,h} + \mathcal{G}_{k_1,k_2}(h),$$

where $\mathcal{G}_{k_1, k_2}(h)$ is $[(m+1)(m+1+h)] \times [(m+1)(m+1+h)]$ block matrix given by

$$\mathcal{G}_{k_1, k_2}(h) = [\mathcal{G}_1 \cdots \mathcal{G}_{m+1}], \quad \mathcal{G}_p := \Gamma_{k_1, k_2}^{0, h} \otimes G_p, \quad G_p := \begin{bmatrix} [\Gamma_{k_1, k_2}^{h_1, h_2}]_{1, p} \\ \vdots \\ [\Gamma_{k_1, k_2}^{h_1, h_2}]_{m+1+h_1, p} \end{bmatrix}, \quad 1 \leq p \leq m+1.$$

Now, from the properties of the Kronecker product and Q_{m+1} ,

$$(\mathcal{E}_{k_1, 1}^\top \mathcal{E}_{k_2, 1})(\mathcal{E}_{k_1, 1+h}^\top \mathcal{E}_{k_2, 1+h}) = (\mathcal{E}_{k_1, 1} \otimes \mathcal{E}_{k_1, 1+h})^\top (\mathcal{E}_{k_2, 1} \otimes \mathcal{E}_{k_2, 1+h}),$$

$$(\mathcal{E}_{k, 1} \otimes \mathcal{E}_{k, 1+h}) = (Q_{m+1} \otimes Q_{m+1})(\mathbf{R}_{k, m+1}^{(1)} \otimes \mathbf{R}_{k, m+1+h}^{(1+h)}),$$

$$(Q_{m+1} \otimes Q_{m+1})^\top (Q_{m+1} \otimes Q_{m+1}) = (Q_{m+1} \otimes Q_{m+1}),$$

so that, by letting $\mathcal{R}_{k, 1, 1+h} := \mathbf{R}_{k, m+1}^{(1)} \otimes \mathbf{R}_{k, m+1+h}^{(1+h)}$,

$$\begin{aligned} E([\mathcal{E}_{k_1, 1}^\top \mathcal{E}_{k_2, 1}][\mathcal{E}_{k_1, 1+h}^\top \mathcal{E}_{k_2, 1+h}]) &= E(\text{tr}(\mathcal{R}_{k_1, 1, 1+h}^\top [Q_{m+1} \otimes Q_{m+1}][Q_{m+1} \otimes Q_{m+1}] \mathcal{R}_{k_2, 1, 1+h})) \\ &= \text{tr}([Q_{m+1} \otimes Q_{m+1}] E(\mathcal{R}_{k_1, 1, 1+h} \mathcal{R}_{k_2, 1, 1+h}^\top)) \\ &= \text{tr}([Q_{m+1} \otimes Q_{m+1}] \mathcal{Y}_{k_1, k_2}(h)). \end{aligned} \quad (41)$$

Hence, the properties of trace, (37), (40) and (41), imply that

$$\text{Cov}(\mathcal{E}_{k_1, 1}^\top \mathcal{E}_{k_2, 1}, \mathcal{E}_{k_1, 1+h}^\top \mathcal{E}_{k_2, 1+h}) = \text{tr}(K_{m+1}^\otimes(h) \Lambda_{k_1, k_2}(h)) - [\text{tr}(K_{m+1} \Gamma_{k_1, k_2})]^2$$

with K_{m+1} and $K_{m+1}^\otimes(h)$ defined in (18) and (19).

Finally, observe that

$$\begin{aligned} \text{tr}([K_{m+1} \otimes K_{m+1}(h)] \text{vec}(\Gamma_{k_1}^{h, 0}) \text{vec}(\Gamma_{k_2}^{h, 0})^\top) &= \text{tr}(\text{vec}(\Gamma_{k_2}^{h, 0})^\top [K_{m+1} \otimes K_{m+1}(h)] \text{vec}(\Gamma_{k_1}^{h, 0})) \\ &= \text{tr}(\Gamma_{k_2}^{0, h} K_{m+1}(h) \Gamma_{k_1}^{h, 0} K_{m+1}) = \text{tr}(K_{m+1} \Gamma_{k_1}^{0, h} K_{m+1}(h) \Gamma_{k_2}^{h, 0}), \\ K_{m+1}(h) \Gamma_{k_1, k_2}^{h, h} &= \begin{bmatrix} 0_{h, h} & 0_{h, m+1} \\ 0_{m+1, h} & K_{m+1} \Gamma_{k_1, k_2}^{0, 0} \end{bmatrix} \implies \text{tr}(K_{m+1}(h) \Gamma_{k_1, k_2}^{h, h}) = \text{tr}(K_{m+1} \Gamma_{k_1, k_2}^{0, 0}) \end{aligned}$$

and

$$\begin{aligned} \text{tr}(K_{m+1}^\otimes(h) \mathcal{G}_{k_1, k_2}(h)) &= \sum_{p=1}^{m+1} \sum_{s=1}^{m+1+h} [K_{m+1} \Gamma_{k_1, k_2}^{0, h}]_{p, s} [K_{m+1}(h) G_p]_s = \sum_{p=1}^{m+1} \sum_{s=1}^{m+1+h} [K_{m+1} \Gamma_{k_1, k_2}^{0, h}]_{p, s} [K_{m+1}(h) \Gamma_{k_1, k_2}^{h, 0}]_{s, p} \\ &= \text{tr}(K_{m+1} \Gamma_{k_1, k_2}^{0, h} K_{m+1}(h) \Gamma_{k_1, k_2}^{h, 0}), \end{aligned}$$

so that

$$\begin{aligned} \text{tr}(K_{m+1}^\otimes(h) \Lambda_{k_1, k_2}(h)) &= \text{tr}(K_{m+1}^\otimes(h) \mathcal{K}_{k_1, k_2}(h)) + \text{tr}(K_{m+1} \Gamma_{k_1}^{0, h} K_{m+1}(h) \Gamma_{k_2}^{h, 0}) \\ &\quad + [\text{tr}(K_{m+1} \Gamma_{k_1, k_2}^{0, 0})]^2 + \text{tr}(K_{m+1} \Gamma_{k_1, k_2}^{0, h} K_{m+1}(h) \Gamma_{k_1, k_2}^{h, 0}). \end{aligned} \quad (42)$$

From (20), (42) implies (21) and (22) when $k_1 = k_2 = k$, and (24) and (25) when $k_1 = 1$ and $k_2 = 2$. $\square$

Lemma 4.2 provides sufficient conditions for the sample mean to converge in probability/almost surely to the process' mean and it is necessary to the proof of Theorem 3.2.

Lemma 4.2. Let $\{Y_t\}_{t \in \mathbb{Z}}$ be a weakly stationary process and $\gamma(h) := \text{Cov}(Y_t, Y_{t+h})$, $h \in \mathbb{Z}$. Then

$$\gamma(h) \rightarrow 0, \quad \text{as } h \rightarrow \infty \implies \frac{1}{n} \sum_{i=1}^n Y_i \xrightarrow{P} E(Y_t), \quad \text{as } n \rightarrow \infty, \quad (43)$$

$$\sum_{h=1}^{\infty} \frac{|\gamma(h)|}{h^q} < \infty, \quad \text{for some } 0 \leq q < 1, \implies \frac{1}{n} \sum_{i=1}^n Y_i \xrightarrow{a.s.} E(Y_t), \quad \text{as } n \rightarrow \infty. \quad (44)$$

Proof of Lemma 4.2. Let $\bar{Y} := \frac{1}{n} \sum_{j=1}^n Y_j$. If $\gamma(h) \rightarrow 0$, for any $\varepsilon > 0$, there exist $h_0 > 0$ and $n_0 > h_0$ such that $|\gamma(h)| < \frac{\varepsilon}{6}$, for all $h > h_0$, $\frac{1}{n} \gamma(0) < \frac{\varepsilon}{3}$, and $\frac{2}{n} \sum_{j=1}^{h_0} |\gamma(j)| < \frac{\varepsilon}{3}$, for all $n > n_0$. Thus,

$$|\text{Var}(\bar{Y})| \leq \frac{1}{n} \gamma(0) + \frac{2}{n} \sum_{h=1}^{h_0} |\gamma(h)| + \frac{2}{n} \sum_{h=h_0+1}^{n-1} |\gamma(h)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon(n-h_0-1)}{3n} \leq \varepsilon.$$

Since ε is arbitrary, $\text{Var}(\bar{Y}) \rightarrow 0$, as $n \rightarrow \infty$, and (43) follows immediately from Chebyshev's inequality. Now, suppose that

$$\sum_{h=1}^{\infty} \frac{|\gamma(h)|}{h^q} < \infty, \quad \text{for some } 0 \leq q < 1,$$

and observe that the stationarity of $\{Y_t\}_{t \in \mathbb{Z}}$ implies that

$$\sum_{n=1}^{\infty} \frac{\text{Var}(Y_n)[\ln(n)]^2}{n^2} = \gamma(0) \sum_{n=1}^{\infty} \frac{[\ln(n)]^2}{n^2} < \infty \quad \text{and} \quad \sup_{n \geq 1} \{\text{Cov}(Y_n, Y_{n+h})\} \leq |\gamma(h)|.$$

Hence, by letting $\rho_h := |\gamma(h)|$, $h \geq 1$ and $b_n := n$, the result follows from theorem 1 in Hu et al. [5]. $\square$

Observe that any stationary processes with absolutely summable covariance function satisfy (44) with $q = 0$. Moreover, any stationary process for which $\gamma(h) \sim \ell h^{-\beta}$, $0 < \beta < 1$, where $\ell \in \mathbb{R}$, satisfy (44) for any $1 - \beta < q < 1$.

Proof of Theorem 3.2. From Corollary 3.1, $\{f_{k,DFA}^2(m, i)\}_{i=1}^{n-m}$ and $\{f_{DCCA}(m, i)\}_{i=1}^{n-m}$, are strictly stationary processes. Hence, the results for $F_{k,DFA}^2(m)$ and $F_{DCCA}(m)$ follow from Lemma 4.2 upon observing that, from (12) and (13),

$$F_{k,DFA}^2(m) = \frac{1}{n-m} \sum_{i=1}^{n-m} f_{k,DFA}^2(m, i) = \frac{1}{N} \sum_{i=1}^N Y_{k,i}, \quad Y_{k,i} = f_{k,DFA}^2(m, i) \quad \text{and} \quad N = n-m,$$

$$F_{DCCA}(m) = \frac{1}{n-m} \sum_{i=1}^{n-m} f_{DCCA}(m, i) = \frac{1}{N} \sum_{i=1}^N Z_i, \quad Z_i = f_{DCCA}^2(m, i) \quad \text{and} \quad N = n-m.$$

The results for $\rho_{DCCA}(m)$ follow upon observing that, from (9),

$$\rho_{DCCA}(m) = \frac{F_{DCCA}(m)}{F_{1,DFA}(m)F_{2,DFA}(m)}, \quad 0 < m < n,$$

and applying the continuous mapping theorem. $\square$

Proof of Proposition 3.1. Notice that, for any $k_1, k_2 \in \{1, 2\}$ and $h > m + 1$,

$$K_{m+1} \Gamma_{k_1, k_2}^{0, h} = \begin{bmatrix} K_{m+1} \Gamma_{k_1, k_2}^{0, h-(m+1)} & K_{m+1} H_{k_1, k_2}(h) \end{bmatrix}, \quad K_{m+1}(h) \Gamma_{k_1, k_2}^{h, 0} = \begin{bmatrix} 0_{h, m+1} \\ K_{m+1} H_{k_1, k_2}(-h) \end{bmatrix},$$

where

$$H_{k_1, k_2}(h) := \begin{bmatrix} \gamma_{k_1, k_2}(h) & \cdots & \gamma_{k_1, k_2}(m+h) \\ \vdots & \ddots & \vdots \\ \gamma_{k_1, k_2}(h-m) & \cdots & \gamma_{k_1, k_2}(h) \end{bmatrix}, \quad h \in \mathbb{Z}.$$

Hence, $\gamma_{k_1, k_2}(h) \rightarrow 0$, as $|h| \rightarrow \infty$, implies

$$\text{tr}(K_{m+1} \Gamma_{k_1, k_2}^{0, h} K_{m+1}(h) \Gamma_{k_1, k_2}^{h, 0}) = \text{tr}(K_{m+1} H_{k_1, k_2}(h) K_{m+1} H_{k_1, k_2}(-h)) \rightarrow 0.$$

Now, observe that $K_{m+1}^{\otimes}(h) \mathcal{K}_{k_1, k_2}(h)$ is a block matrix for which the (p, q) th block is given by

$$\begin{aligned} [K_{m+1}^{\otimes}(h) \mathcal{K}_{k_1, k_2}(h)]^{p, q} &= \sum_{\ell=1}^{m+1} [K_{m+1}]_{p, \ell} K_{m+1}(h) [\mathcal{K}_{k_1, k_2}(h)]^{\ell, q} \\ &= [K_{m+1}]_{p, \ell} \begin{bmatrix} 0_{h, h} & 0_{h, m+1} \\ K_{m+1} \gamma_{k_1, k_2}^{\ell, q}(h) & K_{m+1} c_{k_1, k_2}^{\ell, q}(h) \end{bmatrix}, \quad 1 \leq p, q \leq m+1, \end{aligned}$$

with

$$\begin{aligned} [\gamma_{k_1, k_2}^{\ell, q}(h)]_{\tau, s} &= \mathcal{K}_{k_1, k_2}(\ell, h + \tau, q, s), \quad 1 \leq \tau \leq m+1, \quad 1 \leq s \leq h, \\ [c_{k_1, k_2}^{\ell, q}(h)]_{\tau, s} &= \mathcal{K}_{k_1, k_2}(\ell, h + \tau, q, h + s), \quad 1 \leq \tau, s \leq m+1. \end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{tr}(K_{m+1}^{\otimes}(h)\mathcal{K}_{k_1,k_2}(h)) &= \sum_{p=1}^{m+1} \mathrm{tr}([K_{m+1}^{\otimes}(h)\mathcal{K}_{k_1,k_2}(h)]^{p,p}) = \sum_{p=1}^{m+1} [K_{m+1}]_{p,p} \mathrm{tr}(K_{m+1}C_{k_1,k_2}^{p,p}(h)) \\ &= \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} [K_{m+1}]_{p,p} [K_{m+1}C_{k_1,k_2}^n p, p(h)]_{q,q} \\ &= \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \sum_{\tau=1}^{m+1} [K_{m+1}]_{p,p} [K_{m+1}]_{q,\tau} \mathcal{K}_{k_1,k_2}(p, h + \tau, p, h + q).\end{aligned}$$

Hence $\mathcal{K}_{k_1,k_2}(p, h + \tau, p, h + q) \rightarrow 0$, as $h \rightarrow \infty$, implies

$$\mathrm{tr}(K_{m+1}^{\otimes}(h)\mathcal{K}_{k_1,k_2}(h)) \rightarrow 0, \quad \text{as } h \rightarrow \infty.$$

Therefore, the result follows. $\square$

Proof of Lemma 4.1. Since for any $\nu \geq 0$, $Q_{m+1}1_{m+1} = 0$, it follows that K_{m+1} is a symmetric matrix satisfying $[K_{m+1}]_{r,s} = 0$, if $r = 1$ or $s = 1$. Also, $M_0 = M_0^* = I_{m+1}$, so that $\mathrm{tr}(K_{m+1}M_0) = \mathrm{tr}(K_{m+1}M_0^*) = \mathrm{tr}(K_{m+1})$, while $\mathrm{tr}(K_{m+1}M_m) = \mathrm{tr}(K_{m+1}M_m^*) = 0$. For $1 \leq j < m$, we have

$$\mathrm{tr}(K_{m+1}M_j) = \sum_{r=1}^{m+1} \sum_{s=1}^{m+1} [K_{m+1}]_{r,s} [M_j]_{s,r} = 2 \sum_{r=2}^{m+1-j} [K_{m+1}]_{r,r+j},$$

and, analogously,

$$\alpha_j^{(m)} := \mathrm{tr}(K_{m+1}M_j^*) = \sum_{r=2}^{m+1-j} [K_{m+1}]_{r,r+j} = \frac{1}{2} \mathrm{tr}(K_{m+1}M_j).$$

Consider now the case $\nu = 0$. In this case, for all $r, s > 1$, the (r, s) th element in the matrix K_{m+1} is given by

$$\begin{aligned}[K_{m+1}]_{r,s} &= \sum_{i=r}^{m+1} \sum_{j=s}^{m+1} [Q_{m+1}]_{i,j} = m + 2 - \max\{r, s\} - \sum_{i=r}^{m+1} \sum_{j=s}^{m+1} [P_{m+1}]_{i,j} \\ &= m + 2 - \max\{r, s\} - \frac{(m + 2 - r)(m + 2 - s)[m^2 + 2m + 3(r - 1)(s - 1)]}{m(m + 1)(m + 2)}.\end{aligned}$$

It follows that

$$\alpha_0^{(m)} = \mathrm{tr}(K_{m+1}) = \sum_{r=2}^{m+1} \left\{ m + 2 - r - \frac{(m + 2 - r)^2 [m^2 + 2m + 3(r - 1)^2]}{m(m + 1)(m + 2)} \right\} = \frac{m^2 + 2m - 3}{15}.$$

and

$$\begin{aligned}\alpha_j^{(m)} &= \sum_{r=2}^{m+1-j} \left\{ m + 2 - (r + j) - \frac{(m + 2 - r)(m + 2 - r - j)[m^2 + 2m + 3(r - 1)(r + j - 1)]}{m(m + 1)(m + 2)} \right\} \\ &= \frac{(m - j)(m + 1 - j)(m + 2 - j)}{m(m + 1)(m + 2)} \left[\frac{m^2 + 2m - 3}{15} - \frac{j^2 + 3j(m + 1)}{10} \right] \sim \alpha_0^{(m)}, \quad \text{as } m \rightarrow \infty.\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 4.1. Suppose that $\sum_{h \in \mathbb{Z}} |\gamma_k(h)| < \infty$. Let $a_k, b_k > 0$ be the two real constants satisfying $|\gamma_k(h)| \leq a_k e^{-b_k |h|}$, for all $h \in \mathbb{Z}$. Then, for any $0 < h_0 < m$, from (28),

$$\frac{1}{m} \mathrm{tr}(K_{m+1} \Gamma_k) = \frac{\alpha_0^{(m)}}{m} \left[\gamma_k(0) + 2 \sum_{h=1}^{h_0-1} \beta_h^{(m)} \gamma_k(h) + 2 \sum_{h=h_0}^{m-1} \beta_h^{(m)} \gamma_k(h) \right].$$

Now, since $|\beta_h^{(m)}| \leq 1$ and $|\gamma_k(h)| \leq a_k e^{-b_k h}$,

$$\left| \sum_{h=h_0}^{m-1} \beta_h^{(m)} \gamma_k(h) \right| \leq a_k \sum_{h=h_0}^{\infty} e^{-b_k h} = a_k \frac{e^{-b_k h_0}}{1 - e^{-b_k}} < \frac{a_k}{b_k(1 - e^{-b_k})} \frac{1}{h_0} = O(h_0^{-1}),$$

uniformly in m . From Lemma 4.1, $\beta_h^{(m)} \rightarrow 1$, for all $0 \leq h < h_0$, so that

$$\sum_{h=1}^{h_0-1} \beta_h^{(m)} \gamma_k(h) = [1 - o(1)] \sum_{h=1}^{h_0-1} \gamma_k(h) \sim \sum_{h=1}^{h_0-1} \gamma_k(h), \quad \text{as } m \rightarrow \infty.$$

Now observe that $E(f_{k,DFA}^2(m, 1)) = m^{-1} \text{tr}(K_{m+1} \Gamma_k)$ and, for arbitrary h_0 ,

$$\frac{m^{-1} \text{tr}(K_{m+1} \Gamma_k)}{m/15 \sum_{h \in \mathbb{Z}} \gamma_k(h)} = \frac{m^2 + 2m - 3}{m^2} \left[\frac{\gamma_k(0) + 2[1 - o(1)] \sum_{h=1}^{h_0-1} \gamma_k(h)}{\sum_{h \in \mathbb{Z}} \gamma_k(h)} + O(h_0^{-1}) \right]$$

so that the result follows by letting $h_0 \rightarrow \infty$ at a slower rate than m . A similar argument applies to $E(f_{DCCA}(m, 1)) = \frac{1}{m} \text{tr}(K_{m+1} \Gamma_{1,2})$. $\square$

Proof of Corollary 4.1. From (33),

$$\sum_{h \in \mathbb{Z}} \gamma_k(h) = \tau_k^2 \sum_{h \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \psi_{k,j} \psi_{k,j+h} = \tau_k^2 \left[\sum_{h \in \mathbb{Z}} \psi_{k,j} \right]^2, \quad k \in \{1, 2\}$$

and

$$\sum_{h \in \mathbb{Z}} \gamma_{1,2}(h) = \tau_{1,2} \sum_{h \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \psi_{1,j} \psi_{2,j+h} = \tau_{1,2} \text{sign} \left(\sum_{h \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \psi_{1,j} \psi_{2,j+h} \right) \sqrt{\left[\sum_{j \in \mathbb{Z}} \psi_{1,j} \right]^2 \left[\sum_{j \in \mathbb{Z}} \psi_{2,j} \right]^2}.$$

The result now follows immediately from Theorem 4.1. $\square$

Appendix B. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.jmva.2020.104703>.

References

- [1] J.-M. Bardet, I. Kammoun, Asymptotic properties of the detrended fluctuation analysis of long range dependence processes, *IEEE Trans. Inform. Theory* 54 (5) (2008) 2041–2052.
- [2] D.A.J. Blythe, A rigorous and efficient asymptotic test for power-law cross-correlation, 2013, [arXiv:1309.4073](https://arxiv.org/abs/1309.4073).
- [3] D.A.J. Blythe, V.V. Nikulin, K.-R. Müller, Robust statistical detection of power-law cross-correlation, *Sci. Rep.* 6 (2016) 27089.
- [4] K. Hu, P.C. Ivanov, Z. Chen, P. Carpena, H.E. Stanley, Effect of trends on detrended fluctuation analysis, *Phys. Rev. E* 64 (2001) 011114.
- [5] T.-C. Hu, A. Rosalsky, A. Volodin, On convergence properties of sums of dependent random variables under second moment and covariance restrictions, *Statist. Probab. Lett.* 78 (2008) 1999–2005.
- [6] J.W. Kantelhardt, E. Koscielny-Bunde, H.H.A. Rego, S. Havlin, A. Bunde, Detecting long-range correlations with detrended fluctuation analysis, *Physica A* 295 (3) (2001) 441–454.
- [7] C.K. Peng, S.V. Buldyrev, S. Havlin, M. Simons, H.E. Stanley, A.L. Goldberger, Mosaic organization of DNA nucleotides, *Phys. Rev. E* 49 (2) (1994) 1685–1689.
- [8] B. Podobnik, Z.-Q. Jiang, W.-X. Zhou, H.E. Stanley, Statistical tests for power-law cross-correlated processes, *Phys. Rev. E* 84 (2011) 066118.
- [9] B. Podobnik, H.E. Stanley, Detrended cross-correlation analysis: A new method for analyzing two non-stationary time series, *Phys. Rev. Lett.* 100 (8) (2008) 084102.
- [10] E.L. Siqueira, T. Stošić, L. Bejan, B. Stošić, Correlations and cross-correlations in the Brazilian agrarian commodities and stocks, *Physica A* 389 (14) (2010) 2739–2743.
- [11] Y. Wang, Y. Wei, C. Wu, Cross-correlations between Chinese A-share and B-share markets, *Physica A* 389 (23) (2010) 5468–5478.
- [12] N. Xu, P. Shang, S. Kamae, Modeling traffic flow correlation using DFA and DCCA, *Nonlinear Dynam.* 61 (2010) 207–216.
- [13] G.F. Zebende, DCCA cross-correlation coefficient: quantifying level of cross-correlation, *Physica A* 390 (2011) 614–618.
- [14] G.F. Zebende, A.A. Brito, A.M. Silva Filho, A.P. Castro, ρ_{DCCA} applied between air temperature and relative humidity: An hour/hour view, *Physica A* 494 (2018) 17–26.