JOURNAL OF MULTIVARIATE ANALYSIS 47, 152–162 (1993)

A Mixed Limit Theorem for Stable Random Fields*

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A mixed distributional limit theorem for a stable random field of index $0 < \alpha < 2$ is derived. These random fields are of special interest in pattern analysis, in particular, in pattern synthesis. This paper considers the case when the underlying graph that the random field is defined on is linear. This result is encouraging insofar as it shows that the mixed limit theorems do exist in the stable case. The final limiting distribution can be written in terms of the stable process of index α in $D[0, 1]$.

1. INTRODUCTION AND SUMMARY

A picture or image can be considered to be a collection of gray levels located at the vertices of a set, that is to say, that it can be represented by the collection $\{Y_\alpha : \alpha \in V\}$, where Y_α represents a gray level located at the site α in the set V. For instance, a raster image on a TV screen can be viewed as a collection of gray levels $Y_{(i,j)}$ located at site (i, j) in the set $\{0, ..., n\} \times \{0, ..., n\}$. More structure can be given to the set of sites V by considering a graph $G = (V, e)$ with vertex set V and edge set e. The graph structure can be so chosen as to reflect the relations that we would expect to see between sites in the particular class of images under consideration.

A random image can therefore be modeled by a collection of random variables Y_α located at the vertices α of a graph $G = (V, e)$. Gibbs distributions have been used to specify the distribution of random images because the graph structure G can be tailored to take into account the dependencies intrinsic to the structure of the images arising in practice.

The general pattern analysis problem can be broadly described as follows. There is a true image I which cannot be observed. However, a deformed version $I^{\mathcal{D}}$ of I can be observed. The physical process deforming the true image I is due to the process of observation itself. This physical process could be mathematically modeled as an additive noise or by some other process. The pattern analysis problem is to reconstruct the true

Received July 29, 1992; revised February 26, 1993.

AMS Subject Classifications: 60B12, 60K35.

Key words: mixed limit theorems, random fields, pattern synthesis.

* Research supported by Army Research Office Grant DAAL03-90-G-0103.

0047-259X/93 $5.00

image I based on the observed image $I^{\mathscr{D}}$ by using a statistical procedure that is optimal in some sense.

Bayesian image analysis first formulates a prior distribution for I which incorporates what is known about the structure of the true image I. Gibbs distributions have been used as prior distributions in Bayesian image analysis by Geman and Geman (1984). The next step in Bayesian image analysis is to update the prior notion of I using $I^{\mathscr{D}}$. This gives rise to a posterior distribution on the set of images. Under fairly general conditions, the posterior distribution is a Gibbs distribution. The posterior mean or the posterior mode is usually used as an estimate of the true image I.

Bayesian image analysis has had a good deal of success in dealing with pattern analytic problems. The HANDS project at Brown University (see Grenander *et al.*, 1991) is an application of Bayesian image analysis to recognizing biological shapes like human hands. For Bayesian image analysis, it is necessary to generate an observation from the posterior distribution, and this is commonly done using the Gibbs Sampler; see Geman and Geman (1984), which in turns uses the algorithm in Metropolis *et al.*, (1953). In fact, we can simulate observations from the prior distribution using the same Gibbs Sampler to evaluate its suitability for the problem at hand. This is called pattern synthesis. However, these methods are not very fast and require a good deal of computer time.

A natural question that now arises is whether we can replace simulations with direct mathematical approximations of such distributions when the number of vertices (sites) of the graph is very large. The answer to this question is in the affirmative, for special graphs, provided that the number of vertices increases to infinity and the scale of gray levels present at each site increases at a certain rate. Results that describe this kind of behavior are called mixed limit theorems and have been studied previously by Chow and Grenander (1985), Grenander and Sethuraman (1993), and by Chow (1990).

A graph of interest in pattern analysis is the linear connection graph G_1 whose vertices are $\{0, ..., n\}$ and edge set is $\{(i, i+1): 0 \leqslant i < n\}$.

A random configuration on G_1 is a collection of random variables $\mathbf{Y} = (Y_0, ..., Y_n)$ which reside at the vertices of G_1. Suppose that the joint distribution of $\mathbf{Y}$ has a density (with respect to Lebesgue measure) which is proportional to

$$\prod_{i=1}^{n} A\left(\frac{y_i - y_{i-1}}{\varepsilon^{\beta}}\right) \prod_{i=0}^{n} Q(y_i), \tag{1.1}$$

where $\beta > 0$, $A(\cdot)$ be a symmetric density and $Q(\cdot)$ is a non-negative function on $\mathfrak{R}$.

The function A is called the acceptor function in the pattern analysis literature. The acceptor function models the local dependence between the Y_i's. When the acceptor function A is a symmetric density which is decreasing on $\Re^+$, the gray levels Y_i and Y_j at neighboring sites i and j will tend to be more alike. When the acceptor function A is a symmetric density which is increasing on $\Re^+$, the gray levels Y_i and Y_j at neighboring sites i and j will tend to be more different. In the equation above, the factor of ε^β represents a coupling parameter which controls the global dependence in the configuration, and is often called temperature. When we set $\varepsilon = n^{-1}$, as we do later, it reflects the increase in the scale of gray levels present at each site as the number of sites increase. The joint distribution of the Y_i's is a Gibbs distribution defined on G_1.

We will now normalize the random configuration $\mathbf{Y}$ to obtain the normalized configuration $\mathbf{X} = (X_0, ..., X_n)$ defined by

$$X_i = \frac{Y_i}{\sqrt{\varepsilon}}, \qquad i = 0, ..., n. \tag{1.2}$$

Let P_n be the distribution of $\mathbf{X}$. Consider the mapping, $g_n(\mathbf{y})$, from $\mathscr{R}^{n+1}$ into $D[0, 1]$, which is the usual step function mapping *based on* $\mathbf{y} = (y_0, y_1, ..., y_n)$, defined as follows:

$$g_n(\mathbf{y})(t) = y_{[nt]}, \qquad t \in [0, 1]. \tag{1.3}$$

In this paper we are interested in the limiting distribution of $P_n g_n^{-1}$, that is the limiting distribution of the process $g_n(\mathbf{X}) = \{X_n(t), t \in [0, 1]\}$. Approximation theorems which establish the convergence of the distribution of the process $g_n(\mathbf{X})$ are called mixed limit theorems.

Grenander and Sethuraman (1993) considered random configurations given by the model (1.1) and a cyclical graph which is like G_1, except that 0 is identified with $n+1$. They looked at the case where $\beta = 1$, $\varepsilon = n^{-1}$, $A(\cdot) = \exp(-x^2/2)$, and $Q(x)$ is essentially of the form $\exp(-x^2/2)$, and showed that a process formed from the normalized configuration $\mathbf{X}$ converges to a Gaussian process in $C(M)$, where M is the circle of unit circumference. Chow (1990) considered random configurations given by the model (1.1) with the graph G_1. He looked at the case where $\beta = 1$, $\varepsilon = n^{-1}$, $A(\cdot)$ has a second moment, and $Q(x)$ is essentially of the form $\exp(-x^2/2)$, and showed that a process formed from the normalized configuration $\mathbf{X}$ converges to a Gaussian process in $C[0, 1]$. Both of these papers considered other mixed limit theorems for random configurations, and showed, under some conditions that all the limits were Gaussian processes with a known covariance structure.

All the cases considered so far correspond to the situation in which the acceptor function A has thin tails. In this paper we consider the case where the acceptor function A has thick tails. We show that, under certain conditions, the stochastic process $g_n(\mathbf{X})$ constructed from the random configuration converges to a process which can be expressed in terms of a stable process. This result is found in Theorem 1.

The conclusion of Theorem 1 is an approximation to the limiting distribution of a Gibbs distribution. If we could conveniently simulate observations from this limiting process, then this would result in greater computational efficiency since the Gibbs Sampler would not have to be used. This paper shows the existence of mixed limit theorems in the stable case. In order to directly simulate observations from the limiting process, a certain functional integral will have to be evaluated, and this remains an open problem at the present.

2. Mixed Limit Theorems

Before stating the main theorem of this paper, Theorem 1, we need some notations concerning a stable process. Let $f_\alpha(x)$ be the symmetric stable density with index α, $0<\alpha<2$. Let $T, T_1, ..., T_n$ be i.i.d. random variables with density function $f_\alpha(x)$. Let $\mathbf{T}=(T_1, ..., T_n)$. Let $\mathbf{S}$ be a normalization of $\mathbf{T}$ and be defined by $\mathbf{S}=(S_1, ..., S_n)=(n^{-1/\alpha}T_1, ..., n^{-1/\alpha}T_n)$. Then R_n, the probability distribution of $\mathbf{S}$ has density function proportional to

$$\prod_{i=1}^{n} f_\alpha(n^{1/\alpha}s_i). \tag{2.1}$$

Consider the mapping, $h_n(\mathbf{s})$, from $\mathscr{R}^n$ into $D[0, 1]$, which is the usual step function mapping *based on the partial sums of* $\mathbf{s}=(s_1, ..., s_n)$, defined as follows:

$$h_n(\mathbf{s})(0)=0,\ h_n(\mathbf{s})(t)=\sum_{i=1}^{[nt]} s_i, \qquad \text{for } t\in(0, 1]. \tag{2.2}$$

Then $h_n(\mathbf{S})(t)$ is a process in $D[0, 1]$ with distribution $R_n^* \overset{\text{def}}{=} R_n h_n^{-1}$ and converges to the distribution R^* of a stable process $\{V(t), t\in[0, 1]\}$. See, for instance, Breiman (1968). As usual, we endow the space $D[0, 1]$ with the topology based on the Skorohod metric. This stable process has the properties

$V(0)=0$,

$\{V_n(t), t\in[0, 1]\}$ has independent increments, and

$V_n(t)-V_n(s) \overset{d}{=} |t-s|^\alpha\, T$ for $t, s\in[0, 1]$.

Note that R^* is the distribution of the stable process starting from 0 since $R^*(V(0)=0)=1$. We also need to talk about $R^*_{x_0}$, the distribution of the stable process starting from x_0, defined by $R^*_{x_0}(C)=R^*(C-x_0)$ for each Borel set C in $D[0, 1]$. Clearly, $R^*_{x_0}(V(0)=x_0)=1$.

We specialize the model (1.1) further by assuming that

(I) $Q(x)=\exp(-x^2/2)$,

(II) $A(x)=f_\alpha(x)$ where $0<\alpha<2$, and

(III) $\beta=\dfrac{1}{2}+\dfrac{1}{\alpha}$.

After these preliminaries, we now state the main theorem of this paper.

THEOREM 1. *Let the joint distribution of the random configuration* $\mathbf{Y}$ *be given by the model* (1.1) *satisfying conditions* (I), (II), *and* (III). *Let* $\varepsilon=n^{-1}$. *Let* $\mathbf{X}$ *be the normalized configuration as defined in* (1.2). *Then* $P_n g_n^{-1}$ *the distribution of the step function process constructed from* $\mathbf{X}$, *converges weakly to a distribution* P^*, *as* $n\to\infty$. *The distribution* P^* *can be described in terms of the stable process* $R^*_{x_0}$ *starting from* x_0, *as*

$$P^*(C)=\frac{(1/\sqrt{2\pi})\iint_{y\in C}\exp[-(1/2)\int y(t)^2\,dt]\,dR^*_{x_0}(y)\,dx_0}{\int\exp[-(1/2)\{\int x(t)^2\,dt-(\int x(t)\,dt)^2\}]\,dR^*(x)} \tag{2.3}$$

for C *in the Borel* σ*-field in* $D[0, 1]$.

The proof of this theorem is given after the following two results. The first of these results is from Sethuraman (1961) and is stated as in Lemma 1.

LEMMA 1. *Let* Λ_n *be a sequence of probability measures on* $(\mathscr{X}\times\mathscr{Y}, \mathscr{A}\times\mathscr{B})$, *where* $\mathscr{X}$ *and* $\mathscr{Y}$ *are topological spaces and* $\mathscr{A}$ *and* $\mathscr{B}$ *are the appropriate Borel* σ*-fields. Let* v_n *be the marginal distribution of* Λ_n *on* $\mathscr{X}$ *and* $\Lambda_{n,x}(\cdot)$ *be the conditional probability measure of* Λ_n *given* $X=x$. *Suppose that*

$$v_n(A)\to v(A)\qquad \textit{for all}\quad A\in\mathscr{A} \tag{2.4}$$

and

$$\Lambda_{n,x}(\cdot)\to\Lambda_x(\cdot)\ \textit{weakly},\qquad \textit{for almost all } x \textit{ w.r.t. } v \tag{2.5}$$

Then, $\Lambda_n\to\Lambda$ *weakly where*

$$\Lambda(A\times B)=\int_A \Lambda_x(B)\,dv(x)$$

for each measurable rectangle $A\times B\in\mathscr{A}\times\mathscr{B}$.

An easy way to verify condition (2.4) is to use Scheffé's theorem and verify the sufficient condition

$$\frac{dv_n}{dM}(x) \to \frac{dv}{dM}(x) \tag{2.6}$$

for some measure M on $(\mathscr{X}, \mathscr{A})$ which dominates v_n and v.

The second result that we need is a result on $D[0, 1]$ given below.

LEMMA 2. *Let $f: \mathfrak{R} \to \mathfrak{R}$ be continuous and $x_n \to x \in D[0, 1]$. Then,*

$$\frac{1}{n} \sum_{i=1}^{n} f(x_n(i/n)) \to \int_0^1 f(x(t))\, dt.$$

Proof of Lemma 2. Let $y_n = f(x_n)$ and $y = f(x)$. The $y_n \to y$ in $D[0, 1]$ since f is a continuous map. Define

$$z_n(t) = \begin{cases} y_n(i/n) & \text{if } (i-1)/n < t \leqslant i/n \\ y_n(0) & \text{if } t = 0. \end{cases}$$

Then, $(1/n) \sum_{i=1}^{n} y_n(i/n) = \int_0^1 z_n(t)\, dt$. We now show that $z_n(s) \to y(s)$ a.e. Lebesgue on $[0, 1]$.

Since $x_n \to x$ in $D[0, 1]$, there exist λ_n which are strictly increasing and continuous mappings of $[0, 1] \to [0, 1]$ such that

$$\sup_s |y_n(\lambda_n(s)) - y(s)| \to 0, \qquad \text{and} \qquad \sup_s |\lambda_n(s) - s| \to 0.$$

Let $s \in [0, 1]$ be a continuity point of $y(\cdot)$. There exists i_n such that $(i_n - 1)/n < s \leqslant i_n/n$. Let $s_n^* = \lambda_n^{-1}(i_n/n)$. Then $z_n(s) = y_n(\lambda_n(s_n^*))$ and $s_n^* \to s$. Hence

$$\begin{aligned} |z_n(s) - y(s)| &\leqslant |y_n(\lambda_n(s_n^*)) - y(s_n^*)| + |y(s_n^*) - y(s)| \\ &\leqslant \sup_s |y_n(\lambda_n(s)) - y(s)| + |y(s_n^*) - y(s)| \\ &\to 0 \end{aligned}$$

if s is a continuity point of $y(\cdot)$. Hence $z_n(s) \to y(s)$, as asserted. Since the number of discontinuity points of $y(\cdot)$ is countable, $z_n(s) \to y(s)$ a.e. Lebesgue.

Since y_n, and hence z_n are uniformly bounded, we have from the Dominated Convergence Theorem that

$$\int_0^1 y_n(t)\,dt \to \int_0^1 x(t)\,dt.$$

This proves Lemma 2. ∎

Proof of Theorem 1. For any $x_0 \in \mathscr{R}^1$ and any function $x(\cdot) \in D[0, 1]$, define

$$k_n(x, x_0) = \exp\left[-\frac{1}{2n}\left\{x_0^2 + \sum_{i=1}^n \left(x\left(\frac{i}{n}\right) + x_0\right)^2\right\}\right], \tag{2.7}$$

$$k(x, x_0) = \exp\left[-\frac{1}{2}\int_0^1 (x(t) + x_0)^2\,dt\right], \tag{2.8}$$

$$p_n(x_0) = \int_{D[0,1]} k_n(x, x_0)\,dR_n^*(x), \qquad \text{and} \tag{2.9}$$

$$p(x_0) = \int_{D[0,1]} k(x, x_0)\,dR^*(x). \tag{2.10}$$

Using the fact that the model defined by (1.1) satisfies (I), (II), and (III), and $\varepsilon = n^{-1}$, we see that the pdf of P_n is proportional to

$$\prod_{i=1}^n f_\alpha(n^{1/\alpha}(x_i - x_{i-1}))\exp\left[-\frac{1}{2n}\sum_{i=0}^n x_i^2\right]. \tag{2.11}$$

Define $U_i = X_i - X_{i-1}$, $i = 1, ..., n$ and write $\mathbf{U} = (U_1, ..., U_n)$. Let Q_n be the distribution of $(X_0, \mathbf{U})$. The density function of Q_n is proportional to

$$\prod_{i=1}^n f_\alpha(n^{1/\alpha}u_i)\exp\left[-\frac{1}{2n}\left\{x_0^2 + \sum_{i=1}^n \left(\sum_{j=1}^i u_j + x_0\right)^2\right\}\right]$$
$$= \prod_{i=1}^n f_\alpha(n^{1/\alpha}u_i)\,k_n(h_n(\mathbf{u}), x_0), \tag{2.12}$$

where we have the notations from (2.2) and (2.7).

Consider the product space $\mathscr{R}^1 \times D[0, 1]$. Points in this space are denoted as (x_0, x), (y_0, y), etc. Let Q_n^* be the distribution of $(X_0, h_n(U))$ under Q_n.

Note that

$$g_n(\mathbf{X})(t) = X_0 + h_n(\mathbf{U})(t) \qquad \text{for} \quad t \in [0, 1]. \tag{2.13}$$

Thus to study the limiting properties of $g_n(\mathbf{X})(t)$ under P_n we need to study the limiting properties of $X_0 + h_n(\mathbf{U})(t)$ under Q_n, which is the distribution of $X_0 + X$ under Q_n^*.

Let Q_{n,x_0}^* be the conditional distribution of X given $X_0 = x_0$ under Q_n^* and let $\mu_n = Q_n^* X_0^{-1}$ be the distribution of X_0 under Q_n^*.

By examining the density functions in (2.12) and (2.1), and using the notations in (2.2), (2.7), and (2.9) we see that

$$\frac{dQ_{n,x_0}^*}{dR_n^*}(x) = \frac{k_n(x, x_0)}{p_n(x_0)}, \qquad \text{and} \tag{2.14}$$

$$\frac{d\mu_n}{dx_0} = \frac{p_n(x_0)}{\int_{-\infty}^{\infty} p_n(y_0)\,dy_0}. \tag{2.15}$$

We show that

$$Q_{n,x_0}^* \text{ converges weakly to } Q_{x_0}^*, \tag{2.16}$$

where

$$\frac{dQ_{x_0}^*}{dR^*}(x) = \frac{k(x, x_0)}{p(x_0)} \tag{2.17}$$

and that

$$\mu_n \to \mu, \qquad \text{weakly} \tag{2.18}$$

and, in fact, that

$$\frac{d\mu_n}{dx_0} \to \frac{d\mu}{dx_0} \text{ pointwise}, \qquad \text{where } \frac{d\mu}{dx_0} = \frac{p(x_0)}{\int_{-\infty}^{\infty} p(y_0)\,dy_0}. \tag{2.19}$$

If $x_n \to x$ in $D[0, 1]$, it follows from Lemma 2 that

$$\frac{1}{n} \sum x_n^2(i/n) \to \int_0^1 x^2(t)\,dt, \qquad \text{and} \tag{2.20}$$

$$\frac{1}{n} \sum x_n(i/n) \to \int_0^1 x(t)\,dt \tag{2.21}$$

and thus

$$g(x_n)\,k_n(x_n, x_0) \to g(x)\,k(x, x_0)$$

for any bounded continuous function $g: D[0, 1] \to \mathscr{R}^1$. Since $R_n^* \to R^*$

weakly, we have by an application of Theorem 5.5 of Billingsley (1968), that

$$\int g(x)\, k_n(x, x_0)\, dR_n^*(x) \to \int g(x)\, k(x, x_0)\, dR^*(x) \tag{2.22}$$

and

$$p_n(x_0) = \int k_n(x, x_0)\, dR_n^*(x) \to \int k(x, x_0)\, dR^*(x) = p(x_0), \tag{2.23}$$

which together imply that

$$\int g(x)\, dQ_{n, x_0}(x) \to \frac{\int g(x)\, k(x, x_0)\, dR^*(x)}{p(x_0)},$$

and this proves (2.16). Also

$$\begin{aligned}
&\int_{-\infty}^{\infty} p_n(z_0)\, dz_0 \\
&\quad = \int_{D[0,1]} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2n} \left\{ z_0^2 + \sum_i \left(x\left(\frac{i}{n}\right) + z_0 \right)^2 \right\} \right] dz_0\, dR_n^*(x) \\
&\quad = \iint \exp\left[-\frac{n+1}{2n} \left\{ z_0 + \frac{1}{n+1} \sum_i x\left(\frac{i}{n}\right) \right\}^2 \right] dz_0 \\
&\qquad \times \exp\left[-\frac{1}{2} \left\{ \frac{1}{n} \sum_i x\left(\frac{i}{n}\right)^2 - \frac{1}{n(n+1)} \left(\sum_i x\left(\frac{i}{n}\right) \right)^2 \right\} \right] dR_n^*(x) \\
&\quad \to \sqrt{2\pi} \int \exp\left[-\frac{1}{2} \left\{ \int x(t)^2\, dt - \left(\int x(t)\, dt \right)^2 \right\} \right] dR^*(x)
\end{aligned} \tag{2.24}$$

by another application of (2.20) and (2.21). It can be directly verified that

$$\int p(y_0)\, dy_0 = \sqrt{2\pi} \int \exp\left[-\frac{1}{2} \left\{ \int x(t)^2\, dt - \left(\int x(t)\, dt \right)^2 \right\} \right] dR^*(x). \tag{2.25}$$

Hence the numerator and denominator in the density function of μ_n given in (2.15) converge to $p(x_0)$ and $\int p(y_0)\, dy_0$, respectively. This implies that (2.19) and (2.18).

We now use Lemma 1. Identify $\mathscr{X}$, $\mathscr{Y}$, and A_n in Lemma 1 with $\mathscr{R}^1$, $D[0, 1]$, and Q_n^*, respectively. The assertions in (2.16) and (2.19) verify (2.6) and (2.5). Thus $Q_n^* \to Q^*$ weakly, where

$$Q^*(A \times B) = \int_A Q_{x_0}^*(B)\, d\mu(x_0)$$

$$= \int_{x_0 \in A} \int_{x \in B} \frac{k(x, x_0)\, dR^*(x)}{p(x_0)} \frac{p(x_0)\, dx_0}{\int_{-\infty}^{\infty} p(y_0)\, dy_0}$$

$$= \frac{(1/\sqrt{2\pi}) \int_{x_0 \in A} \int_{x \in B} \exp[-(1/2) \int (x(t) + x_0)^2\, dt]\, dR^*(x)\, dx_0}{\int \exp[-(1/2)\{\int x(t)^2\, dt - (\int x(t)\, dt)^2\}]\, dR^*(x)},$$

where we have used (2.25) in the last step. Thus

$$Q^*(X_0 + X \in C)$$

$$= \frac{(1/\sqrt{2\pi}) \iint_{x_0 + x \in C} \exp[-(1/2) \int (x(t) + x_0)^2\, dt]\, dR^*(x)\, dx_0}{\int \exp[-(1/2)\{\int x(t)^2\, dt - (\int x(t)\, dt)^2\}]\, dR^*(x)}$$

$$= \frac{(1/\sqrt{2\pi}) \iint_{y \in C} \exp[-(1/2) \int y(t)^2\, dt]\, dR_{x_0}^*(y)\, dx_0}{\int \exp[-(1/2)\{\int x(t)^2\, dt - (\int x(t)\, dt)^2\}]\, dR^*(x)},$$

where $R_{x_0}^*$ is the stable process of index α starting from x_0.

Since $P_n g_n^{-1}(C) = Q_n^*(X_0 + X \in C)$, this shows that $P_n g_n^{-1} \to P^*$ weakly where P^* is as defined in (2.3). This completes the proof of Theorem 1. ∎

Remark 1. The assumption made on the functional form of $Q(\cdot)$ can be weakened to include non-Gaussian density functions. This can be done by using the methods in Chow (1990).

Acknowledgments

We thank the referee who pointed out some errors and obscurities in the original version of this paper. This helped us considerably to clarify the proof and make general improvements in our presentation.

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