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Mean residual life order of convolutions of heterogeneous exponential random variables

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ABSTRACT

In this paper, we study convolutions of heterogeneous exponential random variables with respect to the mean residual life order. By introducing a new partial order (reciprocal majorization order), we prove that this order between two parameter vectors implies the mean residual life order between convolutions of two heterogeneous exponential samples. For the 2-dimensional case, it is shown that there exists a stronger equivalence. We discuss, in particular, the case when one convolution involves identically distributed variables, and show in this case that the mean residual life order is actually associated with the harmonic mean of parameters. Finally, we derive the "best gamma bounds" for the mean residual life function of any convolution of exponential distributions under this framework.

1. Introduction

Due to its nice mathematical form and the characterizing memoryless property, the exponential distribution has been widely used in many areas including life-testing, reliability, and operations research. One may refer to [1,2] for an encyclopedic treatment to developments on the exponential distribution. Convolutions of independent exponential random variables often occur naturally in many problems, and especially in reliability theory. Consider a typical reliability scenario in which there is a redundant standby system without repair consisting of n exponential components. At the time of the first failure, one standby component is put into operation; next, at the time of the second failure, another standby component is put into operation, and so on. Finally, the whole system fails at the failure of the last component. It is evident that the lifetime of the system is then a convolution of n exponential lifetimes. When the n exponential components have a common hazard rate λ, then the lifetime of the system is clearly distributed as gamma with parameters (n, λ). Since the distribution theory is quite complicated when the convolution involves non-identical random variables, it will be of great interest to derive bounds and approximations on some characteristics of interest in this setup.

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate λ_i^*, $i = 1, \ldots, n$. Boland et al. [3] then proved that

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{lr} \sum_{i=1}^{n} Y_i;$$

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formal definitions of the orderings stated above will be given in the next section. Bon and Păltănea [4] subsequently showed that

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{hr} \sum_{i=1}^{n} Y_i, \tag{1}$$

and they mainly focused on the special case when one convolution involved identically distributed random variables thereby obtaining characterizations of some classical stochastic orders of convolutions of exponential random variables presented in terms of various means (arithmetic mean, geometric mean and harmonic mean) of their parameters.

Kochar and Ma [5] established that

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{disp} \sum_{i=1}^{n} Y_i. \tag{2}$$

Korwar [6] extended the result in (2) to the case of gamma random variables with different scale parameters but with a common shape parameter (≥ 1). Khaledi and Kochar [7] further strengthened the results of Kochar and Ma [5] and Korwar [6] from usual majorization order to p-larger order (the former implies the latter).

In this paper, we further pursue this problem by examining the mean residual order instead in (1). A new partial order, called the reciprocal majorization order (denoted by $\overset{\mathrm{rm}}{\succeq}$), is introduced which is closely related to majorization and p-larger orders. For the 2-dimensional case, it is proved that

$$X_1 + X_2 \geq_{mrl} X_1^* + X_2^* \Longleftrightarrow (\lambda_1, \lambda_2) \overset{\mathrm{rm}}{\succeq} (\lambda_1^*, \lambda_2^*),$$

and for the general case, we prove that

$$(\lambda_1, \ldots, \lambda_n) \overset{\mathrm{rm}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{mrl} \sum_{i=1}^{n} Y_i. \tag{3}$$

We pay special attention to the case when one convolution involves identically distributed random variables and show in this case that the mean residual life order is actually associated with the harmonic mean of parameters. Finally, we derive the "best gamma bounds" for the mean residual life function of any convolution of exponential distributions under this framework.

2. Definitions

In this section, we first recall some notions of stochastic orders, and majorization and related orders. Then, a new partial order is introduced which is closely related to the main results to be developed in the subsequent sections. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*.

2.1. Stochastic orders

Definition 2.1. For two random variables X and Y with distribution functions F and G, let $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ denote the corresponding survival functions. Then:

(i) X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if X and Y are absolutely continuous with respective densities f and g and $g(x)/f(x)$ is increasing in x;
(ii) X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;
(iii) X is said to be smaller than Y in the stochastic order (denoted by $X \leq_{st} Y$) if $\bar{G}(x) \geq \bar{F}(x)$;
(iv) X is said to be smaller than Y in the mean residual life order (denoted by $X \leq_{mrl} Y$) if $EX_t \leq EY_t$, where $X_t = (X - t | X > t)$ is the residual life at age $t > 0$ of the random lifetime X;
(v) X is said to be smaller than Y in the mean order (denoted by $X \leq_{mn} Y$) if $EX \leq EY$.

From [8], it is known that the hazard rate order implies both the usual stochastic order and the mean residual life order, but neither the usual stochastic order nor the mean residual life order implies the other.

One of the basic criteria for comparing variability in probability distributions is the so-called dispersive order. A random variable X is said to be less dispersed than another random variable Y (denoted by $X \leq_{disp} Y$) if $F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u)$, for $0 \leq u \leq v \leq 1$, where F^{-1} and G^{-1} are the right inverses of the distribution functions F and G of X and Y, respectively.

For a comprehensive discussion on various stochastic orders, one may refer to [8,9].

2.2. A new partial order related to majorization and p-larger orders

It is well known that the notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \le \cdots \le x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 2.2. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

In addition, the vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ weakly (written as $\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \ldots, n$. Clearly,

$$\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}.$$

For extensive and comprehensive discussion on the theory and applications of the majorization order, one may refer to [10]. Bon and Păltănea [4] introduced a pre-order on $\mathbb{R}_+^n$, called p-larger order, which is defined as follows.

Definition 2.3. The vector $\mathbf{x}$ in $\mathbb{R}_+^n$ is said to be p-larger than another vector $\mathbf{y}$ in $\mathbb{R}_+^n$ (written as $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \le \prod_{i=1}^{j} y_{(i)}$$

for $j = 1, \ldots, n$.

Let $\log(\mathbf{x})$ be the vector of logarithms of the coordinates of $\mathbf{x}$. It is then easy to verify that

$$\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y} \Longleftrightarrow \log(\mathbf{x}) \overset{\text{w}}{\succeq} \log(\mathbf{y}).$$

Moreover,

$$\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$. The converse is, however, not true. For example, we have $(1, 5.5) \overset{\text{p}}{\succeq} (2, 3)$, but the weak majorization order clearly does not hold.

We now introduce a new partial order which is closely associated to majorization and p-larger orders.

Definition 2.4. The vector $\mathbf{x}$ in $\mathbb{R}_+^n$ is said to reciprocal majorize another vector $\mathbf{y}$ in $\mathbb{R}_+^n$ (written as $\mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \ge \sum_{i=1}^{j} \frac{1}{y_{(i)}}$$

for $j = 1, \ldots, n$.

Remark 2.5. A natural question that arises here is what relationship exists between reciprocal majorization order and majorization order or p-larger order. In the special case when $n = 2$, it can be easily verified that the following implications hold:

$$(a_1, a_2) \overset{\text{m}}{\succeq} (b_1, b_2) \Longrightarrow (a_1, a_2) \overset{\text{p}}{\succeq} (b_1, b_2) \Longrightarrow (a_1, a_2) \overset{\text{rm}}{\succeq} (b_1, b_2) \tag{4}$$

for any two non-negative vectors (a_1, a_2) and (b_1, b_2). In the general case when $n > 2$, we do not know the nature of the implications. However, the $\overset{\text{rm}}{\succeq}$ order does not imply the $\overset{\text{p}}{\succeq}$ order even in the 2-dimensional case. For example, from the definition of the $\overset{\text{rm}}{\succeq}$ order, it follows that $(1, 4) \overset{\text{rm}}{\succeq} (\frac{4}{3}, 2)$, but it is clear that the $\overset{\text{p}}{\succeq}$ order does not hold between these two vectors.

3. Equivalent characterization for the 2-dimensional case

In this section, we will establish an equivalent characterization for the case when convolutions of two 2-dimensional independent exponential random vectors are ordered in the sense of the mean residual life order.

Let X_1, X_2, Y_1, Y_2 be independent exponential random variables with respective hazard rates $\lambda_1, \lambda_2, \lambda_1^*, \lambda_2^*$. Bon and Păltănea [4] then proved the following equivalence:

$$X_1 + X_2 \geq_{hr} (\geq_{st}) Y_1 + Y_2 \iff (\lambda_1, \lambda_2) \overset{\mathrm{p}}{\succeq} (\lambda_1^*, \lambda_2^*). \tag{5}$$

Recently, Zhao et al. [11] showed further that

$$X_1 + X_2 \geq_{lr} Y_1 + Y_2 \iff (\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*). \tag{6}$$

The theorem established below gives an equivalence similar to those in (5) and (6), which reveals the link between the reciprocal majorization order and the mean residual life order.

Theorem 3.1. *Let (X_1, X_2) be a vector of independent exponential random variables with respective hazard rates λ_1, λ_2, and (X_1^*, X_2^*) be another vector of independent exponential random variables with respective hazard rates λ_1^*, λ_2^*. Then,*

$$X_1 + X_2 \geq_{mrl} X_1^* + X_2^* \iff (\lambda_1, \lambda_2) \overset{\mathrm{rm}}{\succeq} (\lambda_1^*, \lambda_2^*). \tag{7}$$

Proof. $\Longleftarrow$ Suppose the RHS of the equivalence in (7) holds, i.e.,

$$\begin{cases} \min(\lambda_1, \lambda_2) \leq \min(\lambda_1^*, \lambda_2^*), \\ \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} \geq \dfrac{1}{\lambda_1^*} + \dfrac{1}{\lambda_2^*}. \end{cases}$$

Without loss of generality, let us assume $\lambda_1 \leq \lambda_2$ and $\lambda_1^* \leq \lambda_2^*$. Note that if $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} > \frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}$, then there exists some λ_2' such that $\lambda_2 < \lambda_2'$ and $\frac{1}{\lambda_1} + \frac{1}{\lambda_2'} = \frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}$. Let Z_1 and Z_2 be two independent exponential random variables with respective hazard rates λ_1 and λ_2'. By Lemma 2.A.8 of [8], it follows immediately that $X_1 + X_2 \geq_{mrl} Z_1 + Z_2$. Consequently, we find that it is enough to prove the necessity under the following simpler condition:

$$\begin{cases} \lambda_1 \leq \lambda_1^*, \\ \dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} = \dfrac{1}{\lambda_1^*} + \dfrac{1}{\lambda_2^*}. \end{cases}$$

Likewise, if $\lambda_1 = \lambda_1^*$, then $\lambda_2 = \lambda_2^*$ which becomes a trivial case. To obtain the desired result, we now need to distinguish the following two cases.

Case (a) $\lambda_1 < \lambda_1^* < \lambda_2^* < \lambda_2$.

The mean residual life function of $X_1 + X_2$ is, for $t \geq 0$,

$$\varphi_{(X_1,X_2)}(t) = \frac{\int_t^\infty \bar{F}_{(X_1,X_2)}(x)\mathrm{d}x}{\bar{F}_{(X_1,X_2)}(t)} = \frac{\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\frac{1}{\lambda_1^2}\mathrm{e}^{-\lambda_1 t} - \frac{1}{\lambda_2^2}\mathrm{e}^{-\lambda_2 t}\right)}{\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\frac{1}{\lambda_1}\mathrm{e}^{-\lambda_1 t} - \frac{1}{\lambda_2}\mathrm{e}^{-\lambda_2 t}\right)} = \frac{\frac{1}{\lambda_1^2}\mathrm{e}^{-\lambda_1 t} - \frac{1}{\lambda_2^2}\mathrm{e}^{-\lambda_2 t}}{\frac{1}{\lambda_1}\mathrm{e}^{-\lambda_1 t} - \frac{1}{\lambda_2}\mathrm{e}^{-\lambda_2 t}}.$$

Proceeding similarly, we obtain the mean residual life function of $X_1^* + X_2^*$ as, for $t \geq 0$,

$$\varphi_{(X_1^*,X_2^*)}(t) = \frac{\frac{1}{(\lambda_1^*)^2}\mathrm{e}^{-\lambda_1^* t} - \frac{1}{(\lambda_2^*)^2}\mathrm{e}^{-\lambda_2^* t}}{\frac{1}{\lambda_1^*}\mathrm{e}^{-\lambda_1^* t} - \frac{1}{\lambda_2^*}\mathrm{e}^{-\lambda_2^* t}}.$$

To conclude, we then need to show that, for all $t \geq 0$,

$$\varphi_{(X_1,X_2)}(t) = \frac{\frac{1}{\lambda_1^2}\mathrm{e}^{-\lambda_1 t} - \frac{1}{\lambda_2^2}\mathrm{e}^{-\lambda_2 t}}{\frac{1}{\lambda_1}\mathrm{e}^{-\lambda_1 t} - \frac{1}{\lambda_2}\mathrm{e}^{-\lambda_2 t}} \geq \frac{\frac{1}{(\lambda_1^*)^2}\mathrm{e}^{-\lambda_1^* t} - \frac{1}{(\lambda_2^*)^2}\mathrm{e}^{-\lambda_2^* t}}{\frac{1}{\lambda_1^*}\mathrm{e}^{-\lambda_1^* t} - \frac{1}{\lambda_2^*}\mathrm{e}^{-\lambda_2^* t}} = \varphi_{(X_1^*,X_2^*)}(t),$$

which is actually equivalent to showing

$$\varphi(y_1, y_2) = \frac{\frac{1}{y_1^2}\mathrm{e}^{-y_1} - \frac{1}{y_2^2}\mathrm{e}^{-y_2}}{\frac{1}{y_1}\mathrm{e}^{-y_1} - \frac{1}{y_2}\mathrm{e}^{-y_2}} \geq \frac{\frac{1}{(y_1^*)^2}\mathrm{e}^{-y_1^*} - \frac{1}{(y_2^*)^2}\mathrm{e}^{-y_2^*}}{\frac{1}{y_1^*}\mathrm{e}^{-y_1^*} - \frac{1}{y_2^*}\mathrm{e}^{-y_2^*}} = \varphi(y_1^*, y_2^*),$$

where $y_1 < y_1^* < y_2^* < y_2$ and $\frac{1}{y_1} + \frac{1}{y_2} = \frac{1}{y_1^*} + \frac{1}{y_2^*}$. Let $m = \frac{1}{y_1} + \frac{1}{y_2} = \frac{y_1+y_2}{y_1y_2}$. Then, the equation $y^2m - 2y = 0$ has a positive root $y = \frac{2}{m}$, which is in fact the harmonic mean of y_1 and y_2. Denote, for $t \in [y_1, \frac{2}{m})$,

$$g(t) = \frac{t}{mt-1}.$$

Clearly, $g(t)$ is a strictly decreasing function of t with $g(y_1) = y_2$. It is easy to see that

$$g'(t) = -\frac{mg(t)-1}{mt-1} = -\frac{\frac{g(t)}{t}}{\frac{t}{g(t)}}. \tag{8}$$

Now, the problem reduces to showing that the function

$$\varphi(t) = \frac{\frac{1}{t^2}e^{-t} - \frac{1}{g^2(t)}e^{-g(t)}}{\frac{1}{t}e^{-t} - \frac{1}{g(t)}e^{-g(t)}}$$

is decreasing in $t \in [y_1, \frac{2}{m})$. Note that, for $t \geq 0$,

$$\begin{aligned}
&\varphi'(t)\left[\frac{1}{t}e^{-t} - \frac{1}{g(t)}e^{-g(t)}\right]^2 \\
&= \left[\left(-\frac{2}{t^3}e^{-t} - \frac{1}{t^2}e^{-t}\right) - \left(-\frac{2}{g^3(t)}e^{-g(t)} - \frac{1}{g^2(t)}e^{-g(t)}\right)g'(t)\right] \times \left[\frac{1}{t}e^{-t} - \frac{1}{g(t)}e^{-g(t)}\right] \\
&\quad - \left[\left(-\frac{1}{t^2}e^{-t} - \frac{1}{t}e^{-t}\right) - \left(-\frac{1}{g^2(t)}e^{-g(t)} - \frac{1}{g(t)}e^{-g(t)}\right)g'(t)\right] \times \left[\frac{1}{t^2}e^{-t} - \frac{1}{g^2(t)}e^{-g(t)}\right] \\
&= \left[\left(\frac{1}{t^2}e^{-t} + \frac{1}{t}e^{-t}\right)\left(\frac{1}{t^2}e^{-t} - \frac{1}{g^2(t)}e^{-g(t)}\right) - \left(\frac{2}{t^3}e^{-t} + \frac{1}{t^2}e^{-t}\right)\left(\frac{1}{t}e^{-t} - \frac{1}{g(t)}e^{-g(t)}\right)\right] \\
&\quad + \left[\left(\frac{2}{g^3(t)}e^{-g(t)} + \frac{1}{g^2(t)}e^{-g(t)}\right)\left(\frac{1}{t}e^{-t} - \frac{1}{g(t)}e^{-g(t)}\right)\right. \\
&\quad \left. - \left(\frac{1}{g^2(t)}e^{-g(t)} + \frac{1}{g(t)}e^{-g(t)}\right)\left(\frac{1}{t^2}e^{-t} - \frac{1}{g^2(t)}e^{-g(t)}\right)\right]g'(t).
\end{aligned} \tag{9}$$

After some simplification and upon substituting (8) into (9), the RHS of (9) has the same sign as

$$\begin{aligned}
&\frac{t}{g(t)}\left[-\frac{1}{t^4}e^{-2t} + \left\{\frac{1}{g(t)}\left(\frac{2}{t^3} + \frac{1}{t^2}\right) - \frac{1}{g^2(t)}\left(\frac{1}{t^2} + \frac{1}{t}\right)\right\}e^{-(t+g(t))}\right] \\
&\quad - \frac{g(t)}{t}\left[-\frac{1}{g^4(t)}e^{-2g(t)} + \left\{\frac{1}{t}\left(\frac{2}{g^3(t)} + \frac{1}{g^2(t)}\right) - \frac{1}{t^2}\left(\frac{1}{g^2(t)} + \frac{1}{g(t)}\right)\right\}e^{-(t+g(t))}\right] \\
&\overset{\text{sgn}}{=} \left[\frac{1}{g^2(t)}e^{-2g(t)} - \frac{1}{t^2}e^{-2t}\right] + \left[\frac{1-t+g(t)}{t^2} - \frac{1+t-g(t)}{g^2(t)}\right]e^{-(t+g(t))} \\
&\overset{\text{sgn}}{=} \frac{1}{g^2(t)}e^{t-g(t)} - \frac{1}{t^2}e^{g(t)-t} + \frac{1-t+g(t)}{t^2} - \frac{1+t-g(t)}{g^2(t)} \\
&= \frac{1}{g^2(t)}\sum_{i=2}^{\infty}\frac{(t-g(t))^i}{i!} - \frac{1}{t^2}\sum_{i=2}^{\infty}\frac{(g(t)-t)^i}{i!} \\
&\leq \frac{1}{g^2(t)}\left[\sum_{i=2}^{\infty}\frac{(t-g(t))^i}{i!} - \sum_{i=2}^{\infty}\frac{(g(t)-t)^i}{i!}\right] \\
&= \frac{2}{g^2(t)}\sum_{i=1}^{\infty}\frac{(t-g(t))^{2i+1}}{(2i+1)!} \\
&\leq 0,
\end{aligned}$$

which completes the proof for Case (a).

Case (b) $\lambda_1 < \lambda_1^* = \lambda^* = \lambda_2^* < \lambda_2$.

In this case, $\lambda^* = \frac{2}{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}}$ is the harmonic mean of λ_1 and λ_2. It is then readily seen that $X_1^* + X_2^*$ is a gamma random variable with density function $(\lambda^*)^2 t e^{-\lambda^* t}$ for $t \geq 0$; so, the mean residual life function of $X_1^* + X_2^*$ can be written as

$$\varphi_{(X_1^*,X_2^*)}(t) = \frac{t + \frac{2}{\lambda^*}}{\lambda^* t + 1}.$$

Thus, it is sufficient to prove that, for all $t \geq 0$,

$$\varphi_{(X_1,X_2)}(t) = \frac{\frac{1}{\lambda_1^2}e^{-\lambda_1 t} - \frac{1}{\lambda_2^2}e^{-\lambda_2 t}}{\frac{1}{\lambda_1}e^{-\lambda_1 t} - \frac{1}{\lambda_2}e^{-\lambda_2 t}} \geq \frac{t + \frac{2}{\lambda^*}}{\lambda^* t + 1} = \varphi_{(X_1^*,X_2^*)}(t),$$

which is equivalent to proving the inequality

$$\frac{\frac{1}{y_1^2}e^{-y_1} - \frac{1}{y_2^2}e^{-y_2}}{\frac{1}{y_1}e^{-y_1} - \frac{1}{y_2}e^{-y_2}} \geq \frac{1 + \frac{2}{y^*}}{y^* + 1},$$

where $y^* = \frac{2}{\frac{1}{y_1}+\frac{1}{y_2}}$ is the harmonic mean of y_1 and y_2. As shown already in the proof of Case (a), the function

$$\varphi(t) = \frac{\frac{1}{t^2}e^{-t} - \frac{1}{g^2(t)}e^{-g(t)}}{\frac{1}{t}e^{-t} - \frac{1}{g(t)}e^{-g(t)}}$$

is decreasing in $t \in [y_1, y^*)$. Using l'Hospital's rule, we obtain

$$\begin{aligned}
\lim_{t\uparrow y^*} \varphi(t) &= \lim_{t\uparrow y^*} \frac{\frac{1}{t^2}e^{-t} - \frac{1}{g^2(t)}e^{-g(t)}}{\frac{1}{t}e^{-t} - \frac{1}{g(t)}e^{-g(t)}} \\
&= \lim_{t\uparrow y^*} \frac{\left(-\frac{2}{t^3} - \frac{1}{t^2}\right)e^{-t} - \left(-\frac{2}{g^3(t)} - \frac{1}{g^2(t)}\right)g'(t)e^{-g(t)}}{\left(-\frac{1}{t^2} - \frac{1}{t}\right)e^{-t} - \left(-\frac{1}{g^2(t)} - \frac{1}{g(t)}\right)g'(t)e^{-g(t)}} \\
&= \frac{1 + \frac{2}{y^*}}{y^* + 1},
\end{aligned}$$

which then implies

$$\frac{\frac{1}{y_1^2}e^{-y_1} - \frac{1}{y_2^2}e^{-y_2}}{\frac{1}{y_1}e^{-y_1} - \frac{1}{y_2}e^{-y_2}} = \varphi(y_1) \geq \lim_{t\uparrow y^*}\varphi(t) = \frac{1 + \frac{2}{y^*}}{y^* + 1},$$

and this completes the proof of Case (b).

$\Longrightarrow$ Suppose $X_1 + X_2 \geq_{mrl} X_1^* + X_2^*$. Assume that $\lambda_1 \neq \lambda_2$ and $\lambda_1^* \neq \lambda_2^*$. It can then be verified that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\varphi_{(X_1,X_2)}(t)}{\varphi_{(X_1^*,X_2^*)}(t)} &= \lim_{t\to\infty} \frac{\dfrac{\frac{1}{\lambda_1^2}e^{-\lambda_1 t} - \frac{1}{\lambda_2^2}e^{-\lambda_2 t}}{\frac{1}{\lambda_1}e^{-\lambda_1 t} - \frac{1}{\lambda_2}e^{-\lambda_2 t}}}{\dfrac{\frac{1}{(\lambda_1^*)^2}e^{-\lambda_1^* t} - \frac{1}{(\lambda_2^*)^2}e^{-\lambda_2^* t}}{\frac{1}{\lambda_1^*}e^{-\lambda_1^* t} - \frac{1}{\lambda_2^*}e^{-\lambda_2^* t}}} \\
&= \lim_{t\to\infty} \frac{\left[\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}e^{(\lambda_1-\lambda_2)t}\right]\left[\frac{1}{\lambda_1^*} - \frac{1}{\lambda_2^*}e^{(\lambda_1^*-\lambda_2^*)t}\right]}{\left[\frac{1}{(\lambda_1^*)^2} - \frac{1}{(\lambda_2^*)^2}e^{(\lambda_1^*-\lambda_2^*)t}\right]\left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2}e^{(\lambda_1-\lambda_2)t}\right]} \\
&= \frac{\lambda_1^*}{\lambda_1}.
\end{aligned}$$

Since the assumption $X_1 + X_2 \geq_{mrl} X_1^* + X_2^*$ implies that $\varphi_{(X_1,X_2)}(t) \geq \varphi_{(X_1^*,X_2^*)}(t)$ for all $t > 0$, it follows that

$$\lim_{t\to\infty} \frac{\varphi_{(X_1,X_2)}(t)}{\varphi_{(X_1^*,X_2^*)}(t)} \geq 1,$$

and hence we conclude that $\lambda_1 \le \lambda_1^*$. If $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_1^* \ne \lambda_2^*$, we have

$$\lim_{t\to\infty} \frac{\varphi_{(X_1,X_2)}(t)}{\varphi_{(X_1^*,X_2^*)}(t)} = \lim_{t\to\infty} \frac{\frac{t+\frac{2}{\lambda^*}}{\lambda^* t+1}}{\frac{\frac{1}{(\lambda_1^*)^2}e^{-\lambda_1^* t}-\frac{1}{(\lambda_2^*)^2}e^{-\lambda_2^* t}}{\frac{1}{\lambda_1^*}e^{-\lambda_1^* t}-\frac{1}{\lambda_2^*}e^{-\lambda_2^* t}}} = \frac{\lambda_1^*}{\lambda}.$$

From an argument similar to the one above, we get $\lambda_1^* \ge \lambda$. Similarly, in the cases $\lambda_1 \ne \lambda_2$ and $\lambda_1^* = \lambda_2^* = \lambda^*$, and $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_1^* = \lambda_2^* = \lambda^*$, the desired conclusion can be obtained.

On the other hand, for the case $\lambda_1 \ne \lambda_2$ and $\lambda_1^* \ne \lambda_2^*$, by using Taylor's expansion at the origin, we have, for $t > 0$,

$$\varphi_{(X_1,X_2)}(t) = \frac{\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} + o(1)}{\frac{1}{\lambda_1} - \frac{1}{\lambda_2} + o(1)} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + o(1).$$

Similarly,

$$\varphi_{(X_1^*,X_2^*)}(t) = \frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*} + o(1).$$

Clearly, we have $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \ge \frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}$. The remaining cases when either $\lambda_1 = \lambda_2$ or $\lambda_1^* = \lambda_2^*$ can also be handled by using the fact

$$\varphi_{(X,X)}(t) = \frac{t+\frac{2}{\lambda}}{\lambda t + 1} = \frac{2}{\lambda} + o(1)$$

near the origin.

Thus, the equivalence in (7) is proved. ■

4. MRL order between convolutions of exponential random variables

The following result is a natural extension of Theorem 3.1 from the special case of $n = 2$ to the general case.

Theorem 4.1. *Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be independent exponential random variables with hazard rates $\lambda_1, \ldots, \lambda_n, \lambda_1^*, \ldots, \lambda_n^*$, respectively.*

(1) *If $(\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*)$, then $X_1 + \cdots + X_n \ge_{mrl} Y_1 + \cdots + Y_n$;*
(2) *Conversely, if $X_1 + \cdots + X_n \ge_{mrl} Y_1 + \cdots + Y_n$, then*

$$\min\{\lambda_1, \ldots, \lambda_n\} \le \min\{\lambda_1^*, \ldots, \lambda_n^*\} \quad \text{and} \quad \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \ge \frac{1}{\lambda_1^*} + \cdots + \frac{1}{\lambda_n^*}.$$

Proof. Without loss of generality, we may assume that $\lambda_1 \le \cdots \le \lambda_n$ and $\lambda_1^* \le \cdots \le \lambda_n^*$.

Proof of (1) Suppose $(\lambda_1, \ldots, \lambda_n) \overset{\text{rm}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*)$, i.e., $\sum_{i=1}^{j} \frac{1}{\lambda_i} \ge \sum_{i=1}^{j} \frac{1}{\lambda_i^*}$, $j = 1, \ldots, n$. The proof is carried out by induction. The result is obvious for the case $n = 1$ and can also be readily obtained from Theorem 3.1 for the case $n = 2$. Now, let us suppose that the result is true for $n-1$ ($n \ge 3$). To prove the required result for n, let us distinguish two cases.

Case (i) $\lambda_1^* < \lambda_n$.

Based on the fact $\lambda_1 \le \lambda_1^* < \lambda_n$, there exists exactly one integer k ($1 \le k \le n-1$) such that $\lambda_k \le \lambda_1^* < \lambda_{k+1}$. Now, upon observing that $(\lambda_k, \lambda_{k+1}) \overset{\text{rm}}{\succeq} \left(\lambda_1^*, \frac{1}{\frac{1}{\lambda_k}+\frac{1}{\lambda_{k+1}}-\frac{1}{\lambda_1^*}}\right)$, it follows from Theorem 3.1 that

$$X_k + X_{k+1} \ge_{mrl} Y_1 + W, \tag{10}$$

where W is an exponential random variable with hazard rate $\frac{1}{\frac{1}{\lambda_k}+\frac{1}{\lambda_{k+1}}-\frac{1}{\lambda_1^*}}$, independent of X_i and Y_i for all $1 \le i \le n$. On the other hand, it may also be observed that

$$\left(\lambda_1, \ldots \lambda_{k-1}, \frac{1}{\frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}} - \frac{1}{\lambda_1^*}}, \lambda_{k+2}, \ldots, \lambda_n\right) \overset{\text{rm}}{\succeq} (\lambda_2^*, \ldots, \lambda_k^*, \lambda_{k+1}^*, \ldots, \lambda_n^*).$$

Upon using the induction assumption now, we obtain

$$X_1 + \cdots + X_{k-1} + W + X_{k+2} + \cdots + X_n \ge_{mrl} Y_2 + \cdots + Y_k + Y_{k+1} + \cdots + Y_n. \tag{11}$$

It is known from Lemma 2.A.8 of [8] that if the random variables X and Y are such that $X \leq_{mrl} Y$ and if Z is an IFR random variable which is independent of X and Y, then $X+Z \leq_{mrl} Y+Z$. Also, a convolution of distributions with logconcave densities has a logconcave density, and hence possesses IFR property. Upon using these two facts and the inequalities in (10) and (11), we finally can conclude

$$X_1 + \cdots + X_n \geq_{mrl} X_1 + \cdots + X_{k-1} + Y_1 + W + X_{k+2} + \cdots + X_n \geq_{mrl} Y_1 + \cdots + Y_n.$$

Case (ii) $\lambda_1^* \geq \lambda_n$.

In this case, we have $\lambda_i \leq \lambda_i^*$ for all $1 \leq i \leq n$, and the result is then a direct application of Theorem 2.A.9 of [8].

Proof of (2) From Sen and Balakrishnan [12], when the parameters λ_i's are pairwise unequal, i.e., $\lambda_i \neq \lambda_j$ for any $i \neq j$, the density function of $S_n = X_1 + \cdots + X_n$ is given by the formula

$$f_{S_n}(t) = \sum_{i=1}^{n} \lambda_i e^{-\lambda_i t} \prod_{j=1, j\neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right), \quad t \geq 0.$$

Consequently, the mean residual life function of S_n may be written as

$$\varphi_{S_n}(t) = \frac{\int_t^\infty \bar{F}_{S_n}(u) du}{\bar{F}_{S_n}(t)} = \frac{\sum_{i=1}^{n} \frac{1}{\lambda_i} e^{-\lambda_i t} \prod_{j=1, j\neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right)}{\sum_{i=1}^{n} e^{-\lambda_i t} \prod_{j=1, j\neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i} \right)}, \quad t \geq 0, \tag{12}$$

where $\bar{F}_{S_n}$ denotes the survival function of S_n. Using Taylor's expansion at the origin, we obtain, for $t > 0$,

$$\varphi_{S_n}(t) = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} + o(1).$$

In a similar way, the mean residual life function of $S_n^* = X_1^* + \cdots + X_n^*$ is given by

$$\varphi_{S_n^*}(t) = \frac{1}{\lambda_1^*} + \cdots + \frac{1}{\lambda_n^*} + o(1) \quad \text{for } t \geq 0.$$

From the assumption, it follows that $\varphi_{S_n}(t) \geq \varphi_{S_n^*}(t)$ for all $t \geq 0$, and hence

$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} = \lim_{t\to 0, t>0} \varphi_{S_n}(t) \geq \lim_{t\to 0, t>0} \varphi_{S_n^*}(t) = \frac{1}{\lambda_1^*} + \cdots + \frac{1}{\lambda_n^*}.$$

Assume both λ_i and λ_i^* are pairwise unequal. From (12), we have

$$\frac{\varphi_{S_n}(t)}{\varphi_{S_n^*}(t)} = e^{(\lambda_1^* - \lambda_1)t} \frac{\frac{1}{\lambda_1} \prod_{i\neq 1}^{n} \frac{\lambda_i}{\lambda_i - \lambda_1} + \sum_{k=2}^{n} \frac{1}{\lambda_k} e^{-(\lambda_k - \lambda_1)t} \prod_{i\neq k}^{n} \frac{\lambda_i}{\lambda_i - \lambda_k}}{\frac{1}{\lambda_1^*} \prod_{i\neq 1}^{n} \frac{\lambda_i^*}{\lambda_i^* - \lambda_1^*} + \sum_{k=2}^{n} \frac{1}{\lambda_k^*} e^{-(\lambda_k^* - \lambda_1^*)t} \prod_{i\neq k}^{n} \frac{\lambda_i^*}{\lambda_i^* - \lambda_k^*}} \times \frac{\prod_{i\neq 1}^{n} \frac{\lambda_i^*}{\lambda_i^* - \lambda_1^*} + \sum_{k=2}^{n} e^{-(\lambda_k^* - \lambda_1^*)t} \prod_{i\neq k}^{n} \frac{\lambda_i^*}{\lambda_i^* - \lambda_k^*}}{\prod_{i\neq 1}^{n} \frac{\lambda_i}{\lambda_i - \lambda_1} + \sum_{k=2}^{n} e^{-(\lambda_k - \lambda_1)t} \prod_{i\neq k}^{n} \frac{\lambda_i}{\lambda_i - \lambda_k}}.$$

If $\lambda_1 > \lambda_1^*$, then $\lim_{t\to\infty} \varphi_{S_n}(t)/\varphi_{S_n^*}(t) = 0$, which contradicts the hypothesis that $\varphi_{S_n}(t)/\varphi_{S_n^*}(t) \geq 1$ for all $t > 0$. Therefore, we can claim that $\lambda_1 \leq \lambda_1^*$. If either λ_i or λ_i^* are not pairwise unequal, the desired result can be obtained by using a limiting argument.

The theorem is thus proved. ■

In what follows, we focus on the special case when one sum involves identically distributed random variables. Bon and Păltănea [4] summed up the connections between some classical stochastic orders for convolutions of exponential random variables and various means (arithmetic mean, geometric mean and harmonic mean) of their parameters.

Proposition 4.2 (*[4]*). *Let $S_n(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^n X_i$ and $T_n(\lambda, \ldots, \lambda) = \sum_{i=1}^n Y_i$, where X_i $(1 \leq i \leq n)$ are independent exponential random variables with respective parameters λ_i, and Y_i are i.i.d. exponential random variables with a common parameter λ. Then:*

(i) $S_n(\lambda_1, \ldots, \lambda_n) \geq_{mn} T_n(\lambda, \ldots, \lambda) \iff \lambda \geq \frac{n}{\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n}}$;
(ii) $S_n(\lambda_1, \ldots, \lambda_n) \geq_{st} T_n(\lambda, \ldots, \lambda) \iff \lambda \geq \sqrt[n]{\lambda_1 \ldots \lambda_n}$;
(iii) $S_n(\lambda_1, \ldots, \lambda_n) \geq_{hr} T_n(\lambda, \ldots, \lambda) \iff \lambda \geq \sqrt[n]{\lambda_1 \ldots \lambda_n}$;
(iv) $S_n(\lambda_1, \ldots, \lambda_n) \geq_{lr} T_n(\lambda, \ldots, \lambda) \iff \lambda \geq \frac{\lambda_1 + \cdots + \lambda_n}{n}$;
(v) $S_n(\lambda_1, \ldots, \lambda_n) \leq_{lr} (\leq_{hr}, \leq_{st}) T_n(\lambda, \ldots, \lambda) \iff \lambda \leq \min(\lambda_1, \ldots, \lambda_n)$.

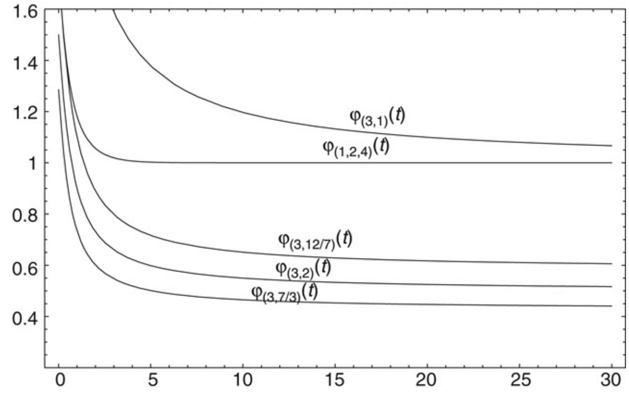

Fig. 1. Plot of mean residual life function of convolution of three exponentials (1, 2, 4) and those of gamma random variables with parameters as arithmetic mean, geometric mean, harmonic mean, and 1, respectively.

As a direct application of Theorem 4.1, we can readily obtain the following corollary which corresponds to the mean residual life order, and thus forms a nice extension of Proposition 4.2.

Corollary 4.3. *Under the assumptions of Proposition 4.2, we have:*

(vi) $S_n(\lambda_1,\ldots,\lambda_n) \geq_{mrl} T_n(\lambda,\ldots,\lambda) \Longleftrightarrow \lambda \geq \frac{n}{\frac{1}{\lambda_1}+\cdots+\frac{1}{\lambda_n}}$;

(vii) $S_n(\lambda_1,\ldots,\lambda_n) \leq_{mrl} T_n(\lambda,\ldots,\lambda) \Longleftrightarrow \lambda \leq \min(\lambda_1,\ldots,\lambda_n)$.

It is well known that $T_n(\lambda,\ldots,\lambda)$ is a gamma random variable with parameters (n,λ) with density function

$$f_{(n,\lambda)}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}\mathrm{e}^{-\lambda t}, \quad t \geq 0.$$

Let $\varphi_{(n,\lambda)}(t)$ be the mean residual life function of the gamma random variable $T_n(\lambda,\ldots,\lambda)$, and let $\varphi_{(\lambda_1,\ldots,\lambda_n)}(t)$ be the mean residual life function of the convolution $S_n(\lambda_1,\ldots,\lambda_n)$. Then, from Corollary 4.3, the best gamma bounds for the mean residual life function $\varphi_{(\lambda_1,\ldots,\lambda_n)}(t)$ can be derived as follows:

$$\varphi_{\left(n,\frac{n}{\frac{1}{\lambda_1}+\cdots+\frac{1}{\lambda_n}}\right)}(t) \leq \varphi_{(\lambda_1,\ldots,\lambda_n)}(t) \leq \varphi_{(n,\min\limits_{1\leq i\leq n}\lambda_i)}(t)$$

for all $t \geq 0$.

We now present an example to illustrate the best gamma bounds. For the case $n = 3$, we have

$$\varphi_{(\lambda_1,\lambda_2,\lambda_3)}(t) = \frac{\frac{1}{\lambda_1}\mathrm{e}^{-\lambda_1 t}\left[\frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\right] + \frac{1}{\lambda_2}\mathrm{e}^{-\lambda_2 t}\left[\frac{\lambda_1\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\right] + \frac{1}{\lambda_3}\mathrm{e}^{-\lambda_3 t}\left[\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right]}{\mathrm{e}^{-\lambda_1 t}\left[\frac{\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\right] + \mathrm{e}^{-\lambda_2 t}\left[\frac{\lambda_1\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\right] + \mathrm{e}^{-\lambda_3 t}\left[\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right]}$$

and

$$\varphi_{(3,\lambda)}(t) = \frac{\lambda^2t^2+4\lambda t+6}{\lambda^3t^2+2\lambda^2t+2\lambda}.$$

Let us choose $(\lambda_1,\lambda_2,\lambda_3) = (1,2,4)$. Then, the arithmetic mean, geometric mean and harmonic mean are $\frac{\lambda_1+\lambda_2+\lambda_3}{3} = \frac{7}{3}$, $\sqrt[3]{\lambda_1\lambda_2\lambda_3} = 2$, and $\frac{3}{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}} = \frac{12}{7}$, respectively, and $\min\{\lambda_1,\lambda_2,\lambda_3\} = 1$. Then, Fig. 1 illustrates the following inequality:

$$\varphi_{(3,\frac{7}{3})}(t) \leq \varphi_{(3,2)}(t) \leq \varphi_{(3,\frac{12}{7})}(t) \leq \varphi_{(1,2,4)}(t) \leq \varphi_{(3,1)}(t).$$

Clearly, for the mean residual life function $\varphi_{(1,2,4)}(t)$, the best gamma lower bound $\varphi_{(3,\frac{12}{7})}(t)$ is the best approximation near the origin, while the best gamma upper bound $\varphi_{(3,1)}(t)$ is the best approximation in the right tail.

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