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# Sequences of elliptical distributions and mixtures of normal distributions

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**Abstract**

Two conditions are shown under which elliptical distributions are scale mixtures of normal distributions with respect to probability distributions. The issue of finding the mixing distribution function is also considered. As a unified theoretical framework, it is also shown that any scale mixture of normal distributions is always a term of a sequence of elliptical distributions, increasing in dimension, and that all the terms of this sequence are also scale mixtures of normal distributions sharing the same mixing distribution function. Some examples are shown as applications of these concepts, showing the way of finding the mixing distribution function.


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## 1. Introduction

In this paper, two conditions for an absolutely continuous elliptical distribution to be a scale mixture of normal distributions, with respect to a probability distribution, are shown, and the issue of finding the mixing distribution function is approached; also, it is shown that any scale mixture of normal distributions can be viewed as a term of a sequence of elliptical distributions. The proof of the main results relies on the concept of an expansive sequence of elliptical distributions, that we state previously, and on the existence of some distributions of higher dimensions shown by Gupta and Varga [8] and Eaton [4].

Scale mixtures of normal distributions are an important class of elliptical distributions. They share good properties with normal distributions, are easy to work with, and are useful to robustify statistical procedures usually based on normal distributions. In a Bayesian framework, they can be used in simulation methods. So, a characterization and a way to reduce elliptical distributions to scale mixtures of normal distributions is very valuable.

There are many publications on elliptical distributions. For a comprehensive study of elliptical distributions, Kelker [10], Chu [3], Cambanis et al. [2], Fang et al. [5], Fang and Zhang [6] and Gupta and Varga [8] can be seen. A survey about absolutely continuous elliptical distributions can be found in [7].

Some results on scale mixtures of normal distributions and their relationship with elliptical distributions can be found in [3–5,8,9]. Some of these approaches do not keep to probability mixing distributions (as we do in this paper) but allow "weighting functions" that can turn out to take negative values. Andrews and Mallows [1] study conditions for a unidimensional symmetrical distribution to be a scale mixture of normal distributions.

The first condition that we show for an absolutely continuous elliptical distribution to be a scale mixture of normal distribution is based upon the successive derivatives of its functional parameter $g$, and includes the elliptical distribution in a sequence. In this way, these results give an interpretation of derivatives of $g$ in a probabilistic framework, extend the theorem of Andrews and Mallows [1], reinterpret, in the framework of mixtures of normals, Proposition 1 of Eaton [4] and supplement, in some sense, Theorem 4.1.3 of Gupta and Varga [8].

The second condition refers to Laplace transforms, and also shows the mixing distribution function. These results particularize some aspects of a theorem of Chu [3] (see also [8]) and extend the lemma in Andrews and Mallows [1].

In Section 2, we introduce the concepts of expansive and semi-expansive sequences of elliptical distributions and establish conditions under which an elliptical distribution can be a term of an expansive sequence. These concepts provide a general theoretical framework for the subsequent study of mixtures of normal distributions.

In Section 3 the two conditions are shown. Finally, in Section 4, some examples and applications to the study of elliptical distributions are shown.

## 2. Sequences of elliptical distributions

We show the definition and some properties of elliptical distributions, introduce the concepts of expansive and semi-expansive sequences of elliptical distributions, and study some of their properties.

### 2.1. Some properties of elliptical distributions

We deal with elliptical distributions that are absolutely continuous.

**Definition 1** (*Elliptical distribution*). If $\mu$ is an $n$-dimensional vector, $\Sigma$ is an $n \times n$ positive definite symmetric matrix and $g$ is a non-negative Lebesgue measurable function on $[0, \infty)$ such that

$$0 < \int_0^\infty t^{\frac{n}{2}-1} g(t)\, dt < \infty, \tag{2.1}$$

then the $n$-dimensional density $f$ given by

$$f(x; \mu, \Sigma, g) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}} \int_0^\infty t^{\frac{n}{2}-1} g(t)\, dt} |\Sigma|^{-\frac{1}{2}} g\left((x-\mu)' \Sigma^{-1} (x-\mu)\right) \tag{2.2}$$

is said to be *elliptical* with parameters $\mu$, $\Sigma$ and $g$. If vector $X$ has density (2.2), we say that $X$ has the elliptical distribution (e.d.) $\mathrm{E}_n(\mu, \Sigma, g)$ and write $X \sim \mathrm{E}_n(\mu, \Sigma, g)$.

We will refer to function $g$ by the name of *functional parameter*.

The parametrization of an elliptical distribution is not strictly unique. Suppose that $X \sim \mathrm{E}_n(\mu, \Sigma, g)$; then $X \sim \mathrm{E}_n\left(\mu^*, \Sigma^*, g^*\right)$ iff there exist two positive numbers $a$ and $b$, such that $\mu^* = \mu$, $\Sigma^* = a\Sigma$ and $g^*(t) = bg(at)$ for almost all $t \geqslant 0$ [7]; see also [6]).

We will say that two real functions $g$ and $g^*$ are *equivalent* if $g = bg^*$ a.e. for some $b > 0$; we will denote $g \equiv g^*$. Thus, the functional parameter $g$ of an e.d. $\mathrm{E}_n(\mu, \Sigma, g)$ can be replaced with another $g^*$ (keeping the same parameters $\mu$ and $\Sigma$) iff $g^* \equiv g$.

For each vector $x = (x_1, \ldots, x_n)'$, we will denote $x_{(p)} = \left(x_1, \ldots, x_p\right)'$, for $p \leqslant n$; also, $\Sigma_{(p)}$ will denote the upper-left $p \times p$ submatrix of the $n \times n$ matrix $\Sigma$.

If $X \sim \mathrm{E}_n(\mu, \Sigma, g)$ and $X_{(p)} = \left(X_1, \ldots, X_p\right)'$, with $p < n$, then (see [6,7])

$$X_{(p)} \sim \mathrm{E}_p\left(\mu_{(p)}, \Sigma_{(p)}, g_{(p)}\right), \tag{2.3}$$

where $g_{(p)}$ is the function given by

$$g_{(p)}(t) = \int_0^\infty w^{\frac{n-p}{2}-1} g(t+w)\, dw \tag{2.4}$$

$$= \int_t^\infty (w-t)^{\frac{n-p}{2}-1} g(w)\, dw. \tag{2.5}$$

In particular, if $p = n-2$, then

$$g_{(n-2)}(t) = \int_t^\infty g(w)\, dw \tag{2.6}$$

and, thence, $g'_{(n-2)}(t) = -g(t)$ for each continuity point $t$ of $g$. The following lemma develops this subject. We denote by $0_{n\times m}$ the null $n \times m$ matrix and by $I_n$ the identity $n \times n$ matrix.

**Lemma 2** (*Derivative of marginal parameter*). *Let $X \sim \mathrm{E}_n(0_{n\times 1}, I_n, g)$ and $Y \sim \mathrm{E}_{n+2}\left(0_{(n+2)\times 1}, I_{n+2}, h\right)$ for some $n \geqslant 1$. Equality in distribution $X \overset{\mathrm{d}}{=} Y_{(n)}$ holds iff $g' \equiv -h$.*

**Proof.** $Y_{(n)} \sim \mathrm{E}_n\left(0_{n\times 1}, I_n, h_{(n)}\right)$, where from (2.6), $h_{(n)}(t) = \int_t^\infty h(w)\,dw$; therefore, $h'_{(n)} = -h$ a.e.

If $X \stackrel{\mathrm{d}}{=} Y_{(n)}$, then $g \equiv h_{(n)}$ and $g' \equiv h'_{(n)} \equiv -h$.

Reciprocally, if $g' \equiv -h$, then there is a positive number $b$ such that $bg' = -h = h'_{(n)}$ a.e. and, consequently, $bg = h_{(n)} + c$ a.e., where $c$ is a constant. But $c = 0$ because $bg$ and $h_{(n)}$ are functional parameters of $n$-dimensional e.d.'s, and then, from (2.1), $\int_0^\infty t^{\frac{n}{2}-1} bg(t)\,dt < \infty$ and $\int_0^\infty t^{\frac{n}{2}-1} h_{(n)}(t)\,dt < \infty$; therefore,

$$\int_0^\infty t^{\frac{n}{2}-1} c\,dt = \int_0^\infty t^{\frac{n}{2}-1} bg(t)\; dt - \int_0^\infty t^{\frac{n}{2}-1} h_{(n)}(t)\,dt < \infty$$

and it is possible only if $c = 0$. Thence $bg = h_{(n)}$ a.e. and $X \sim \mathrm{E}_n(0_{n\times 1}, I_n, h_{(n)})$. □

## 2.2. Expansive and semi-expansive sequences

The concepts of expansive and semi-expansive sequences of elliptical distributions are introduced. A sequence is expansive (alternatively: semi-expansive) if each one of its terms is equivalent to a marginal distribution of the next term (alt.: of the term following the next term). These concepts allow us to establish relations between the derivatives of the functional parameter of a distribution and the functional parameters of some distributions posterior to it in the sequence. And this permits us to derive conditions under which e.d.'s are scale mixtures of normal distributions.

Theorem 4 shows a condition for a sequence of e.d.'s to be semi-expansive and Theorem 6 shows a condition for terms of expansive sequences.

When a term of a sequence is a vector or a matrix we use superscripts to denote its position in the sequence and subscripts to denote its components.

**Definition 3** (*Expansive and semi-expansive sequences*). For all $m \in \{1, 2, \ldots\}$, let $\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)$ be an $m$-dimensional elliptical distribution and let $X^m = \left(X_1^m, \ldots, X_m^m\right)' \sim \mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)$.

(i) The sequence of distributions $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$ is said to be *expansive* if, for all $m \in \{1, 2, \ldots\}$,

$$X^m \stackrel{\mathrm{d}}{=} X_{(m)}^{m+1} = \left(X_1^{m+1}, \ldots, X_m^{m+1}\right)', \tag{2.7}$$

namely, if the distribution of vector $X^m$ is equal to the (marginal) distribution of a subvector of vector $X^{m+1}$.

(ii) The sequence $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$ is said to be *semi-expansive* if, for all $m \in \{1, 2, \ldots\}$,

$$X^m \stackrel{\mathrm{d}}{=} X_{(m)}^{m+2} = \left(X_1^{m+2}, \ldots, X_m^{m+2}\right)'. \tag{2.8}$$

Clearly, all expansive sequences are semi-expansive and a semi-expansive sequence is expansive if, in addition, $X^m \stackrel{\mathrm{d}}{=} X_{(m)}^{m+1}$ for all odd $m$.

The next theorem shows a condition, based on the functional parameter $g$, for a sequence to be semi-expansive.

(On notation: we use sometimes signs such as $\dot{g}$ or $\ddot{g}$ simply to denote functions, with no reference to derivatives.)

**Theorem 4** (*Condition for semi-expansivity*). *The sequence of e.d.'s* $\{\mathrm{E}_m(0_{m\times 1}, I_m, g_m)\}$ *is semi-expansive iff there exists a sequence* $\{g_m^*\}$ *of functions such that*

$$g_m^* \equiv g_m, \tag{2.9}$$

$$g_{m+2}^* = -\left(g_m^*\right)' \tag{2.10}$$

*for all* $m \in \{1, 2, \dots\}$ *and* $t > 0$.

**Proof.** (*The if part*): For all $m$, let $X^m = \left(X_1^m, \dots, X_m^m\right)' \sim \mathrm{E}_m(0_{m\times 1}, I_m, g_m)$. From (2.9) and (2.10) we have $g_m' \equiv -g_{m+2}$; hence, by virtue of Lemma 2, $X^m \overset{\mathrm{d}}{=} X_{(m)}^{m+2}$.

(*The only if part*): For all $m \in \{1, 2, \dots\}$, let $X^m \sim \mathrm{E}_m(0_{m\times 1}, I_m, g_m)$.

First, we are going to obtain differentiable functions $\ddot{g}_m$ such that $\ddot{g}_m \equiv g_m$. For all $m$, by virtue of (2.8) and (2.3) (see also (2.6)), $X^m \sim \mathrm{E}_m(0_{m\times 1}, I_m, \dot{g}_m)$, with $\dot{g}_m(t) = \int_t^\infty g_{m+2}(w)\,dw$; hence $\dot{g}_m \equiv g_m$; and function $\dot{g}_m$ is continuous. Again by (2.8) and (2.3), we obtain that $\dot{g}_m \equiv \ddot{g}_m$, where $\ddot{g}_m(t) = \int_t^\infty \dot{g}_{m+2}(w)\,dw$. Thus $g_m \equiv \ddot{g}_m$, and function $\ddot{g}_m$ is differentiable.

For all $m$, by obtaining again another equivalent function by means of (2.8) and (2.3), we find that there exists a constant $b_{m+2} > 0$ such that $\ddot{g}_m(t) = b_{m+2}\int_t^\infty \ddot{g}_{m+2}(w)\,dw$ and, consequently, $\ddot{g}_m' = -b_{m+2}\ddot{g}_{m+2}$. We add $b_1 = b_2 = 1$, and for all $k \in \{0, 1, 2, \dots\}$ we define functions $g_{1+2k}^*$ and $g_{2+2k}^*$ as follows:

$$g_{1+2k}^* = \left(\prod_{j=0}^{k} b_{1+2j}\right)\ddot{g}_{1+2k},$$

$$g_{2+2k}^* = \left(\prod_{j=0}^{k} b_{2+2j}\right)\ddot{g}_{2+2k}.$$

All the elements of sequence $\{g_m^*\}$ satisfy (2.9), because $g_m^* \equiv \ddot{g}_m \equiv g_m$. We check that they also satisfy (2.10). For all $k \in \{0, 1, 2, \dots\}$ we have that

$$\begin{aligned}\left(g_{1+2k}^*\right)' &= \left(\prod_{j=0}^{k} b_{1+2j}\right)\ddot{g}_{1+2k}' = \left(\prod_{j=0}^{k} b_{1+2j}\right)\left(-b_{1+2k+2}\ddot{g}_{1+2k+2}\right)\\ &= -\left(\prod_{j=0}^{k+1} b_{1+2j}\right)\ddot{g}_{1+2(k+1)} = -g_{1+2k+2}^*.\end{aligned}$$

Similarly, it is proved that $\left(g_{2+2k}^*\right)' = -g_{2+2k+2}^*$. $\square$

By applying iteratively (2.10) for $m = 1, 3, 5, \ldots$ and then for $m = 2, 4, 6, \ldots$ we see that a sequence $\{\mathrm{E}_m\,(0_{m\times 1}, I_m, g_m)\}$ is semi-expansive iff there exist functions $g_m^* \equiv g_m$ such that

$$g_{1+2k}^* = (-1)^k \left(g_1^*\right)^{(k)}, \tag{2.11}$$

$$g_{2+2k}^* = (-1)^k \left(g_2^*\right)^{(k)} \tag{2.12}$$

for $k \in \{1, 2, \ldots\}$.

We notice, by (2.11) and (2.12), that a semi-expansive sequence $\{\mathrm{E}_m\,(0_{m\times 1}, I_m, g_m)\}$ is determined by $g_1$ and $g_2$. It is also determined by any pair of terms $g_{n_1}$ and $g_{n_2}$, with $n_1$ even and $n_2$ odd, of the sequence $\{g_m\}$, because from them equivalent functions to $g_1$ and $g_2$ can be obtained by applying any of the expressions (2.4) or (2.5).

An expansive sequence $\{\mathrm{E}_m\,(0_{m\times 1}, I_m, g_m)\}$ is determined by any term, $g_n$, of the sequence $\{g_m\}$, because from this one equivalent functions to $g_1$, $g_3$ and $g_2$ can be obtained successively by applying (2.4) or (2.5), (2.11), and (2.4) or (2.5), respectively.

To prove Theorem 6 we need to establish first the following lemma.

**Lemma 5** (*Distributions of higher dimension*). *Let* $\mathrm{E}_n\,(\mu, \Sigma, g)$ *be an elliptical distribution, with* $n \geqslant 1$. *If*

$$(-1)^k\, g^{(k)}(t) \geqslant 0 \tag{2.13}$$

*for* $k \in \{1, 2, \ldots\}$ *and* $t > 0$, *then, for all* $k \in \{0, 1, 2, \ldots\}$,

$$0 < \int_0^\infty t^{\frac{n+2k}{2}-1} (-1)^k\, g^{(k)}(t)\, dt < \infty \tag{2.14}$$

*and, hence, the distribution* $\mathrm{E}_{n+2k}\left(\mu^{n+2k}, \Sigma^{n+2k}, g_{n+2k}\right)$, *with*

$$g_{n+2k} = (-1)^k\, g^{(k)}$$

*does exist for each vector* $\mu^{n+2k} \in \mathbb{R}^{n+2k}$ *and each positive definite symmetric* $(n+2k) \times (n+2k)$ *matrix* $\Sigma^{n+2k}$.

**Proof.** We prove (2.14) by induction on $k$. From (2.1), the statement is true for $k = 0$. Let, now, $k > 0$; we suppose that the statement is true for $k - 1$ and we prove it for $k$. By integrating by parts we have

$$\int_0^\infty t^{\frac{n+2k}{2}-1} (-1)^k\, g^{(k)}(t)\, dt \tag{2.15}$$

$$= \lim_{t\to 0} t^{\frac{n+2k}{2}-1} (-1)^{k-1}\, g^{(k-1)}(t) \tag{2.16}$$

$$- \lim_{t\to \infty} t^{\frac{n+2k}{2}-1} (-1)^{k-1}\, g^{(k-1)}(t) \tag{2.17}$$

$$+ \left(\frac{n+2k}{2} - 1\right) \int_0^\infty t^{\frac{n+2(k-1)}{2}-1} (-1)^{k-1}\, g^{(k-1)}(t)\, dt. \tag{2.18}$$

The integrals in (2.15) and (2.18) exist and are non-negative. Hence, there exist the limits (2.16) and (2.17). Clearly, they are not negative. We prove that both limits are zero by reduction to absurd.

If the limit (2.16) would be equal to $c > 0$, there should be a point $t_1 > 0$ such that for every $t \in (0, t_1)$ it would be $t^{\frac{n+2k}{2}-1}\,(-1)^{k-1}\,g^{(k-1)}(t) > \frac{c}{2}$; thence it would be

$$\int_0^\infty t^{\frac{n+2(k-1)}{2}-1}\,(-1)^{k-1}\,g^{(k-1)}(t)\,dt \geqslant \frac{c}{2}\int_0^{t_1} \frac{1}{t}\,dt = \infty,$$

which is false, by the recurrence hypothesis.

Similarly, if the limit (2.17) would be equal to $c > 0$, it would be a $t_1$ such that

$$\int_0^\infty t^{\frac{n+2(k-1)}{2}-1}\,(-1)^{k-1}\,g^{(k-1)}(t)\,dt \geqslant \frac{c}{2}\int_{t_1}^{\infty} \frac{1}{t}\,dt = \infty.$$

Therefore, the integral (2.15) is equal to the addend (2.18), which, by the recurrence hypothesis, is positive and finite. □

The next theorem shows that an e.d. $\mathrm{E}_n(\mu, \Sigma, g)$ can be a term of an expansive sequence iff the successive derivatives of its functional parameter $g$ are alternatively positive and negative.

**Theorem 6** (*Condition for terms of expansive sequences*). *Let* $\mathrm{E}_n(\mu, \Sigma, g)$ *be an elliptical distribution, with* $n \geqslant 1$.

*There is an expansive sequence of e.d.'s whose nth term is the distribution* $\mathrm{E}_n(\mu, \Sigma, g)$ *iff there is a function* $\ddot{g} = g$ *a.e. such that*

$$(-1)^k\,\ddot{g}^{(k)}(t) \geqslant 0 \tag{2.19}$$

*for* $k \in \{0, 1, 2, \ldots\}$ *and* $t > 0$.

*In this case, one such sequence is the sequence* $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$ *given by the following specifications:*

$$\mu^m = \begin{cases} \mu_{(m)} & \text{if } m \leqslant n, \\ \left(\mu' \;\; 0'_{(m-n)\times 1}\right)' & \text{if } m > n, \end{cases} \tag{2.20}$$

$$\Sigma^m = \begin{cases} \Sigma_{(m)} & \text{if } m \leqslant n, \\ \begin{pmatrix} \Sigma & 0_{n\times m} \\ 0_{m\times n} & I_{m-n} \end{pmatrix} & \text{if } m > n. \end{cases} \tag{2.21}$$

*For* $m < n$

$$g_m(t) = \int_t^\infty (w-t)^{\frac{n-m}{2}-1}\,\ddot{g}(w)\,dw \tag{2.22}$$

*and for* $k \in \{0, 1, 2, \ldots\}$

$$g_{n+2k} = (-1)^k\,\ddot{g}^{(k)}, \tag{2.23}$$

$$g_{n+2k+1}(t) = (-1)^{k+1}\int_t^\infty (w-t)^{-\frac{1}{2}}\,\ddot{g}^{(k+1)}(w)\,dw. \tag{2.24}$$

**Proof.** First we prove that if condition (2.19) is satisfied then (2.20)–(2.24) really describe a distribution $\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)$ for every $m$, and that sequence $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$ is expansive and its $n$th term is the distribution $\mathrm{E}_n(\mu, \Sigma, g)$.

Let $X^n = \left(X_1^n, \ldots, X_n^n\right)' \sim \mathrm{E}_n(\mu, \Sigma, g)$. Obviously, $X^n \sim \mathrm{E}_n(\mu, \Sigma, \ddot{g})$. For any $m < n$ there exists the distribution $\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)$ because this is just the distribution of subvector $X^n_{(m)}$. There exists also the distribution $\mathrm{E}_n\left(\mu^n, \Sigma^n, g_n\right)$; it is the same as $\mathrm{E}_n(\mu, \Sigma, g)$.

For $k \in \{1, 2, \ldots\}$, by virtue of Lemma 5, there exists the distribution $\mathrm{E}_{n+2k}\left(\mu^{n+2k}, \Sigma^{n+2k}, g_{n+2k}\right)$, specified by (2.20), (2.21) and (2.23). We also see that for $k \in \{0, 1, 2, \ldots\}$ there exits the distribution $\mathrm{E}_{n+2k+1}\left(\mu^{n+2k+1}, \Sigma^{n+2k+1}, g_{n+2k+1}\right)$, specified by (2.20), (2.21) and (2.24): if $X^{n+2k+2} \sim \mathrm{E}_{n+2k+2}\left(\mu^{n+2k+2}, \Sigma^{n+2k+2}, g_{n+2k+2}\right)$, then, from (2.3), the distribution of subvector $X^{n+2k+2}_{(n+2k+1)} = \left(X_1^{n+2k+2}, \ldots, X^{n+2k+2}_{n+2k+1}\right)$ is just $\mathrm{E}_{n+2k+1}\left(\mu^{n+2k+1}, \Sigma^{n+2k+1}, g_{n+2k+1}\right)$.

Hence, there exists the sequence $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$, and its $n$th term is the distribution $\mathrm{E}_n(\mu, \Sigma, g)$.

Let us see that sequence $\left\{\mathrm{E}_m\left(\mu^m \Sigma^m, g_m\right)\right\}$ is expansive. For all $m$, let $X^m \sim \mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)$. For $m < n$ we have $X^m \overset{\mathrm{d}}{=} X^n_{(m)} \overset{\mathrm{d}}{=} X^{m+1}_{(m)}$; and for $k \in \{0, 1, 2, \ldots\}$ we have that $X^{n+2k+1} \overset{\mathrm{d}}{=} X^{n+2k+2}_{(n+2k+1)}$. To see that $X^{n+2k} \overset{\mathrm{d}}{=} X^{n+2k+1}_{(n+2k)}$ we may suppose, without loss of generality, that $\mu = 0_{n\times 1}$ and $\Sigma = I_n$; then, by virtue of (2.23) and Lemma 2, we have that $X^{n+2k} \overset{\mathrm{d}}{=} X^{n+2k+2}_{(n+2k)}$, and $X^{n+2k+2}_{(n+2k)} \overset{\mathrm{d}}{=} X^{n+2k+1}_{(n+2k)}$.

Now we prove the reciprocal. Let $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$ be an expansive sequence whose $n$th term is the distribution $\mathrm{E}_n(\mu, \Sigma, g)$. Then the sequence $\{\mathrm{E}_m(0_{m\times 1}, I_m, g_m)\}$ is expansive, too, and, by virtue of Theorem 4, there exists a sequence of functions $\left\{g_m^*\right\}$ satisfying (2.9) and (2.10) for every $m$. Since (2.9) is verified for $m = n$, there exists a constant $b > 0$ such that $g = g_n = b g_n^*$ a.e. Let $\ddot{g} = b g_n^*$ and, for all $m$, let $\ddot{g}_m = b g_m^*$. It is obvious that $\ddot{g} = g$ a.e. and, besides, for all $k$, $(-1)^k \ddot{g}^{(k)} = (-1)^k \left(b g_n^*\right)^{(k)} = b(-1)^k \left(g_n^*\right)^{(k)} = b g^*_{n+2k} \geqslant 0$, where the last equality is obtained by induction on $k$, starting at $k = 0$ and by applying (2.10). □

Note that if function $g$ mentioned in Theorem 6 is continuous on $(0, \infty)$, then we need not use function $\ddot{g}$, and we can put $g$ instead of $\ddot{g}$ in expressions (2.19) and (2.22)–(2.24).

Note how expressions (2.23) and (2.24) allow us to interpret the functional parameters $g_m$, for $m \geqslant n$, in the sequence $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$, in terms of the successive derivatives of parameter $g$.

## 3. Elliptical distributions as scale mixtures of normal distributions

We approach the issue of characterizing the e.d.'s that are scale mixtures of normal distributions and the one of finding the corresponding mixing distribution function.

Firstly, some basic definitions and results concerning scale mixtures of normal distributions are shown. Then, we show a characterization of elliptical distributions that are scale mixtures of normal distributions and put them in relation with expansive sequences of e.d.'s. Next, we show a second characterization and deal with the subject of finding the mixing distribution function.

### *3.1. Basics on scale mixtures of normal distributions and related elliptical distributions*

We denote $\mathrm{N}_n\left(\,\cdot\,;\mu,\Sigma\right)$ the $n$-dimensional normal density with parameters $\mu$ and $\Sigma$; we write $X \sim \mathrm{N}_n\left(\mu,\Sigma\right)$ if vector $X$ has density $\mathrm{N}_n\left(\,\cdot\,;\mu,\Sigma\right)$.

**Definition 7** (*Scale mixture of normal distributions*). If $\mu$ is an $n$-dimensional vector, $\Sigma$ is an $n \times n$ positive definite symmetric matrix and $H$ is a (unidimensional) probability distribution function such that $H(0) = 0$, then the $n$-dimensional density $f$ given by

$$\begin{aligned} f(x;\mu,\Sigma,H) \\ &= \int_0^\infty \mathrm{N}_n\left(x;\mu,v^2\Sigma\right) dH(v) \qquad (3.1)\\ &= (2\pi)^{-\frac{n}{2}}\,|\Sigma|^{-\frac{1}{2}} \int_0^\infty v^{-n}\,\exp\left\{-\frac{1}{2}v^{-2}\,(x-\mu)'\,\Sigma^{-1}\,(x-\mu)\right\} dH(v) \qquad (3.2)\end{aligned}$$

is said to be a *scale mixture of normal densities* $\left\{\mathrm{N}_n\left(x;\mu,v^2\Sigma\right) \mid v \in (0,\infty)\right\}$ with mixing distribution function $H$. If vector $X$ has density (3.1), we say that the distribution of $X$ is the scale mixture of normal distributions (s.m.n.d.) $\mathrm{SMN}_n\left(\mu,\Sigma,H\right)$ and write $X \sim \mathrm{SMN}_n\left(\mu,\Sigma,H\right)$.

Note that in the above definition we use only probability distribution functions as mixing distributions. Some wider approaches to the idea of mixture (see for instance [3]) allow the use of another kind of weighting function.

For each vector $x \neq \mu$, we have $f(x;\mu,\Sigma,H) < \infty$, since function $\mathrm{N}_n(x;\mu,v^2\Sigma)$, as a function of $v$, is bounded, because it is continuous and both its limits, at 0 and at $\infty$, are 0. On the contrary, the value

$$f(\mu;\mu,\Sigma,H) = (2\pi)^{-\frac{n}{2}}\,|\Sigma|^{-\frac{1}{2}} \int_0^\infty v^{-n}\,dH(v), \qquad (3.3)$$

which is proportional to the moment $\int_0^\infty v^{-n}\,dH(v)$, may be $\infty$.

A vector $X = (X_1,\ldots,X_n)'$ has the distribution $\mathrm{SMN}_n\left(\mu,\Sigma,H\right)$ iff it admits the following equality in distribution:

$$X \stackrel{\mathrm{d}}{=} \mu + VZ, \qquad (3.4)$$

where $Z \sim \mathrm{N}_n\left(0,\Sigma\right)$ and $V$, the mixture variable, is a unidimensional random variable independent of $Z$ and having distribution function $H$.

Then, the distribution of $X$ conditional to $V = v$ is $(X \mid V = v) \sim \mathrm{N}_n\left(\mu,v^2\Sigma\right)$.

If $X_{(p)} = \left(X_1,\ldots,X_p\right)'$, with $p \leqslant n$, then the distribution of $X_{(p)}$ conditional to $V$ is $\left(X_{(p)} \mid V = v\right) \sim \mathrm{N}_p\left(\mu_{(p)},v^2\Sigma_{(p)}\right)$. Hence,

$$X_{(p)} \sim \mathrm{SMN}_p\left(\mu_{(p)},\Sigma_{(p)},H\right), \qquad (3.5)$$

where the mixing distribution function is $H$, the same as for $X$.

From (3.2) we see that any scale mixture of normal distribution is an elliptical distribution (this subject will be developed in Corollary 9). The reciprocal is not true; some counterexamples are shown in Section 4. A first general characterization of e.d.'s that are s.m.n.d.'s can be stated in terms of the usual quadratic form as follows.

Let $X \sim \mathrm{E}_n(\mu, \Sigma, g)$. It is known that $X$ has a stochastic representation of the form

$$X \stackrel{\mathrm{d}}{=} \mu + A' Q^{\frac{1}{2}} U^{(n)} \tag{3.6}$$

(see details in [6,7]) with $Q = (X-\mu)' \Sigma^{-1} (X-\mu)$. If we compare (3.6) with (3.4) we deduce that $X \sim \mathrm{SMN}_n(\mu, \Sigma, H)$ iff

$$Q \stackrel{\mathrm{d}}{=} V^2 J^2, \tag{3.7}$$

where $V$ is the mixture variable, whose distribution function is $H$, and $J$ is a variable independent of $V$, with $J^2 \sim \chi_n^2$; this is equivalent to saying that the density of $Q$ is a mixture of gamma densities $\left\{ G\left(\frac{1}{2}v^{-2}, \frac{n}{2}\right) \mid v \in (0, \infty) \right\}$ with mixing distribution function $H$. (See also [8, Corollary 4.1.4.2]).

In this case, from (3.7), the moments of the modular variable $R \stackrel{\mathrm{d}}{=} Q^{\frac{1}{2}}$ and those of vector $X$ (see [7]) can be expressed as functions of the moments of the mixture variable $V$:

$$\begin{aligned}
\mathrm{E}[R^s] &= 2^{\frac{s}{2}} \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n}{2})} \mathrm{E}[V^s], \\
\mathrm{Var}[X] &= \mathrm{E}[V^2]\Sigma, \\
\gamma_2[X] &= n(n+2) \frac{\mathrm{E}[V^4]}{(\mathrm{E}[V^2])^2},
\end{aligned}$$

where $\mathrm{Var}[X]$ and $\gamma_2[X]$ denote the covariance matrix and the kurtosis coefficient (see [11]) of vector $X$, respectively.

Two more practical characterizations of e.d.'s that are s.m.n.d.'s are shown in the following sections.

### 3.2. A necessary and sufficient condition

Elliptical distributions that are scale mixtures of normal distributions are characterized and put in relation with expansive sequences of elliptical distributions.

The next theorem shows that a necessary and sufficient condition for an e.d. $\mathrm{E}_n(\mu, \Sigma, g)$ to be a s.m.n.d. is the alternation of sign of the successive derivatives of its functional parameter $g$; this is just the same condition established in Theorem 6 for being a term of an expansive sequence. The particularization of Theorem 8 to $n = 1$ coincides with the theorem established in [1] for univariate symmetrical distributions.

**Theorem 8** (*Condition for being a mixture*). *Let* $\mathrm{E}_n(\mu, \Sigma, g)$ *be an elliptical distribution, with* $n \geqslant 1$, *and let* $X = (X_1, \ldots, X_n)' \sim \mathrm{E}_n(\mu, \Sigma, g)$. *There is a mixing distribution function H such that*

$$X \sim SMN_n(\mu, \Sigma, H)$$

*iff there exists a function* $\ddot{g} = g$ *a.e. such that*

$$(-1)^k \, \ddot{g}^{(k)}(t) \geqslant 0 \tag{3.8}$$

*for* $k \in \{0, 1, 2, \ldots\}$ *and* $t > 0$.

**Proof.** (i) (*The if part*): By virtue of Theorem 6, there exists an expansive sequence of e.d.'s $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$, whose $n$th term is the distribution $\mathrm{E}_n(\mu, \Sigma, g)$. Therefore, there exists, by virtue both of Theorem 4.1.3 of Gupta and Varga [8] and Theorem 2 of Eaton [4], a mixture function $H$, such that $X \sim \mathrm{SMN}_n(\mu, \Sigma, H)$.

(*The only if part*): If distribution $\mathrm{E}_n(\mu, \Sigma, g)$ is a s.m.n.d., then the cited theorem of Gupta and Varga [8] implies that there exists an expansive sequence of e.d.'s $\left\{\mathrm{E}_m\left(\mu_m, \Sigma_m, g_m\right)\right\}$ whose $n$th term is the distribution $\mathrm{E}_n(\mu, \Sigma, g)$. The result follows now from Theorem 6. $\square$

Note that, once again, if function $g$ is continuous we can do without the function $\ddot{g}$ and directly put $g$ in (3.8).

The next corollary shows that any s.m.n.d. $\mathrm{SMN}_n(\mu, \Sigma, H)$ is also an e.d. $\mathrm{E}_n(\mu, \Sigma, g)$, included in a sequence of e.d.'s having functional parameters that are functions of the successive derivatives of $g$, and which are also s.m.n.d.'s, all with the same mixing distribution function $H$.

**Corollary 9** (*Mixtures inside sequences*). *Let* $\mathrm{SMN}_n(\mu, \Sigma, H)$ *be a scale mixture of normal distributions, with* $n \geqslant 1$.

(i) *A vector X satisfies* $X \sim \mathrm{SMN}_n(\mu, \Sigma, H)$ *iff* $X \sim \mathrm{E}_n(\mu, \Sigma, g)$, with

$$g(t) = \int_0^\infty v^{-n} \exp\left\{-\frac{1}{2} v^{-2} t\right\} dH(v). \tag{3.9}$$

(ii) *The sequence* $\left\{\mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)\right\}$ *of elliptical distributions described by* (2.20)–(2.24), *replacing* $\ddot{g}$ *with* $g$, *is expansive, its nth term is the distribution* $\mathrm{E}_n(\mu, \Sigma, g)$ *and, in addition, it verifies that if* $X^m \sim \mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)$ *then*

$$X^m \sim \mathrm{SMN}_m\left(\mu^m, \Sigma^m, H\right),$$

*where H is the same mixture function as in* $\mathrm{SMN}_n(\mu, \Sigma, H)$.

**Proof.** (i) It follows immediately from (3.2).

(ii) Function $g$ satisfies (3.8), therefore, by virtue of Theorem 6, the sequence $\{\mathrm{E}_m(\mu^m, \Sigma^m, g_m)\}$ is expansive and its $n$th term is the distribution $\mathrm{E}_n(\mu, \Sigma, g)$. Besides, for all $m$, if $X^m \sim \mathrm{E}_m\left(\mu^m, \Sigma^m, g_m\right)$ then there exists, by virtue of Theorem 4.1.3 of Gupta and Varga [8], a mixing function $H_m$ such that $X^m \sim \mathrm{SMN}_m\left(\mu^m, \Sigma^m, H_m\right)$. Now, the mixing function $H_m$ is the same for all $m$, because if $m < r$ then $X^m$ is distributed as a sub-vector of $X^r$ and (see (3.5)) $H_m = H_r$. Hence, for all $m$, $H_m = H_n = H$ and $X^m \sim \mathrm{SMN}_m\left(\mu^m, \Sigma^m, H\right)$. $\square$

**Remark.** Let $\mathrm{E}_n(\mu, \Sigma, g)$ be an e.d. equivalent to $\mathrm{SMN}_n(\mu, \Sigma, H)$. The functional parameters $g_m$ of the distributions of the sequence alluded in Corollary 9(ii) can be obtained from $g_1$ and $g_2$ in this way

$$g_{1+2k} = (-1)^k g_1^{(k)},$$

$$g_{2+2k} = (-1)^k g_2^{(k)}$$

for $k \in \{1, 2, \ldots\}$. As for the starting points, $g_1$ and $g_2$, if $n > 2$ then $g_1(t) = \int_t^\infty (w-t)^{\frac{n-1}{2}-1} g(w)\,dw$ and $g_2(t) = \int_t^\infty (w-t)^{\frac{n-2}{2}-1} g(w)\,dw$; if $n = 1$, and we suppose that $g = g_1$ is continuous (in any case, we can replace $g$ with anyone of its equivalents: the function $\ddot{g}$ of Theorem 8 or the function defined by (3.9)), we make $g_3 = -g'$ and calculate $g_2$ as $g_2(t) = \int_t^\infty (w-t)^{-\frac{1}{2}} g_3(w)\,dw = -\int_t^\infty (w-t)^{-\frac{1}{2}} g'(w)\,dw$; and if $n = 2$, we make $g_1(t) = \int_t^\infty (w-t)^{-\frac{1}{2}} g(w)\,dw$ and, if $g$ is not continuous, we can calculate $g_2$ as $g_2(t) = -\int_t^\infty (w-t)^{-\frac{1}{2}} g_1'(w)\,dw$.

### *3.3. A second condition: Calculation of the mixing distribution function*

If a given e.d. $\mathrm{E}_n(\mu, \Sigma, g)$ is known to be a s.m.n.d. $\mathrm{SMN}_n(\mu, \Sigma, H)$ the issue arises of finding the corresponding mixing distribution function $H$. According to Definition 7, this should be a probability distribution function. We obtain $H$ from the inverse Laplace transform of the functional parameter $g$. In this way, the corresponding s.m.n.d. is fully determined.

Theorem 10, based upon Theorem 8, shows first that a condition for an e.d. $\mathrm{E}_n(\mu, \Sigma, g)$ to be a s.m.n.d., in the sense of Definition 7, is that its functional parameter is the Laplace transform of a distribution function $M$ of a measure with support in $(0, \infty)$. The theorem also shows the relation between the mixing distribution function $H$ and the distribution function $M$. Thus, the theorem particularizes some aspects of a theorem of Chu [3] (see also [8]) and extends the lemma in [1].

**Theorem 10** (*Condition: mixing distribution function*). *Let* $\mathrm{E}_n(\mu, \Sigma, g)$ *be an elliptical distribution, with* $n \geqslant 1$, and let $X \sim \mathrm{E}_n(\mu, \Sigma, g)$. *There is a mixing distribution function* $H$ *such that* $X \sim \mathrm{SMN}_n(\mu, \Sigma, H)$ *iff there exists a measure distribution function* $M$, *with* $M(x) = 0$ *for* $x \leqslant 0$, *such that*

$$g(t) = \int_0^\infty e^{-ty}\,dM(y) \tag{3.10}$$

*for almost all* $t$, *namely, such that* $g$ *is equivalent to the Laplace transform of* $M$, *a.e. In this case, the mixing distribution function* $H$ *is related to the distribution function* $M$ *as follows*:

$$M(y) = \frac{\int_0^\infty t^{\frac{n}{2}-1} g(t)\,dt}{\Gamma\left(\frac{n}{2}\right)} \int_{(0,y]} \left(-u^{\frac{n}{2}}\right) dH\left((2u)^{-\frac{1}{2}}\right). \tag{3.11}$$

**Proof.** If (3.10) holds then, for $k \in \{0, 1, 2, \ldots\}$ and $t > 0$,

$$(-1)^k g^{(k)}(t) = \int_0^\infty y^k e^{-ty}\, dM(y) \geqslant 0$$

and $X \sim \mathrm{SMN}_n(\mu, \Sigma, H)$ for some mixing distribution function $H$.

If $X \sim \mathrm{SMN}_n(\mu, \Sigma, H)$, then, by equating (2.2)–(3.2) we obtain that

$$g(t) = \frac{\int_0^\infty s^{\frac{n}{2}-1} g(s)\, ds}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty v^{-n} \exp\left\{-\frac{1}{2} v^{-2} t\right\} dH(v) \tag{3.12}$$

for almost all $t$, and now, by making the change $u = \frac{1}{2} v^{-2}$, we obtain

$$\begin{aligned} g(t) &= \frac{\int_0^\infty s^{\frac{n}{2}-1} g(s)\, ds}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \exp\{-tu\} \left(-u^{\frac{n}{2}}\right) dH\left((2u)^{-\frac{1}{2}}\right) \\ &= \int_0^\infty e^{-ty}\, dM(y) \end{aligned} \tag{3.13}$$

for almost all $t$, where $M$ is as in (3.11). □

If we know the inverse Laplace transform $M$ of $g$, we may use (3.11) to find $H$, as we will show in Section 4. Furthermore, the next corollary shows an explicit expression for the density $h$ of $H$, whenever it exists. It is obtained easily by differentiating in (3.11).

**Corollary 11** (*Mixing density function*). *Suppose that* (3.10) *and* (3.11) *holds. Then distribution H is absolutely continuous, with density h, iff M is absolutely continuous. In this case*

$$g(t) = \int_0^\infty e^{-ty} m(y)\, dy$$

*with* $m(y) = M'(y)$ *a.e., for almost all t, and*

$$m(y) = \frac{\int_0^\infty s^{\frac{n}{2}-1} g(s)\, ds}{2^{\frac{3}{2}} \Gamma\left(\frac{n}{2}\right)} y^{\frac{n-3}{2}} h\left((2y)^{-\frac{1}{2}}\right), \tag{3.14}$$

*a.e. and, therefore,*

$$h(v) = \frac{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{\int_0^\infty t^{\frac{n}{2}-1} g(t)\, dt} v^{n-3} m\left(\frac{1}{2} v^{-2}\right). \tag{3.15}$$

**Remark.** Some results about the moment $\int_0^\infty v^{-n}\, dH(v)$ can be obtained. If (3.10) (and (3.11)) holds then (see (3.12) and also (3.3))

$$\begin{aligned} \int_0^\infty v^{-n}\, dH(v) &= \frac{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{\int_0^\infty s^{\frac{n}{2}-1} g(s)\, ds} \times g(0) \\ &= (2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}} f(\mu; \mu, \Sigma, g). \end{aligned}$$

It will be $\int_0^\infty v^{-n}\, dH(v) < \infty$ iff $g(0) < \infty$ or, equivalently, if $f(\mu; \mu, \Sigma, g) < \infty$.

**Remark.** Function $M$, given by (3.11), is a distribution function of a measure. Its limit at infinity is $\lim_{y\to\infty} M(y) = g(0)$ (put $t = 0$ in (3.13) and compare with (3.11)). Therefore, $M$ is a distribution function of a finite measure iff $g(0) < \infty$ (or $\int_0^\infty v^{-n}\, dH(v) < \infty$ or $f(\mu; \mu, \Sigma, g) < 0$). In this case, if we have chosen the functional parameter $g$ so that $g(0) = 1$ (the reparametrization $g^*(t) = g(t)/g(0)$ can do it), function $M$ is a probability distribution function.

## 4. Some examples and applications

First, in light of the obtained results, we are going to realize the utility of the previous concepts for a comprehensive treatment of the relationship between e.d.'s and s.m.n.d.'s.

Next, we consider three families of elliptical distributions and study their possible representation as scale mixtures of normal distributions.

We put $q(x) = (x-\mu)'\,\Sigma^{-1}\,(x-\mu)$.

### 4.1. Examples of mixtures

*A degenerate mixture*: If we make $H(v) = I_{[v_0,\infty)}(v)$ (the distribution function of a probability degenerate in $v_0$), for some $v_0 \in (0,\infty)$, in (3.1), then $f(\,\cdot\,; \mu, \Sigma, H)$ is the normal density $\mathrm{N}_n\left(\cdot; \mu, v_0^2\Sigma\right)$. This is an elliptical density with functional parameter $g(t) = \exp\left\{-\frac{1}{2}v_0^{-2}t\right\}$. We may check that $g$, in fact, satisfies (3.8): $(-1)^k\, g^{(k)}(t) = 2^{-k}v_0^{-2k}e^{-\frac{1}{2}v_0^{-2}t} \geqslant 0$. Also, the results of Section 3.3 permit us to get back $H$ from $g$. In this case, $g$ is the Laplace transform of distribution function $M(y) = I_{\left[2^{-1}v_0^{-2},\infty\right)}(y)$. From (3.11) we have that

$$I_{\left[\frac{1}{2v_0^2},\infty\right)}(y) = 2^{\frac{n}{2}}v_0^n \int_{(0,y]} \left(-u^{\frac{n}{2}}\right) dH\left((2u)^{-\frac{1}{2}}\right) = v_0^n \int_{\left[(2y)^{-\frac{1}{2}},\infty\right)} v^{-n}\, dH(v),$$

and hence $\int_{[z,\infty)} v^{-n}\, dH(v) = v_0^{-n} I_{(-\infty,v_0]}(z)$. This implies that $P_H((-\infty, v_0)) = P_H((v_0,\infty)) = 0$ and $P_H(\{v_0\}) = 1$, where $P_H$ is the probability whose distribution function is $H$. Therefore, $H(v) = I_{[v_0,\infty)}(v)$.

*A uniform mixture*: If $n = 3$ and $H$ has density $h(v) = I_{(0,1)}(v)$, then $f(x; \mu, \Sigma, H) = (2\pi)^{-\frac{3}{2}}\,|\Sigma|^{-\frac{1}{2}}\,(q(x))^{-1}\exp\left\{-\frac{1}{2}q(x)\right\}$ for $x \neq \mu$ and $f(\mu; \mu, \Sigma, H) = \infty$. This is an elliptical density with functional parameter $g(t) = t^{-1}\exp\left\{-\frac{1}{2}t\right\}$. Again we may check: $(-1)^k\, g^{(k)}(t) = k!t^{-(k+1)}e^{-\frac{t}{2}}\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{t}{2}\right)^j \geqslant 0$. And we also can get back $H$ from $g$: by making use of the inverse Laplace transform we obtain that $m(y) = I_{\left(\frac{1}{2},\infty\right)}(y)$, and from (3.15) we get $h(v) = I_{(0,1)}(v)$. Note that in this case $M(y) = yI_{\left[\frac{1}{2},\infty\right)}(y)$, and then $\lim_{y\to\infty} M(y) = f(\mu; \mu, \Sigma, H) = \infty$.

*A Pareto mixture*: If $n = 3$ and $H$ has density $h(v) = v^{-2}I_{[1,\infty)}(v)$, then the mixture density (3.1) is just the $\mathrm{E}_3(\mu, \Sigma, g)$ density with $g(t) = t^{-2}\left(2 - (2+t)\exp\left\{-\frac{1}{2}t\right\}\right)$

for $t > 0$. In this case we have $f(\mu;\mu,\Sigma,H) = 2^{-\frac{7}{2}}\pi^{-\frac{3}{2}}|\Sigma|^{-\frac{1}{2}} < \infty$. Again we check: $(-1)^k g^{(k)}(t) = 2\frac{(k+1)!}{t^{k+2}}\left(1 - e^{-\frac{t}{2}}\sum_{j=0}^{k+1}\frac{1}{j!}\left(\frac{t}{2}\right)^j\right) \geqslant 2\frac{(k+1)!}{t^{k+2}}\left(1 - e^{-\frac{t}{2}}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{t}{2}\right)^j\right) = 0$. Now we obtain $H$ from $g$: by making use of the inverse Laplace transform we obtain that $m(y) = 2yI_{\left(0,\frac{1}{2}\right)}(y)$, and now (3.15) yields $h(v) = v^{-2}I_{[1,\infty)}(v)$. In this case $\lim_{y\to\infty} M(y) < \infty$ and $f(\mu;\mu,\Sigma,H) < \infty$, too.

### 4.2. Application to the study of elliptical distributions

*Pearson-type VII distribution*: For any positive integer $n$, let

$$f(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{(m\pi)^{\frac{n}{2}}\,\Gamma\left(\frac{m}{2}\right)}\,|\Sigma|^{-\frac{1}{2}}\left(1 + m^{-1}q(x)\right)^{-\frac{m+n}{2}} \tag{4.1}$$

for some $m \in (0,\infty)$. This is a $\mathrm{E}_n(\mu,\Sigma,g)$ density with $g(t) = \left(1+\frac{t}{m}\right)^{-\frac{m+n}{2}}$. We have

$$(-1)^k g^{(k)}(t) = \left(1+\frac{t}{m}\right)^{-\frac{m+n}{2}-k}\frac{\Gamma\left(\frac{m+n}{2}+k\right)}{m^{\frac{m+n}{2}}\Gamma(\frac{m+n}{2})} \geqslant 0;$$

therefore, (4.1) is a $\mathrm{SMN}_n(\mu,\Sigma,H)$ density for some $H$. We obtain $H$ from $g$: the use of Laplace transform yields $m(y) = m^{\frac{m+n}{2}}\left(\Gamma\left(\frac{m+n}{2}\right)\right)^{-1}e^{-my}y^{\frac{m+n}{2}-1}$, and from (3.15) we get $h(v) = m^{\frac{m}{2}}\left(\Gamma\left(\frac{m}{2}\right)\right)^{-1}2^{-\left(\frac{m}{2}-1\right)}\exp\left\{-\frac{1}{2}mv^{-2}\right\}v^{-m-1}$.

If $V$ is the mixture variable, then $W = mV^{-2}$ has the gamma $G\left(\frac{1}{2},\frac{m}{2}\right)$ distribution. If $m$ is an integer, then (4.1) is a *Student's t* density, and variable $W$ has the $\chi^2_m$ distribution. If $m = 1$, (4.1) is a Cauchy density.

*Pearson-type II distribution*: For any $n$, let

$$f(x) = \frac{\Gamma\left(\frac{n}{2}+m+1\right)}{\pi^{\frac{n}{2}}\Gamma(m+1)}\,|\Sigma|^{-\frac{1}{2}}(1-q(x))^m I_{[0,1)}(q(x))$$

for some $m \in (-1,\infty)$. This an instance of an e.d. that is not a s.m.n.d., because its functional parameter, $g(t) = (1-t)^m I_{[0,1)}(t)$, does not satisfy condition (3.8), for the lack of continuity at 1 of some of its derivatives $g^{(k)}$. Actually, $g^{(k)}(1^+) = 0$ for all $m \in (-1,\infty)$; but, for $m < 0$, we have $g(1^-) = \infty$; for any non-negative integer $m$, we have $g^{(m)}(1^-) = (-1)^m m!$; and for positive non-integer $m$, $g^{(\lfloor m\rfloor)}(1^-)$ is $\infty$ if the integer part, $\lfloor m\rfloor$, of $m$ is even, and is $-\infty$ if $\lfloor m\rfloor$ is odd.

*Logistic distribution*: Here a density is shown that, as opposed to the previous distribution, is positive everywhere, but is not a scale mixture of normal densities either. For any $n$ we consider the density (see [5])

$$f(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}\int_0^\infty t^{\frac{n}{2}-1}\frac{e^{-t}}{(1+e^{-t})^2}\,dt}\,|\Sigma|^{-\frac{1}{2}}\,\frac{\exp\{-q(x)\}}{(1+\exp\{-q(x)\})^2}.$$

Here $g(t) = e^{-t}\left(1+e^{-t}\right)^{-2}$ and condition (3.8) is not satisfied, because $(-1)^2 g^{(2)}(t) = e^{-t}\left(1+e^{-2t}-4e^{-t}\right)\left(1+e^{-t}\right)^{-4} < 0$ for $t \in \left(0, -\ln\left(2-\sqrt{3}\right)\right)$.

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