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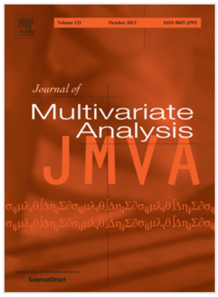

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On an interaction function for copulas

Dorota Kurowicka and Wim T. van Horssen

Delft University of Technology, The Netherlands
D.Kurowicka@tudelft.nl, W.T.vanHorssen@tudelft.nl

Abstract

We study properties of a local dependence function of Wang for copulas. In this paper this dependence function is called the mixed derivative measure of interactions as it is a mixed derivative of a log of a density function. It is stressed that this measure is not margin free in the sense that the interaction function of a density and the corresponding copula are not equal. We show that there is no Archimedean copula with constant interactions. The interaction function is positive (negative) for an Archimedean copula density whose second derivative of the generator is log convex (log concave). Moreover, the only Archimedean copula with interactions proportional to its density is Frank's copula. We obtain some preliminary results concerning the connection between the behaviour of the interaction function and the tail dependence of the distribution. Moreover, the notion of an interaction function has been extended to the more than two dimensional case, and we study its properties for a canonical Archimedean copula.

Keywords: Local dependence function, copulae, Archimedean copulae, complete monotonic functions, d-monotonic functions, tail dependence, tail order.

1 Introduction

Many measures of dependence as well as dependence functions have been introduced in the literature to study dependence properties of distributions. A good overview can be found in e.g. [1] for bivariate distributions. A very natural dependence function introduced in [7] is the mixed derivative of the log of a density function. The density of the independent random variables is a product of the marginal densities. The logarithm of this product gives a sum of functions each dependent on only one variable. Hence the mixed derivatives of this sum will be equal to zero. The opposite is also true. If the support of a joint density is a rectangle then the interaction function equal to zero ensures independence. This natural concept was used in [24] to study independencies and conditional independencies in joint distributions. Following [24] we call this dependence function the *mixed derivative measure of interaction* or the *interaction function*.

Let $\mathbf{X} = (X_1, X_2)$ be an absolutely continuous random vector and its density f_{12} be twice differentiable. We assume that the support of f_{12} is a rectangle. We denote $\frac{\partial^2}{\partial x_1 \partial x_2} f_{12}$ as $D^2_{12} f_{12}$. Then the mixed derivative measure of interaction can be defined as follows:

Definition 1.1 *The mixed derivative measure of interaction for* $\mathbf{X}$ *with density* f_{12}*, which is twice differentiable, is*

$$i_{12}(x_1, x_2) = D^2_{12} \log f_{12}(x_1, x_2).$$

In [7] it is shown that $i_{12}(x_1, x_2)$ is the natural continuous analogue of the collection of local cross-product ratios which characterize the dependency structure of bivariate discrete data. A second interpretation of the mixed derivative measure of interactions was given in [13]. Jones showed, using kernel methods, that i_{12} is a local version of the linear correlation coefficient.

Properties of $i_{12}(x_1, x_2)$ ([7, 25]) include:

- it is finite everywhere;
- it is everywhere zero if and only if X_1 and X_2 are independent;
- it is constant if f_{12} is the bivariate normal density;
- it remains unchanged for densities constructed as follows: $h_1(x_1, x_2) = f(x_1, x_2)g_1(x_1)/f_1(x_1)$, $h_2(x_1, x_2) = f(x_1, x_2)g_2(x_2)/f_2(x_2)$ where f_1, f_2 are margins of f, and g_i is a one dimensional density with the same support as $f_i, i = 1, 2$;
- i_{12} is a function only of the conditional distribution of X_2 given X_1, or of X_1 given X_2;

There is a unique distribution for given margins and given interaction function: [7]. Proof of this result uses an analogous result of [23] for contingency tables with prescribed row and column sums and positive entries.

$i_{12}(x_1, x_2) \geq 0$ for all x_1, x_2 is equivalent with f_{12} being TP_2 (Total Positive of order 2) or LRD (Likelihood Ratio Dependent) (see e.g. [1]) which is a very strong notion of positive dependence between X_1 and X_2.

Definition 1.2 *A non-zero function f is TP_2 if for all $x_1 < y_1, x_2 < y_2$ where (x_1, x_2) and (y_1, y_2) belong to the domain of f*

$$f(x_1, x_2)f(y_1, y_2) \geq f(x_1, y_2)f(y_1, x_2).$$

The interaction function $i_{12}(x_1, x_2)$ describes the local dependence of f_{12} in the neighbourhood of a point (x_1, x_2). Therefore if $f_{12}(x_1, x_2)$ behaves as the product of margins $f_1(x_1)f_2(x_2)$ for x_1, x_2 which are in the tail of the marginal densities then one could expect that the interaction function is equal to zero in this region. Hence we expect some relationship between the behaviour of the interaction function on the boundary of its domain and the usual concept of tail dependence. In this paper we will make a first step to investigate properties of the interaction function and search for connections between i_{12} and the tail dependence for the elliptical and the Archimedean copulas. For these distributions the properties of the interaction function are determined by the properties of their generating functions.

We first look at the relationship between the interaction function of a density and the corresponding copula.

2 Interactions for copulas

A bivariate copula is a distribution on a unit square with uniform margins (see e.g. [19, 12]). The density function f_{12} of $\mathbf{X}$ can be written as the product of marginal densities f_1, f_2 and the corresponding copula density c:

$$f_{12}(x_1, x_2) = f_1(x_1)f_2(x_2)c(F_1(x_1), F_2(x_2)), \tag{1}$$

where F_1, F_2 are marginal cumulative distribution functions of $\mathbf{X}$. Conversely if F_1^{-1}, F_2^{-1} exist then the copula density corresponding to f_{12} is:

$$c(u_1, u_2) \quad = \quad \frac{f_{12}(F_1^{-1}(u_1), F_2^{-1}(u_2))}{f_1(F_1^{-1}(u_1))f_2(F_2^{-1}(u_2))}. \tag{2}$$

We denote the mixed derivative measure of interaction of the density function and the copula function as follows:

$$\begin{aligned} i^f_{12}(x_1,x_2) &= D^2_{12}\log f_{12}(x_1,x_2), \\ i^c_{12}(u_1,u_2) &= D^2_{12}\log c(u_1,u_2) \end{aligned}$$

where $D^2_{12}c$ means $\frac{\partial^2}{\partial u_1 \partial u_2}c$. The support of c is the unit square.

From the above and the relationships (1) and (2) we find that the interaction function of the density is related to the interaction function of the corresponding copula:

$$\begin{aligned} i^f_{12}(x_1,x_2) &= i^c_{12}(F_1(x_1),F_2(x_2))f_1(x_1)f_2(x_2), & (3) \\ i^c_{12}(u_1,u_2) &= \frac{i^f_{12}(F_1^{-1}(u_1),F_2^{-1}(u_2))}{f_1(F_1^{-1}(u_1))f_2(F_2^{-1}(u_2))}. & (4) \end{aligned}$$

The interactions of a density and the corresponding copula differ by a factor which is the product of marginal distributions. They are not equal as is suggested in [1] but they have the same sign. From the above we see that even though the interaction function is affected by the marginal transformations the linear transformations of the margins have no influence on the interaction function of the copula. Hence to study properties of the interaction function of the copula we can concentrate on the distributions with the standardized margins.

For meta-gaussian distributions, that is for distributions with standard normal margins and a copula density c, the interaction function is given by:

$$i^{c,z}_{12}(z_1,z_2) = i^c_{12}\left(\Phi^{-1}(z_1),\Phi^{-1}(z_2)\right)\phi(z_1)\phi(z_2), (z_1,z_2)\in \mathbf{R}^2, \tag{5}$$

where ϕ and Φ (Φ^{-1}) denote density and (inverse) cumulative distribution functions (cdf) of the standard normal distribution, respectively.

2.1 Elliptical copulas

A random vector $\mathbf{X}=(X_1,X_2)$ is said to have an elliptically symmetric density $f(\mathbf{x})$ if

$$f(\mathbf{x}) = |\Sigma|^{-1/2} g\left(\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right) \tag{6}$$

for some function g and Σ positive definite with entries $\Sigma_{11}=\Sigma_{22}=1$ and $\Sigma_{12}=\Sigma_{21}=\rho,\ -1<\rho<1$. Alternatively, we may say that

$$\mathbf{X} \stackrel{d}{=} RA\mathrm{U}$$

where the radial random variable $R\geq 0$ is independent of $\mathbf{U}$. $\mathbf{U}$ is a bivariate random vector distributed uniformly on the unit sphere in $\mathbf{R}^2$, and $A=\begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2}\end{pmatrix}$.

The properties of elliptical distributions and the corresponding copulas depend on the properties of g or equivalently on the properties of R. In [21] the properties of g have been established that lead to an elliptically symmetric TP_2 density. It has been shown that the only elliptically symmetric TP_2 density for all $\rho,\ 0<\rho<1$, is the gaussian distribution. Moreover, a bivariate t-distribution has been shown not to be a TP_2 density for any values of its parameters ρ and ν.

Another line of research concerns the tail dependence properties of elliptical distributions. The coefficients of lower and upper tail dependence are defined in [12]. They give information about dependence in the lower and upper tail of a joint distribution, respectively. Tail dependence coefficients are invariant under strictly increasing transformations of margins. Hence they are functions of the corresponding copula. These coefficients are defined as follows:

$$\lambda_L = \lim_{q\to 0} \frac{C(q,q)}{q}, \quad \lambda_U = \lim_{q\to 1} \frac{1-2q+C(q,q)}{1-q}$$

provided that the limits exist.

The positive values of the tail dependence coefficients indicate the strength of the association of margins in the tails. For elliptical distributions the upper and lower tail dependence coefficients are equal. In [22] it is proven that if the radial random variable R has a regularly varying tail (see [20]: R is regularly varying with index $-\alpha$, denoted as $\mathrm{RV}_{-\alpha}$, means that the tail of the survival function of R behaves at infinity as $x^{-\alpha}$) then X_1 and X_2 are tail dependent. The heaviness of tails is often studied using a concept of Maximum Domain of Attraction (MDA) of a univariate extreme value distribution (Φ_α - Frehet, Ψ_α - Weibull and Λ- Gumbel). R has a regularly varying tail with index $-\alpha$ if $R \in MDA(\Phi_\alpha)$. If $R \in MDA(\Lambda)$ then $\lambda_L = \lambda_U = 0$. In the case when $\lambda_L = 0$ one might want to distinguish the speed of convergence to zero of copula $C(q,q)$ as $q \to 0$. In [9] the concept of tail order was introduced. The decay of a copula as $q \to 0$ can be described as being of order of $C(q,q) \sim q^{\kappa_L} l(q)$ where l is a slowly varying function (see e.g [20]) and κ_L is called the *lower tail order.* $\kappa_L = 1$ corresponds to the usual lower tail dependence and $1 < \kappa_L < 2$ describes so called *intermediate tail dependence.* If $\kappa_l = 2$ then C behaves in the neighbourhood of $(0,0)$ as the independent copula. In [11] the concept has been extended to the negative tail dependence which can occur when $\kappa > 2$. Similarly the *upper tail order* κ_U is defined as $\overline{C}(1-q,1-q) \sim q^{\kappa_U} l(q)$, where $\overline{C}$ denotes the survival function of C.

In [4] the coefficient of the tail dependence of the second kind is introduced to distinguish different speeds of convergence of a copula on the corners of the unit square in case when $R \in MDA(\Lambda)$. They are defined as follows:

$$\mu_L = \lim_{q\to 0} \frac{2\log q}{\log C(q,q)}, \quad \mu_U = \lim_{q\to 1} \frac{2\log(1-q)}{1-2q+C(q,q)} - 1.$$

If $\lambda_L = 1$ then $\mu_L > -1$ and $\mu_L = 1$ whenever $\lambda_L > 0$. Similar behaviour is observed for μ_U.

If g is twice differentiable then the interaction function for an elliptically symmetric density f given by (6), can be easily calculated using definition 1.1:

$$i_{12}^f(x_1,x_2) \quad = \quad \frac{4}{(1-\rho^2)^2} \frac{d^2}{dt^2} \left[\log g(t)\right]_{t=\xi_{12}} (x_1-\rho x_2)(x_2-\rho x_1) - \frac{2\rho}{1-\rho^2} \frac{d}{dt} \left[\log g(t)\right]_{t=\xi_{12}} \quad (7)$$

where $\xi_{12} = \mathbf{x}^T\Sigma^{-1}\mathbf{x} = \frac{1}{1-\rho^2}(x_1^2 + x_2^2 - 2\rho x_1 x_2)$.

In the next two examples we will investigate the behaviour of interaction functions for gaussian and Student-t densities and corresponding copulas as well as the meta-gaussian distribution corresponding to these copulas.

Example 2.1 *The interaction function for a bivariate gaussian density with standard normal margins and correlation ρ is:*

$$i_{12}^n(z_1,z_2) = \frac{\rho}{1-\rho^2}.$$

Notice that the interaction function is constant. For the gaussian distribution the interaction function is positive (negative) if and only if $\rho > 0$ ($\rho < 0$). It is obvious that

$$id_L = id_U = \lim_{z\to\pm\infty} i_{12}^n(z,z) = \frac{\rho}{1-\rho^2}.$$

$id_U > (<)0$ for $\rho > (<)0$, $id_U = 0$ for $\rho = 0$ and $id_U \to \infty(-\infty)$ when $\rho \to 1(-1)$.

The interaction function of the gaussian copula with parameter ρ is:

$$i_{12}^c(u_1, u_2) = \frac{\frac{\rho}{1-\rho^2}}{\phi(\Phi^{-1}(u_1))\phi(\Phi^{-1}(u_2))}.$$

In contrast to the interaction function of the normal distribution the interaction function of the normal copula is not constant.

In [10] is was shown that the tail order of an elliptical distribution with a Kotz type density generator g given below is equal to $\kappa = [2/(1+\rho)]^{\xi}$.

$$g(x) \quad = \quad K x^{N-1} \exp\left(-\beta x^{\xi}\right), \quad x > 0, \ \beta, \xi, N > 0 \tag{8}$$

where K is a normalizing constant. The gaussian distribution belongs to this class with $\xi = 1$. Hence the tail order of the gaussian copula is $\kappa = 2/(1+\rho)$. In case when $0 < \rho < 1$ we see that $1 < \kappa < 2$. Hence the gaussian copula possesses the intermediate tail dependence. The interaction function of a distribution with gaussian copula and standard gaussian margins is constant (or equivalently tends to a constant as $z_1 = z_2 = z \to \infty$).

Example 2.2 *The interaction function for a bivariate Student-t density with correlation ρ and degrees of freedom ν can be calculated from (7) with $g(t) = (1 + t/\nu)^{-(\nu+2)/2}$. We get:*

$$i_{12}^t(x_1, x_2) \quad = \quad (\nu+2)\left[\frac{\rho}{\nu(1-\rho^2)+\xi_{12}} + \frac{(x_1-\rho x_2)(x_2-\rho x_1)}{(\nu(1-\rho^2)+\xi_{12})^2}\right], \tag{9}$$

where $\xi_{12} = x_1^2 + x_2^2 - 2\rho x_1 x_2$. Notice when $\rho \geq 0$ and $x_1 = x_2 = x \to \infty$, (9) is positive and of order x^{-2}. In case when $x_1 = -x_2 = x \to \infty$, (9) can be negative.

Since for the generating function of the Student-t density we have $(\log g(t))' = -\frac{\nu+2}{2(\nu+t)} \to -1/2$ and $(\log g(t))'' = \frac{\nu+2}{2(\nu+t)^2} \to 0$ as $\nu \to \infty$, then we can see that the interaction function of the Student-t distribution tends to $\rho/(1-\rho^2)$ as $\nu \to \infty$, which is the interaction function of the gaussian distribution.

The interaction function for the t-copula can be found with formula (4), where $f_1, f_2 = f$ and $F_1 = F_2 = F$ are univariate Student-t densities and cdfs, respectively. In [10] one finds that the t-copula with parameters ρ and ν has the tail order $\kappa = 1$ as the survival function $\overline{F}_R$ of the radial variable R is regularly varying with index $-\nu$. The margins of the bivariate Student-t density are univariate t and their survival functions are $RV_{-\nu}$. Hence the cdf is $RV_{-\nu}$ at 0 (see e.g [20] for properties of the regularly varying functions). The inverse cdfs are $RV_{-1/\nu}$ and the densities are $RV_{-\nu-1}$. Hence, $f \circ F^{-1}$ is $RV_{(\nu+1)/\nu}$.

We will look now at the behaviour of the interaction function of a meta gaussian density with Student-t copula. First we notice that since $i_{12}^t(x,x) \sim x^{-2}$ as $x \to \infty$ then

$$i_{12}^c(u,u) = i_{12}^t(F^{-1}(u), F^{-1}(u))/(f(F^{-1}(u)))^2$$

is of order u^{-2} as $u \to 0$. From the above and using Mill's ratio $i_{12}^{c,z}(z,z) = i_{12}^c(\Phi(z), \Phi(z))(\phi(z))^2$ is of order z^2 when $z \to -\infty$. This means that the interaction function of a meta-gaussian density with t-copula tends to infinity along the diagonal.

$$id_U = \lim_{z\to\infty} i_{12}^{c,z}(z,z) = \infty.$$

Similarly one can show that along the anti-diagonal the interaction function of a meta-gaussian density with t-copula tends to minus infinity.

In the next section we will study properties of another family of copulas that are constructed with the help of a univariate generating function, called Archimedean copulas.

2.2 Archimedean Copulas

Archimedean copulas have been studied extensively by many authors (see e.g [12, 15, 16, 14]) for basic properties and more references). This family of copulas is constructed with help of the generating function $\psi : [0, \infty] \to [0, 1]$ with continuous first two order derivatives on $(0, \infty)$ satisfying

$$\psi(0) = 1, \quad \lim_{x\to\infty} \psi(x) = 0, \quad \psi^{'}(x) < 0, \quad \psi^{''}(x) > 0 \quad \text{for all } x \in (0, \infty). \tag{10}$$

Then the function

$$C^A(u_1, u_2) \quad = \quad \psi[\psi^{-1}(u_1) + \psi^{-1}(u_2)] \tag{11}$$

is a strict Archimedean copula. C^A is strictly positive except when $u_1 = 0$ and $u_2 = 0$. The density function c^A is nonzero on $(0, 1]^2$ and is equal to:

$$c^A(u_1, u_2) = \frac{\psi^{''}(\psi^{-1}(u_1) + \psi^{-1}(u_2))}{\psi'(\psi^{-1}(u_1)) \cdot \psi'(\psi^{-1}(u_2))}. \tag{12}$$

We consider here copula densities for which the mixed derivative exists. Hence such that the fourth derivative of the generator, denoted as $\psi^{(4)}$, exists. With these assumptions the interactions for Archimedean copulae are:

$$i^A_{12}(u_1, u_2) \quad = \quad \frac{1}{\psi'(\psi^{-1}(u_1))\psi'(\psi^{-1}(u_2))} \frac{d^2}{dt^2} [\log \psi''(t)]_{t=\psi^{-1}(u_1)+\psi^{-1}(u_2)} . \tag{13}$$

From the above we see that the interaction function can be zero only if the second derivative of ψ is exponential. After incorporating conditions on ψ as stated above we see that the interaction function is equal to zero if and only if

$$\psi(x) = \exp(-\theta x), \theta > 0.$$

Notice that due to (11) for all $\theta > 0$ this is the generator of the independent copula which was to be expected due to properties of the interaction function as discussed in Section 1.

For Archimedean copulas Kendall's tau correlation can be calculated from the generator (see e.g.[12]) as:

$$\tau \quad = \quad 1 - 4 \int_0^\infty s\,(\psi'(s))^2\, ds. \tag{14}$$

In [12] the Archimedean copulas with generators which are the Laplace transforms (LT) of a positive random variables Y have been studied. The generators which are Laplace transforms of a positive variable are completely monotone:

Definition 2.1 *A real- valued function g which has continuous derivatives of all orders is completely monotone if $(-1)^k g^{(k)}(x) \geq 0$ for $x > 0$ and for $k = 0, 1, 2, ...$*

The tail dependence and the tail order of such LT-copulas have been investigated in [9]. The tail order of a LT-copula is determined by the maximum non-negative moment of the variable Y.

A complete characterization of Archimedean copulas has been given in [15]. The authors showed that any Archimedean copula is the survival copula of a random vector $\mathbf{X}$ following an l_1-norm symmetric distribution, i.e. $\mathbf{X} \stackrel{d}{=} R\mathbf{S}$, where R is a positive random variable that places no mass at zero and $\mathbf{S}$ is distributed uniformly on the simplex and independent of R.

The Archimedean generator ψ is the Williamson d-transform [27] of the distribution F_R with radial part R.

Definition 2.2 *If R is a non-negative random variable with distribution function F_R satisfying $F_R(0) = 0$, and $d \geq 2$ is an integer, then the Williamson d-transform of F_R is a real function on $[0, \infty)$ given by*

$$\psi(x) \quad = \quad \int_x^\infty \left(1 - \frac{x}{y}\right)^{d-1} dF_R(y), \; x \geq 0. \tag{15}$$

Generators which are the Williamson d-transform of a positive function are d-monotone.

Definition 2.3 *A real-valued function g is $d-$monotone if it is differentiable up to order $d-2$ on $(0, \infty)$ with derivatives satisfying*

$$(-1)^k g^{(k)}(x) \geq 0, \;\; k = 0, 1, ..., d-2, \tag{16}$$

and if $(-1)^{d-2} f^{(d-2)}$ is non-decreasing and convex on $(0, \infty)$.

The distribution function of R is uniquely given by its Williamson d-transform, and the distribution function can be recovered through an inversion formula. If ψ is d-times differentiable F_R has a density given by

$$f_R(x) \quad = \quad \frac{(-1)^d x^{d-1} \psi^{(d)}(x)}{(d-1)!}, \;\; x \geq 0. \tag{17}$$

The Laplace transform can be thought of as a limiting Williamson d-transform as $d \to \infty$.

Similarly to the copulas of elliptically symmetric distributions the properties of an Archimedean copula (with a generator that is the Williamson d-transform of F_R) are determined by the properties of the distribution of the radial part R.

Extremal behaviour of an Archimedean copula has been studied in [2] and [14]. In [14] the authors showed that if $R \in MDA(\Phi_\alpha)$, hence $\psi \in \mathrm{RV}_{-\alpha}$, then the Archimedean copula has a lower tail dependence coefficient $\lambda_L = 2^{-\alpha}$ and the lower tail order $\kappa_L = 1$. Moreover, when $1/R \in MDA(\Phi_\alpha)$ for $\alpha \in (0, 1)$, hence $1 - \psi(x^{-1}) \in \mathrm{RV}_{-\alpha}$ then the copula has an upper tail dependence $\lambda_U = 2 - 2^\alpha$ and $\kappa_U = 1$. Additional conditions concerning intermediate upper and lower tail orders of Archimedean copulas have been presented in [10]. An Archimedean copula has a lower intermediate tail order when $R \in MDA(\Lambda)$ with auxiliary function $a(.) \in \mathrm{RV}_\beta$ for some $0 < \beta < 1$, and an upper intermediate tail order when $1/R \in MDA(\Phi_\alpha)$ with $1 < \alpha < 2$. In [11] the last result has been extended. When $1/R \in MDA(\Phi_\alpha)$ and the expectation of $1/R$ is finite then $\kappa_U = \alpha$. In case when the tail order is larger than 2 then there is the negative dependence in the tail.

We first calculate interactions for a few families of Archimedean copulae that is for Frank, Clayton and Gumbel copulas and then we study in more details properties of the interaction function for Archimedean copulas.

2.2.1 Frank's copula

The generating function and the density of Frank's copula [5] are:

$$\begin{aligned} \psi^F(x) &= -\theta^{-1} \log(1 - (1 - e^{-\theta}) e^{-x}), \\ c_\theta^F(u_1, u_2) &= \frac{\theta(1 - e^{-\theta}) e^{-\theta(u_1 + u_2)}}{[1 - e^{-\theta} - (1 - e^{-\theta u_1})(1 - e^{-\theta u_2})]}. \end{aligned}$$

The tail dependence coefficients and tail orders of Frank's copula are $\lambda_L = \lambda_U = 0$ and $\kappa_L = \kappa_U = 2$, respectively. The interaction function for Frank's copula is proportional to its density function.

$$i_{12}^F(u_1, u_2) = 2\theta c_\theta^F(u_1, u_2).$$

The properties of local dependence can be easily observed from the behaviour of the density. This is very specific for Frank's copula.

From the above and because the corners of the density of Frank's copula are known to be finite we get

$$id_L = id_U = \lim_{z\to\pm\infty} i_{12}^{F,z}(z,z) = 0.$$

2.2.2 Clayton copula

The generating function and the density of Clayton's copula [3] are:

$$\begin{aligned}\psi^{Cl}(x) &= (1+x)^{-1/\theta},\ \theta \geq 0\\ c_\theta^{Cl}(u_1,u_2) &= (1+\theta)(u_1u_2)^{-1-\theta}(u_1^{-\theta}+u_2^{-\theta}-1)^{-\frac{1}{\theta}-2}.\end{aligned}$$

For the Clayton copula the tail dependence coefficients are: $\lambda_L = 2^{-1/\theta}, \lambda_U = 0$ and $\kappa_L = 1$ and $\kappa_U = 2$. The interaction function for this copula is:

$$i_{12}^{Cl}(u_1,u_2) = \theta(1/\theta+2)\frac{\frac{1}{(u_1^\theta+u_2^\theta-1)^2}}{u_1^{1+\theta}u_2^{1+\theta}}.$$

From the above it follows that $i_{12}^{Cl}(u,u)$ is of order $u^{-2-4\theta}$ for u close to zero. Hence applying Mill's ratio we get:

$$id_L = \lim_{z\to-\infty} i_{12}^{Cl,z}(z,z) = \infty.$$

Additionally we see that:

$$id_U = \lim_{z\to\infty} i_{12}^{Cl,z}(z,z) = 0.$$

2.2.3 Gumbel-Hougaard copula

The generating function of the Gumbel-Hougaard copula[6, 8] is:

$$\psi^G(x) = \exp(-x^{1/\theta}),\ \theta \geq 1.$$

For the Gumbel copula the tail dependence coefficients are: $\lambda_L = 0, \lambda_U = 2-2^{1/\theta}$ and $\kappa_L = 2^{-1/\theta}$ and $\kappa_U = 1$. The interaction function has quite a long functional form but we can easily calculate that $\psi^{-1}(x) = (-\log x)^\theta$, $\psi'(x) = -\frac{1}{\theta}x^{1/\theta-1}\exp(-x^{1/\theta})$ and

$$[\log\psi''(x)]'' = \frac{\theta-1}{\theta^2}\frac{x^{3/\theta}+2(\theta-1)x^{2/\theta}+(5\theta^2-5\theta+1)x^{1/\theta}+\theta(\theta-1)(2\theta-1)}{x^2(\theta-1+x^{1/\theta})^2}.$$

Since $1/[\psi'(\psi^{-1}(u))]^2 = \frac{\theta^2(-\log u)^{2(\theta-1)}}{u^2}$ then

$$i_{12}^c(u,u) = \frac{[\log\psi''(x)]''\,|_{x=2\psi^{-1}(u)}}{[\psi'(\psi^{-1}(u))]^2} \to \begin{cases}(\theta-1)2^{-2+1/\theta}\frac{1}{(-\log u)u^2} & u\to 0,\\ (2\theta-1)\theta 2^{-2}\frac{1}{(-\log u)^2u^2} & u\to 1.\end{cases}$$

This gives

$$\begin{aligned}id_L &= \lim_{z\to-\infty} i_{12}^{c,z}(z,z) \to (\theta-1)2^{-2+1/\theta}\frac{1}{(-\log\phi(z)+\log|z|)\frac{\phi^2(z)}{z^2}}\phi^2(z)\\ &= (\theta-1)2^{-2+1/\theta}\frac{z^2}{\frac{z^2}{2}+\log|z|} = (\theta-1)2^{-1+1/\theta},\end{aligned}$$

and

$$id_U = \lim_{z\to\infty} i_{12}^{c,z}(z,z) \to (2\theta-1)\theta 2^{-2}\frac{\phi^2(z)}{(-\log\Phi(z))^2\Phi^2(z)} = \infty$$

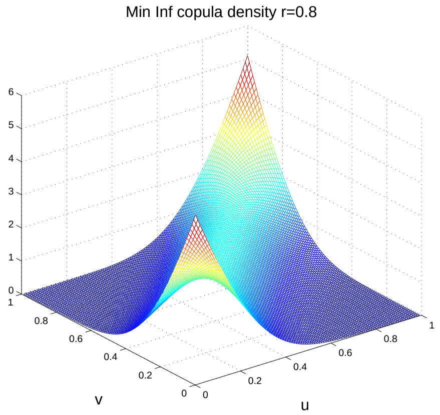

Figure 1: The density function of a minimum information copula with rank correlation $r = 0.8$.

since $\lim_{z\to\infty} \frac{\phi(z)}{-\log\Phi(z)} = \infty$.

Notice that similar to the gaussian distribution the interaction function of a meta gaussian distribution for Gumbel copula with an intermediate lower tail dependence converges to a constant along the diagonal as $z \to -\infty$.

To conclude this section we point out that the examples above as well as the results presented earlier for Gaussian and Student-t copulas suggest that the tail behaviour of the interaction function of copulas with the standard gaussian margins allows to quantify how strong the tail dependence of copulas is. We observed that when a copula is tail independent then $i_{12}^{c,z}$ tends to zero. In case of the intermediate tail dependence $i_{12}^{c,z}$ converges to a constant and its behaviour is comparable to the Gaussian copula. When strong tail dependence is present the limit of $i_{12}^{c,z}$ becomes infinite.

More research is needed to understand better properties of the interaction function and the connections of this local dependence measure with other dependence concepts.

2.3 Interactions for bivariate Archimedean copulae

In this section we study in more detail properties of the interaction function for Archimedean copulae. As mentioned earlier for this family of copulae the properties of the interaction function are determined by their generator.

2.3.1 Constant interaction function

In Section 2.2 we presented the form of the generator that led to a zero interaction function.

Property 1. $i_{12} = 0$ if and only if $\psi(t) = \exp(-\theta t), \theta > 0$.

The minimum information copula was introduced and studied in [17]. This is a distribution on the unit square with uniform marginal distributions which minimizes the relative information (Kullback-Leibler divergence) with respect to a uniform distribution subject to a correlation constraint. In Figure 1 the density of a minimum information copula with a correlation of 0.8 is shown.

In [17] it is shown that the density of the minimum information copula is of the following form:

$$c^{MI}(u_1, u_2) = a(u_1, \theta) \cdot a(u_2, \theta) e^{\theta u_1 u_2},$$

where $a : [0, 1] \times R \to R$ is continuous and the parameter θ corresponds to correlation. The function a has been represented as the Taylor expansion $a(x, \theta) = \sum_{m=1}^{\infty} \alpha_m(x)\theta^{2m}$. For the coefficients $\alpha_m(x)$ the recurrence relationship is known (see [17] for details).

From the above we can see that the interaction function of the minimum information copula is constant, that is,

$$i_{12}^{MI}(u_1, u_2) = \theta.$$

The minimum information copula is the unique copula with constant interaction function. This follows from the result mentioned before [7] that the interaction function and margins determine the density function. This copula does not belong to the Archimedean family because it is radially symmetric and the only radially symmetric copula in this family is Frank's copula [5]. Hence the following property can be added:

Property 2. There is no Archimedean copula with constant (not equal to zero) interaction function.

2.3.2 Sign of the interaction function

The interaction function for Archimedean copulae given in (13) leads immediately to the following property:

Property 3. $i_{12}^{A} > 0 (< 0)$ if and only if ψ'' is a log-convex (log-concave) function.

The generators that are Laplace transforms of a positive function, are completely monotone and log-convex [26]. The second order derivatives of such generators are also Laplace transforms. Hence, they are also log-convex which means that $i_{12}^{A} \geq 0$ for LT-Archimedean copulas. However, the complete monotonicity is not necessary for the interaction function to be positive as is shown in the example below. This example presents the family of Archimedean generators constructed with the help of the $d-$Williamson transform in [16] which generates so called gamma-simplex copula.

Example 2.3 *The generator $\psi_{\theta,d}(x)$ for $x \geq 0$ and $\theta > 0$ is given below.*

$$\psi_{\theta,d}(x) = \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(-1)^{d-1-k} x^{d-1-k}}{\Gamma(\theta)} \Gamma(k - d + \theta + 1, x),$$

where $\Gamma(k, x) = \int_x^{\infty} t^{k-1} e^{-t} dt$ denotes the upper incomplete gamma function. For $\theta = d$, $\psi_{d,d}(x) = e^{-x}$, which is the generator of the independent copula. For $d = 2$ the generator is:

$$\psi_{\theta,2}(x) = \frac{1}{\Gamma(\theta)} (\Gamma(\theta, x) - x\Gamma(\theta - 1, x)).$$

We find that

$$(\log \psi''_{\theta,2}(x))'' = -\frac{\theta - 2}{x^2}$$

which is positive when $\theta < 2$ and negative when $\theta > 2$. Hence, the complete monotonicity of the generator is not necessary for the interaction function to become positive at least for some values of its parameters.

Clayton's copula presented is Section 2.2 has a positive interaction function. The interaction function for Frank's copula is positive when $\theta > 0$, and is negative for $\theta < 0$.

The sign of the interaction function is determined by the sign of the second order derivative of $\log \psi^{''}(t)$

$$\frac{d^2}{dt^2}(\log(\psi^{''}(t)) \quad = \quad \frac{\psi^{(4)}(t)\psi^{''}(t) - (\psi^{'''}(t))^2}{(\psi''(t))^2}. \tag{18}$$

From the above we see immediately that if the fourth order derivative of ψ exists and is negative then the interaction function is negative.

Property 4. $i_{12} < 0$ if $\psi^{(4)}(x) < 0$.

The next example presents the generator 4.2.17 as given in [19] with the property that the second order derivative of this generator is neither log-convex nor log-concave for certain values of its parameters.

Example 2.4 *The generator is of the form*

$$\psi(x) \quad = \quad \left((2^{-\theta} - 1)\exp(-x) + 1\right)^{-\frac{1}{\theta}} - 1 \text{ for } \theta \in (-\infty, \infty) \setminus \{0\}.$$

For $\theta = -1$ we get the generator of the independent copula and when θ tends to infinity we obtain the upper Freshet bound. The general form of the interaction function leads to a long formula. In Figure 2 we show a plot of this generator for $\theta = -6$, its first and second order derivatives and the $(\log \psi^{''}(x))^{''}$ which determines the sign of the interaction function of the copula generated by ψ. It can be observed that $(\log \psi^{''}(x))^{''}$ is positive up to about $x = 3$ and then stays negative. For $\theta > -1$ the interaction function is positive.

Property 5. There exists an Archimedean copula with an interaction function that changes sign for different regions of the unit square.

2.3.3 Proportionality to the density

In Section 2.2.1 we observed that Frank's copula has an interaction function which is proportional to the density. We will now show that Frank's copula is the only Archimedean copula with this property.

Property 6. Frank's copula is the only Archimedean copula with interaction function proportional to the density.

PROOF of Property 6 can be found in Appendix A.

2.3.4 Relationship with tail dependence

Similar to the discussion in Example 2.1 and by using the properties of the regularly varying function, we see that if we assume that $\psi \in \mathrm{RV}_{-\alpha}$ then (since ψ is d-monotone) $\psi' \in \mathrm{RV}_{-\alpha-1}$ and $\psi'' \in \mathrm{RV}_{-\alpha-2}$. Moreover, $\psi^{-1} \in \mathrm{RV}_{-1/\alpha}$. The logarithm of a regularly varying function is slowly varying, denoted as RV_0, hence $\log \psi'' \in \mathrm{RV}_0$. If the first and second derivatives of $\log \psi''$ are eventually monotonic then the monotonic density theorem [20] can be used and we get $(\log \psi'')'' \in \mathrm{RV}_{-2}$. Hence the interaction function $i_{12}^c(u, u)$ of an Archimedean copula with generator ψ is of order u^{-2} as $u \to 0$. Using Mill's ratio we can conclude that

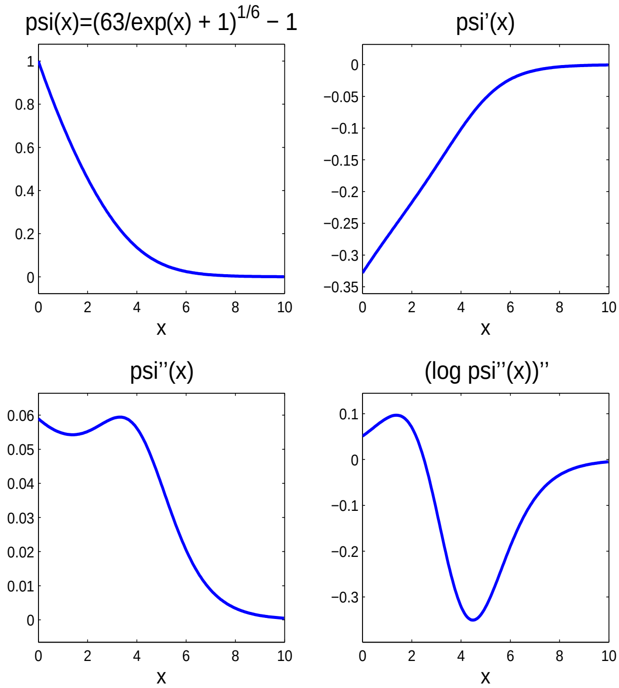

Figure 2: Generator $\psi(x)$ for $\theta = -6$ and its first and second order derivatives as well as part of the interaction function which determines its sign.

$$id_L = \lim_{z \to -\infty} i_{12}^{c,z}(z,z) = \infty.$$

Analogously, the regular variation of $1-\psi$ at zero and the monotonicity of the derivatives of $\log \psi''$ would lead to infinite behaviour of $i_{12}^{c,z}(z,z)$ at $z \to -\infty$.

Property 7. If $\psi \in \mathrm{RV}_{-\alpha}$ and the first and second derivatives of $\log \psi''$ are eventually monotone then $id_L = \infty$. Similarly if $1-\psi \in \mathrm{RV}_{-\alpha}$ at zero then $id_U = \infty$.

In the next section we extend the definition of the bivariate interaction function to higher dimensions and further study their properties for canonical d-dimensional Archimedean copulas.

3 k-dimensional interaction function

Let $D = \{1, 2, ..., d\}$ and let $\mathbf{X}_D = (X_1, ..., X_d)$ be an absolutely continuous random vector with density f_D. We assume that the support of f_D is a rectangle in $\mathbf{R}^d$ and that f_D is d times differentiable. Denote by $S_k = \{i_1, ..., i_k\}$, where $i_j \in D, j = 1, ..., k$ and $2 \leq k \leq d$ are subsets of D with k different elements (notice $S_d = D$). Moreover let $\mathbf{X}_{S_k} = (X_{i_1}, ..., X_{i_k})$ and f_{S_k} be the k-dimensional margin of f_D.

Then the k-dimensional interaction function for f_D can be defined as follows:

Definition 3.1 *For a given S_k, the k-dimensional interaction function for the density f_D, which is k times differentiable, is*

$$i_{S_k}(x_1, ..., x_d) = \frac{\partial^k}{\partial x_{i_1} ... \partial x_{i_k}} \log f_D(x_1, ..., x_d).$$

All two dimensional interactions equal to zero means that the joint density can be represented as a product of functions depending on only one variable. This happens only if the random variables $X_i, i = 1, ..., d$ are mutually independent. If all higher than two order interactions are zero then the density can be rewritten as a product of functions depending on two variables. This is a departure from mutual independence but the joint density has still quite an easy form e.g. the multivariate gaussian distribution is the exponential function of a quadratic form, hence all higher than 2-dimensional interactions are equal to zero. The knowledge that some higher order interactions of a density are equal to zero indicates the complexity of the dependance structure of this density. In other words the maximal order of interactions not equal to zero might be a measure of departure from independence in a density.

We now study properties of k- dimensional interaction functions for canonical Archimedean copulas. Similarly as in the two dimensional case, properties of the k-interaction function for canonical Archimedean copulas can be studied by examining properties of its generator.

A copula is called canonical d-Archimedean if it has the form

$$C_\psi(u_1, ..., u_d) = \psi\left(\sum_{i=1}^{d} \psi^{-1}(u_i)\right), \tag{19}$$

where the following properties (a),(b) and (c) are satisfied

(a) $\psi : [0, \infty] \to [0, 1]$ is d-monotone, and

(b) $\psi(0) = 1$, $\lim_{x \to \infty} \psi(x) = 0$.

Moreover, we assume that C_ψ is $d \times k$ times differentiable with $k \leq d$, and C_ψ is strictly positive except for $u_i = 0, i = 1, .., d$, which can be translated to the additional condition:

(c) ψ is $d \times k$ times differentiable with $k \leq d$.

Then the density function c_ψ which is nonzero on $(0,1]^d$ is equal to:

$$c_\psi(u_1, ..., u_d) = \frac{\psi^{(d)}\left(\sum_{j=1}^d \psi^{-1}(u_j)\right)}{\prod_{j=1}^d \psi'(u_j)}, \tag{20}$$

and the k-interaction function is:

$$i_k(u_1, ..., u_d) = i_{S_k}(u_1, ..., u_d) = \frac{1}{\prod_{j=1}^k \psi'(u_{i_j})} \frac{d^k}{dt^k} \log\left((-1)^d \psi^{(d)}(t)\right), \tag{21}$$

where $t = \sum_{j=1}^d \psi^{-1}(u_j)$.

When $d = 2$ the interaction of Archimedean copula with generator ψ is equal to zero if and only if ψ is of the form $\exp(-\theta t)$ where $\theta > 0$. We will now investigate what form the generator can have in case of a d-dimensional Archimedean copula with the k-interaction functions equal to zero.

3.1 k-interactions equal to zero

The canonical Archimedean copula (19) describes exchangeable dependence, hence each of its k-dimensional ($1 \leq k \leq d$) marginal distributions are the same. When $\psi(x) = \exp(-\theta x), \theta > 0$ the density of the canonical Archimedean copulas is equal to 1 at every point of the unit hypercube. This is the density of mutually independent random variables for which all k-interactions ($2 \leq k \leq d$) are equal to zero. We will investigate if there are generators for which the k-interactions are zero but the lower order ones are not necessarily zero. The answer to this question is given in Theorem 3.1.

Theorem 3.1 *There are $k-1$ functionally independent solutions of the differential equation*

$$\frac{d^k}{dx^k} \log\left((-1)^d \psi^{(d)}(x)\right) \quad = \quad 0 \tag{22}$$

satisfying the conditions (a),(b) and (c). These solutions are of the following form:

$$\psi_m(x) \quad = \quad \frac{\int_x^\infty (y-x)^{d-1} \exp(P_m(y)) dy}{\int_0^\infty y^{d-1} \exp(P_m(y)) dy}, \tag{23}$$

where $m = 1, 2, ..., k-1$ and $P_i(y)$ denotes a polynomial in y of degree i:

$$P_i(y) \quad = \quad c_i y^i + c_{i-1} y^{i-1} + ... + c_1 y$$

with $c_i < 0$ and $c_{i-1}, ..., c_1$ arbitrary.

Notice that $\psi_1(x) = \exp(c_1 x)$ with $c_1 < 0$ is the generator of mutually independent copula. For this generator all interactions of size 2 to d are equal to zero. Hence $\psi_1(x)$ is certainly one of the possible solutions of (22) for $k = 2, ..., d$. The $\psi_2(x)$ is of the following form

$$\psi_2(x) \quad = \quad \frac{\int_x^\infty (y-x)^{d-1} \exp(c_2 y^2 + c_1 y) dy}{\int_0^\infty y^{d-1} \exp(c_2 y^2 + c_1 y) dy},$$

where $c_2 < 0$ and c_1 arbitrary. We can see that $\psi_2(x)$ is d- monotone and $\log\left((-1)^d \psi_2^{(d)}(x)\right)$ is a second order polynomial, hence its third and higher order derivatives are zero. The second order derivative of $\log\left((-1)^d \psi_2^{(d)}(x)\right)$ does not have to vanish. This means that $\psi_2(x)$ is one of the

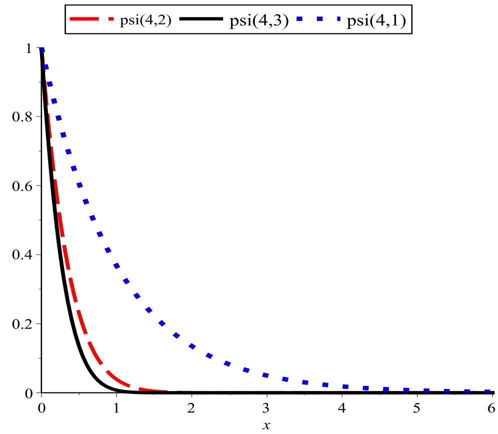

Figure 3: Generators of canonical Archimedean copula with 4-interactions equal to 0. psi(4,1)=exp($-x$) has 2,3, and 4-interactions equal to zero (for each bivariate margins of the copula Kendall's $\tau = 0$), for psi(4,2), given by (23) with $c_2 = -1, c_1 = 0$ we get 3 and 4-interactions equal to zero ($\tau = -0.0686$) and for psi(4,3), given by (23) with $c_3 = -1, c_2 = c_1 = 0$ only 4-interactions are zero ($\tau = -0.0954$).

possible solutions of (22) for $k \geq 3$. $\psi_1(x)$ and $\psi_2(x)$ are functionally independent etc.

PROOF of Theorem 3.1 that follows the above line of reasoning can be found in Appendix B.

The result above can be also obtained through the relationship between canonical Archimedean copulas and simplex distributions as discussed in [15]. Notice that (23) can be rewritten as:

$$\psi_m(x) \quad = \quad \int_x^\infty \left(1 - \frac{x}{t}\right)^{d-1} \frac{t^{d-1}\exp(P_m(t))}{\int_0^\infty y^{d-1}\exp(P_m(y))dy} dt$$

which is the Williamson d-transform of a non-negative random variable with density

$$f_m(x) \quad = \quad \frac{x^{d-1}\exp(P_m(x))}{\int_0^\infty y^{d-1}\exp(P_m(y))dy}, \; x \geq 0, \; m = 1, 2, ..., k-1.$$

Alternative PROOF of Theorem 3.1 using the relationship between the generator and the radial density can also be found in Appendix B.

We see that the solutions ψ_m correspond to simplex distributions with Kotz type distributed radial densities (a simple form of this density is given in (8)). The radial random variable R_m with density f_m belongs to $MDA(\Lambda)$ with auxiliary function $a(x) = 1/x^{m-1} \in \mathrm{RV}_{-(m-1)}$. Hence we get for bivariate margins of this Archimedean copula $\lambda_L = 0, \mu_L = 2^{1/(m-1)}$ and the copula does not have the intermediate tail dependence. Similarly the upper tail $\lambda_U = 0$ and since $1/R$ is regularly varying at zero with index larger than d there is no intermediate tail dependence but there is a negative dependence in the bivariate tails.

Figure 3 shows plots of three generators (psi(d,m) for $d = 4$, $m = 1, 2, 3$) of canonical Archimedean copula with 4-interactions equal to zero. The function psi(4,1) is a generator of the independent copula. The functions psi(4,2) and psi(4,3) lead to three and four and only four order interactions equal to zero, respectively. The Kendall's tau for each of their bivariate margins are calculated numerically.

3.2 Positivity of k-interactions

From the definition of the k-interaction function one sees that its sign depends on the sign of the derivative of the order k of the function $\log((-1)^d\psi^{(d)}(x))$.

Comparing the result in [18], the positivity of 2-interactions (log convexity of $(-1)^d\psi^{(d)}$) is equivalent with C_ψ being multivariate totally positive of order 2 (MTP$_2$) which is the generalization of the TP$_2$ concept of positive dependence.

Definition 3.2 *f_D is MTP$_2$ if*

$$f_D(\mathbf{x}\vee\mathbf{y})f_D(\mathbf{x}\wedge\mathbf{y}) \geq f_D(\mathbf{x})f_D(\mathbf{y})$$

where $\mathbf{x}\vee\mathbf{y} = (\max\{x_1,y_1\},...,\max\{x_n,y_n\})$ and $\mathbf{x}\wedge\mathbf{y} = (\min\{x_1,y_1\},...,\min\{x_n,y_n\})$.

The completely monotone generators (Laplace transforms) are log convex and their d-th derivatives multiplied by $(-1)^d$ are also completely monotone, hence are also log convex. This means that the generators of the LT-Archimedean copulas have positive 2-interactions. Positivity of 2-interactions, however, does not guarantee that also the higher order interactions are positive.

Example 3.1 *By simple calculations we get that the d-th derivative of the generator of Clayton's copula is*

$$\frac{d^d\psi^{Cl}}{dx^d} = (-1)^d\frac{1}{\theta}\left(\frac{1}{\theta}+1\right)\cdot ... \cdot\left(\frac{1}{\theta}+d-1\right)(1+x)^{-\frac{1}{\theta}-d}.$$

Hence the sign of the k-interactions is determined by

$$-\left(\frac{1}{\theta}+d\right)\frac{d^k}{dx^k}\log(1+x).$$

This means that the k-th order interactions for Clayton's copula are positive when k is even and negative for odd k.

3.3 Proportionality of d-interactions to the density

The interesting property has been observed in Section 2.3. We showed that the Frank's copula is the unique copula for which the interaction function is proportional to the density. Hence we can see that the local dependence is high in points when the density is high. The significance of this result is rather of theoretical than practical importance. In this section we examine whether the higher order interactions can be proportional to the density for canonical Archimedean copula.

Comparing (20) and (21) we see that the k-interaction function can be proportional to the density only in case $k = d$. We want to find a generator that satisfies conditions (a), (b) and (c) in Section 3.2 and such that

$$\frac{d^d}{dx^d}\log\left((-1)^d\psi^{(d)}(x)\right) = A\psi^{(d)}(x) \tag{24}$$

where A is a non-zero constant.

Theorem 3.2 *There are $d-1$ functionally independent solutions of the differential equation (24) satisfying conditions (a),(b) and (c). These solutions can be represented as solutions of the integral equations*

$$\psi_m(x) = \frac{\int_x^\infty (y-x)^{d-1}\exp(A\psi_m(y)+P_m(y))dy}{\int_0^\infty y^{d-1}\exp(A\psi_m(y)+P_m(y))dy}, \tag{25}$$

where $m = 1, 2, ..., d-1$ *and* $P_i(y)$ *denotes a polynomial in* y *of degree* i*:*

$$P_i(y) \quad = \quad c_i y^i + c_{i-1} y^{i-1} + ... + c_1 y$$

with $c_i < 0$ *and* $c_{i-1}, ..., c_1$ *arbitrary.*

Notice that (25) simplifies to (23) when $A = 0$. For fixed parameters c_i and A each integral equation has a unique solution since the associated to (25) map $T : \psi \to T_\psi$ with

$$T_\psi = \frac{\int_x^\infty (y-x)^{d-1} \exp(A\psi + P_m(y))dy}{\int_0^\infty y^{d-1} \exp(A\psi + P_m(y))dy}$$

is a contracting map in the space of continuous functions on $[0, \infty)$, and consequently T has a unique fixed points in this space. The integral equations are therefore well suited to be solved numerically. For $A \neq 0$ it might be impossible to construct (exact) analytical solutions for $d > 2$ but numerical approximation can always be constructed. For $d = 2$ (25) has an analytical solution.

PROOF of Theorem 3.2 can be found in Appendix C. Afterwards we show how the integral equation can be solved for the case $d = 2$ and leads to one solution corresponding to the generator of Frank's copula.

4 Conclusions

In this paper we studied properties of the interaction functions for densities and corresponding copulas. The interaction function of a copula is not equal to the interaction function of the corresponding distribution. They, however, have the same sign.

The interaction function gives information about local dependence in a density. It allows to study independencies and conditional independencies in a joint distribution as well as the complexity of the dependence structure. This is done by investigating how the density can be factorized into the products of lower dimensional functions.

In this paper we looked at the interaction function of the elliptical and Archimedean copulas as they are determined by the one dimensional function. The properties of the interaction function can be translated into the properties of generators of these densities. This naturally simplifies the problem. Much more research is needed to be able to interpret properties of general distributions from the behaviour of corresponding interaction functions.

Since the interaction function provides information about local dependence then it is not surprising that its extreme behaviour would be related to the extreme behaviour of a copula. We made a first step in investigating the relationship between the extremes of the iteration function $i_{12}^{c,z}$ for meta gaussian distributions, that is, for distributions with standard normal margins and a copula density c. We noticed that if c has tail dependence then $i_{12}^{c,z}(z,z)$ is infinite as $z \to \infty$. In the case of a copula with tail independence the interaction function of the meta gaussian density seems to go to zero and when the copula with density c has intermediate tail dependence one can expect that $i_{12}^{c,z}(z,z)$ converges to a constant that is not equal to zero (see the gaussian distribution and the Gumbel copula). More research is needed to establish full classification of extremal behaviour of the interaction function.

Appendix A

PROOF of Property 6
We present here only the main steps of the proof and refer the reader to the full solution of this problem in Appendix C formulated in Section 3.3 for the general case.

Comparing equations (12) and (13) it is easy to see that the interaction function is proportional to its density function if

$$\frac{d^2}{dx^2}(\log(\psi^{''}(x)) = \frac{\psi^{(4)}(x)\psi^{''}(t) - (\psi^{'''}(x))^2}{(\psi^{''}(x))^2} = A\psi^{''}(x),\ A \neq 0.$$

This is a second order differential equation for $h = \psi^{''}$ that can be solved by substituting $y = h^{'}$. This substitution leads to a Bernoulli equation which can be simplified to a first order linear equation. Solving this equation and incorporating the conditions (10), leads to the desired result.

Appendix B

PROOF of Theorem 3.1
The condition (b) is trivially satisfied. The functions (23) are differentiable and their derivatives up to $d-1$-th order are of the form

$$\frac{d^s}{dx^s}\psi_m(x) = (-1)^s(d-1)\cdot(d-2)\cdot\ldots\cdot(d-s)\frac{\int_x^\infty (y-x)^{d-1-s}\exp(P_m(y))dy}{\int_0^\infty y^{d-1}\exp(P_m(y))dy}$$

$s = 1, ..., d-1$ and the d-th order derivative calculated with Leibnitz rule is equal to:

$$\frac{d^d}{dx^d}\psi_m(x) = (-1)^d(d-1)!\frac{\exp(P_m(x))}{\int_0^\infty y^{d-1}\exp(P_m(y))dy}.$$

Hence ψ_m are d-monotone and sufficiently differentiable. $\log\left((-1)^d\psi_m^{(d)}(x)\right)$ is the m-th order polynomial for which the $m+1$-th and all higher order derivatives are zero. The lower order derivatives do not have to be equal to zero. Moreover the functions are functionally independent.

PROOF (Alternative) of Theorem 3.1

Since ψ is $k \times d$ times differentiable then the corresponding radial density f_R exists. Using Leibnitz rule we can differentiate ψ d-times and find $\psi^{(d)}$. We can also use the relationship (17) and get that

$$\psi^{(d)}(x) = (-1)^d(d-1)!\frac{f_R(x)}{x^{d-1}}. \tag{26}$$

Hence the radial densities of solutions of the differential equation (22) must be of the form

$$f_R(x) \propto x^{d-1}\exp(c_0 + c_1x + \ldots + c_{k-1}x^{k-1}).$$

Only $k-1$ functionally independent solutions are available as the constant part of the polynomial disappears through the normalization of the density. The constant corresponding to the highest power of the polynomial in each solution has to be negative as otherwise this would not be a proper density.

Appendix C

PROOF of Theorem 3.2
The differential equation (24) can be integrated d times, yielding

$$\log\left((-1)^d\psi^{(d)}(x)\right) = A\psi(x) + c_{d-1}x^{d-1} + \ldots + c_1x + c_0 = A\psi(x) + Q_{d-1}(x) \tag{27}$$

To get a solution satisfying the conditions a),b) and c) we must impose $c_{d-1} < 0$ or $c_{d-1} = 0$ and $c_{d-2} < 0$ or, ... , or $c_{d-1} = ... = c_2 = 0$ and $c_1 < 0$. This will give $d-1$ possible solutions of (24).

The equation (27) can be rewritten as

$$\left((-1)^d \psi^{(d)}(x)\right) = \exp\left(A\psi(x) + Q_{d-1}(x)\right).$$

Integrating the above from 0 to x gives

$$\psi^{(d-1)}(x) \quad = \quad \psi^{(d-1)}(0) + (-1)^d \int_0^x \exp\left(A\psi(y) + Q_{d-1}(y)\right) dy$$

Noticing that due to the restrictions on $c_i, i = 1, ..., d-1$ we have $\lim_{x\to\infty} \psi^{(d-1)}(x) = 0$, and so

$$\psi^{(d-1)}(0) = (-1)^{d-1} \int_0^\infty \exp\left(A\psi(y) + Q_{d-1}(y)\right) dy$$

which leads to

$$\psi^{(d-1)}(x) \quad = \quad (-1)^{d-1} \int_x^\infty \exp\left(A\psi(y) + Q_{d-1}(y)\right) dy. \tag{28}$$

Integrating (28) again from 0 to x and including the information that $\lim_{x\to\infty} \psi^{(d-2)}(x) = 0$, the following can be obtained

$$\psi^{(d-2)}(x) = (-1)^{d-2} \int_x^\infty \int_\tau^\infty \exp\left(A\psi(y) + Q_{d-1}(y)\right) dy d\tau$$

which by changing the order of integration gives

$$\psi^{(d-2)}(x) = (-1)^{d-2} \int_x^\infty (y-x) \exp\left(A\psi(y) + Q_{d-1}(y)\right) dy.$$

Repeating the above steps, integrating from 0 to x and rearranging the order of integrals, we get

$$\psi^{(d-3)}(x) = (-1)^{d-3} \frac{1}{2} \int_x^\infty (y-x)^2 \exp\left(A\psi(y) + Q_{d-1}(y)\right) dy.$$

Finally after d such steps

$$\psi(x) \quad = \quad \frac{1}{(d-1)!} \int_x^\infty (y-x)^{d-1} \exp\left(A\psi(y) + Q_{d-1}(y)\right) dy. \tag{29}$$

Since $\psi(0) = 1$ then we also have

$$1 = \frac{1}{(d-1)!} \int_0^\infty y^{d-1} \exp\left(A\psi(y) + Q_{d-1}(y)\right) dy$$

which gives that

$$e^{c_0} \quad = \quad \frac{1}{\frac{1}{(d-1)!} \int_0^\infty y^{d-1} \exp\left(A\psi(y) + c_{d-1}y^{d-1} + ... + c_1 y\right) dy}. \tag{30}$$

From (29) and (30) it follows that ψ, that is, the possible solutions of the differential equation (24) have to satisfy (25).

PROOF (Solution of (24) for the case of $d = 2$)

In case $d = 2$ an analytical solution of (24) can be found. In this case the equation can be rewritten as:

$$\psi^{(2)}(x) = e^{A\psi(x)+c_1x+c_0} \text{ with } c_1 < 0.$$

Denoting as $v(x) = A\psi(x) + c_1x + c_0$ and noticing that $A\psi^{(2)}(x) = v^{(2)}(x)$ we get

$$v^{(2)}(x) = Ae^{v(x)}.$$

Multiplying the above by $v'(x)$ and integrating both sides from 0 to x leads to:

$$v'(x)^2 \quad = \quad 2Ae^{v(x)} - 2Ae^{v(0)} + v'(0)^2 = 2Ae^{v(x)} + k. \tag{31}$$

If $A > 0$ then $k = -2Ae^{A+c_0} + (A\psi'(0) + c_1)^2$ should be non-negative. Thus two cases have to be studied i) $A > 0$ and $k = 0$ and ii) $A > 0$ and $k > 0$. If $A < 0$ then k is always positive. Hence we need to consider also the case iii) $A < 0$ and $k > 0$.

Case i: $A > 0$ and $k = 0$.

Since $v'(x) = A\psi'(x) + c_1 < 0$ then equation (31) can be written as follows

$$v'(x) = -\sqrt{2A}e^{v(x)/2}.$$

This is a separable equation which leads to the solution:

$$v(x) = -2\log\left(\sqrt{\frac{A}{2}}x + e^{-\frac{A+c_0}{2}}\right).$$

Hence ψ is

$$\psi(x) = -\frac{2}{A}\log\left(\sqrt{\frac{A}{2}}x + e^{-\frac{A+c_0}{2}}\right) - \frac{c_1}{A}x - \frac{c_0}{A}.$$

This solution does not tend to zero for x tending to infinity.

Case ii: $A > 0$ and $k > 0$.

In this case the following equation has to be solved[1]

$$v'(x) = -\sqrt{2Ae^{v(x)} + k}.$$

Substituting $w(x) = \sqrt{2Ae^{v(x)} + k}$ we get $w^2(x) - k = 2Ae^{v(x)}$ and the equation above reduces to the separable equation

$$\frac{1}{k - w^2}w'(x) = \frac{1}{2}$$

which leads to the solution of the form

$$w(x) = \sqrt{k}\frac{\frac{\sqrt{k}+w(0)}{\sqrt{k}-w(0)}e^{\sqrt{k}x} - 1}{\frac{\sqrt{k}+w(0)}{\sqrt{k}-w(0)}e^{\sqrt{k}x} + 1}.$$

Since $w^2(x) = 2Ae^{v(x)} + k$ and $v(x) = A\psi(x) + c_1x + c_0$ we find after some algebra that

$$A\psi(x) = A - (c_1 + \sqrt{k})x + 2\log(2\sqrt{k}) - 2\log(\sqrt{k} + w(0) + (\sqrt{k} - w(0))e^{-\sqrt{k}x}). \tag{32}$$

Incorporating the condition $\lim_{x\to\infty}\psi(x) = 0$ we get that

$$c_1 = -\sqrt{k} \tag{33}$$

[1] Notice that similar to Case i) only the negative sign has to be taken into account.

and

$$A + 2\log(2\sqrt{k}) - 2\log(\sqrt{k} + w(0)) = 0. \tag{34}$$

Since $w(0) = \sqrt{2Ae^{A+c_0} + k}$ then from (33) and (34) we can find that

$$e^{c_0} = \frac{2}{A}c_1^2(1 - e^{-A/2}).$$

The above can be included into (32) and we get

$$\psi(x) = -\frac{2}{A}\log\left(1 - (1 - e^{-A/2})e^{c_1 x}\right)$$

with $c_1 < 0$, which is the generator of Frank's copula.

Case iii: $A < 0$ and $k > 0$

The solution procedure presented in case ii) can be repeated analogously in this case yielding

$$\psi(x) = -\frac{2}{A}\log\left(1 - (1 - e^{-A/2})e^{c_1 x}\right) \text{ with } c_1 < 0.$$

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