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Estimation in singular linear models with stepwise inclusion of linear restrictions

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ABSTRACT

In this paper, we consider the general linear model $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}\}$, without any rank assumptions to the model matrix $\mathbf{X}$ and covariance matrix $\boldsymbol{\Sigma}$, and its two restricted models $\mathscr{M}_{r_1} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}|\mathbf{A}_1\boldsymbol{\beta} = \mathbf{r}_1, \boldsymbol{\Sigma}\}$ and $\mathscr{M}_{r_{12}} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}|\mathbf{A}\boldsymbol{\beta} = \mathbf{r}, \boldsymbol{\Sigma}\}$, where $\mathbf{r} = (\mathbf{r}_1', \mathbf{r}_2')'$ and $\mathbf{A} = (\mathbf{A}_1', \mathbf{A}_2')'$. We give the necessary and sufficient conditions for the BLUEs to equal under $\mathscr{M}$ and $\mathscr{M}_{r_1}$, as well as under $\mathscr{M}_{r_1}$ and $\mathscr{M}_{r_{12}}$. We also derive that the BLUEs under $\mathscr{M}_{r_1}$ are superior to the BLUEs under $\mathscr{M}$, and that the BLUEs under $\mathscr{M}_{r_{12}}$ are superior to the BLUEs under $\mathscr{M}_{r_1}$ in the sense of the covariance matrix.

1. Introduction

Consider a general linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \qquad \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}, \qquad \mathrm{cov}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}, \quad \text{or in compact form } \mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}\}, \tag{1.1}$$

where $\mathbf{y}$ is an $n \times 1$ observable random vector, $\mathbf{X}$ is given $n \times p$ matrix of arbitrary rank, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, $\boldsymbol{\Sigma}$ is a known nonnegative definite matrix of arbitrary rank. Also assume

$$\mathbf{r}_1 = \mathbf{A}_1\boldsymbol{\beta} \tag{1.2}$$

and

$$\mathbf{r}_2 = \mathbf{A}_2\boldsymbol{\beta} \tag{1.3}$$

to be disjoint sets of m_1 and m_2 linear restrictions, respectively, with $m_1 + m_2 = m$. We denote by

$$\mathbf{r} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \boldsymbol{\beta} = \mathbf{A}\boldsymbol{\beta}, \tag{1.4}$$

the full set of restrictions with $r(\mathbf{A}) = m$. Such information may arise from different sources like past experience or long association of the experimenter with the experiment, similar kind of experiments conducted in the past, etc.; see Section 5.1 in [11]. The model (1.1) subject to (1.2) can be written in the following form

$$\mathscr{M}_{r_1} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}|\mathbf{A}_1\boldsymbol{\beta} = \mathbf{r}_1, \boldsymbol{\Sigma}\}. \tag{1.5}$$

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Furthermore, by unifying the model (1.5) and linear restrictions (1.3), we have

$$\mathscr{M}_{r_{12}} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}|\mathbf{A}\boldsymbol{\beta} = \mathbf{r}, \boldsymbol{\Sigma}\}. \tag{1.6}$$

Corresponding to (1.5) and (1.6), by utilization of the method of mixed estimation suggested by H. Theil and A.S. Goldberger [15], we have

$$\mathscr{M}_{c_1} = \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{r}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\} \tag{1.7}$$

and

$$\mathscr{M}_{c_{12}} = \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{r} \end{pmatrix}, \begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}. \tag{1.8}$$

When $\mathbf{X}$ has full column rank, some authors showed that the OLSE (ordinary least squares estimator) of $\boldsymbol{\beta}$ under $\mathscr{M}_{r_{12}}$ is superior to the OLSE of $\boldsymbol{\beta}$ under $\mathscr{M}_{r_1}$, and that the OLSE of $\boldsymbol{\beta}$ under $\mathscr{M}_{r_1}$ is superior to the OLSE of $\boldsymbol{\beta}$ under $\mathscr{M}$ in the sense of the covariance matrix; see Section 5.4 in [11,2]. As is well known, OLSEs and BLUEs of unknown parameters in a general linear model are two of the most useful estimations and defined according to different optimal criteria. This prompts us to compare the superiority of BLUEs of $\boldsymbol{\beta}$ under $\mathscr{M}$, $\mathscr{M}_{r_1}$ and $\mathscr{M}_{r_{12}}$. Further, on the assumption that full column rank of $\mathbf{X}$ is removed, we consider the superiority of BLUEs of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$, $\mathscr{M}_{r_1}$ and $\mathscr{M}_{r_{12}}$.

Another purpose of this article is to investigate the relations among the BLUEs of the mean vector $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$, $\mathscr{M}_{r_1}$ and $\mathscr{M}_{r_{12}}$. There is a rich literature on equality of BLUEs under $\mathscr{M}$ and $\mathscr{M}_{r_1}$. For example, T. Mathew [7] investigated the equality of the BLUEs of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ and $\mathscr{M}_{r_1}$ and a necessary and sufficient condition was given. Recently, Y. Tian [17] and X. Ren [13] revisited the equivalence of the BLUEs of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ and $\mathscr{M}_{r_1}$ and obtained some new equivalent conditions by rank of matrix. Y. Tian [18] derived the equality of the BLUEs under $\mathscr{M}_{r_1}$ and $\mathscr{M}_{c_1}$. The related work in this area can be found in [1,8,3,12,5].

Now we recall that an unbiased linear estimator $\mathbf{Gy}$ of $\mathbf{X}\boldsymbol{\beta}$ is the BLUE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ if

$$\mathrm{cov}(\mathbf{Gy}) \leq_{\mathrm{L}} \mathrm{cov}(\mathbf{Ly}) \quad \forall \mathbf{L} : \mathbf{LX} = \mathbf{X}, \tag{1.9}$$

where "$\leq_{\mathrm{L}}$" refers to the Löwner ordering. It is well known that the general expression for BLUE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$ can be written in some closed forms through generalized inverses of matrices. The following lemma was given by C.R. Rao [10].

Lemma 1.1. *A linear estimator* $\mathbf{Gy}$ *is the BLUE of* $\mathbf{X}\boldsymbol{\beta}$ *under* $\mathscr{M}$ *if and only if the matrix* $\mathbf{G}$ *satisfied the following equation:*

$$\mathbf{G}(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}) = (\mathbf{X}, \mathbf{0}). \tag{1.10}$$

This equation is always consistent, that is, $\mathscr{C}((\mathbf{X}, \mathbf{0})') \subseteq \mathscr{C}((\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})')$ *holds. In this case, the general solution to* (1.10)*, denoted by* $\mathbf{P}_{\mathbf{X};\boldsymbol{\Sigma}}$*, can be expressed in the following parametric form*

$$\mathbf{P}_{\mathbf{X};\boldsymbol{\Sigma}} = (\mathbf{X}, \mathbf{0})(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})^{\dagger} + \mathbf{U}\mathbf{E}_{(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})}, \tag{1.11}$$

where $\mathbf{U} \in \mathbb{R}^{n\times n}$ *is arbitrary, and the general expression for the BLUE of* $\mathbf{X}\boldsymbol{\beta}$ *under* $\mathscr{M}$ *can be written as*

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathbf{P}_{\mathbf{X};\boldsymbol{\Sigma}}\mathbf{y}. \tag{1.12}$$

In particular,

(a) $\mathrm{E}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} = \mathbf{X}\boldsymbol{\beta}$ *and* $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} = \mathbf{P}_{\mathbf{X};\boldsymbol{\Sigma}}\boldsymbol{\Sigma}\mathbf{P}'_{\mathbf{X};\boldsymbol{\Sigma}}$.
(b) $\mathscr{C}(\mathbf{X}, \boldsymbol{\Sigma}) = \mathscr{C}(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})$ *and* $r(\mathbf{X}, \boldsymbol{\Sigma}) = r(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})$.
(c) *The product* $\mathbf{P}_{\mathbf{X};\boldsymbol{\Sigma}}\boldsymbol{\Sigma}$ *can uniquely be written as* $\mathbf{P}_{\mathbf{X};\boldsymbol{\Sigma}}\boldsymbol{\Sigma} = (\mathbf{X}, \mathbf{0})(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})^{\dagger}\boldsymbol{\Sigma}$.

Remark 1.1. Recently, it was shown that the above result can also be proved by inertia of matrix, see Y. Tian [4].

In addition to (1.12), we may recall here three general representations for the $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})$:

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathbf{y} - \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})^{-}\mathbf{E}_{\mathbf{X}}\mathbf{y}, \tag{1.13}$$

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathbf{P}_{\mathbf{X}}\mathbf{y} - \mathbf{P}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})^{-}\mathbf{E}_{\mathbf{X}}\mathbf{y}, \tag{1.14}$$

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathbf{X}(\mathbf{X}'\mathbf{W}^{\dagger}\mathbf{X})^{-}\mathbf{X}'\mathbf{W}^{\dagger}\mathbf{y}, \tag{1.15}$$

where

$$\mathbf{W} = \boldsymbol{\Sigma} + \mathbf{XUX}', \tag{1.16}$$

and $\mathbf{U}$ is an arbitrary matrix such that

$$\mathscr{C}(\mathbf{W}) = \mathscr{C}(\mathbf{X}, \boldsymbol{\Sigma}), \tag{1.17}$$

see C.R. Rao [10].

The paper is organized as follows. In Section 2, we present general expressions of BLUEs of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}_{r_1}$, $\mathscr{M}_{c_1}$, $\mathscr{M}_{r_{12}}$ and $\mathscr{M}_{c_{12}}$, and give the necessary and sufficient conditions for the equality of the BLUEs of $\mathbf{X}\boldsymbol{\beta}$ and the equality of the BLUEs of $\boldsymbol{\beta}$ under $\mathscr{M}_{r_1}$ and $\mathscr{M}_{r_{12}}$ to hold, respectively, which is motivated by the work of [7,17,13]. In Section 3, we discuss the superiority of the BLUEs of $\mathbf{X}\boldsymbol{\beta}$, as well as the superiority of the BLUEs of $\boldsymbol{\beta}$ under the models mentioned above.

Throughout this paper, we will use the symbol $\mathbb{R}^{m\times n}$ to denote the collection of all $m\times n$ real matrices. The symbols $\mathbf{A}'$, $\mathbf{A}^-$, $\mathbf{A}^\dagger$, $r(\mathbf{A})$ and $\mathscr{C}(\mathbf{A})$ denote, respectively, the transpose, the generalized inverse, the Moore–Penrose inverse, the rank and the range (column space) of a real matrix $\mathbf{A}$. $\mathbf{A} > \mathbf{B}$ ($\geq \mathbf{0}$, $< \mathbf{0}$, $\leq \mathbf{0}$) means that $\mathbf{A}-\mathbf{B}$ is positive definite (positive semi-definite, negative definite, negative semi-definite), and by $(\mathbf{A}, \mathbf{B})$ we refers to the partitioned matrix with $\mathbf{A}$ and $\mathbf{B}$ as submatrices. Moreover, let $\mathbf{P_A}$, $\mathbf{E_A}$ and $\mathbf{F_A}$ stand for the three orthogonal projectors $\mathbf{P_A} = \mathbf{AA}^\dagger$, $\mathbf{E_A} = \mathbf{I}_m - \mathbf{AA}^\dagger$ and $\mathbf{F_A} = \mathbf{I}_n - \mathbf{A}^\dagger\mathbf{A}$. Both $i_+(\mathbf{A})$ and $i_-(\mathbf{A})$, called the partial inertia of $\mathbf{A} = \mathbf{A}'$, are defined to be the number of the positive and negative eigenvalues of $\mathbf{A}$ with multiplicities, respectively.

As is shown in the latter sections, estimations of parametric functions under $\mathscr{M}$, $\mathscr{M}_{r_1}$ and $\mathscr{M}_{r_{12}}$ involve some complicated operations of matrices and their generalized inverse. In order to simplify various matrix expressions including the generalized inverses of matrices, we need the following four lemmas concerning inertia formulas for partitioned matrices given by Y. Tian [16].

Lemma 1.2. *Let* $\mathbf{A} = \mathbf{A}' \in \mathbb{R}^{m\times m}$, $\mathbf{B} = \mathbf{B}' \in \mathbb{R}^{n\times n}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{P} \in \mathbb{R}^{m\times m}$. *Then*

$$i_\pm(\mathbf{PAP}') \leq i_\pm(\mathbf{A}), \qquad i_\pm(-\boldsymbol{A}) = i_\mp(\mathbf{A}), \tag{1.18}$$

$$i_\pm(\mathbf{PAP}') = i_\pm(\mathbf{A}), \quad \text{if } \mathbf{P} \text{ is nonsingular}, \tag{1.19}$$

$$i_\pm\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{pmatrix} = i_\pm(\mathbf{A}) + i_\pm(\mathbf{B}), \tag{1.20}$$

$$i_\pm\begin{pmatrix}\mathbf{0} & \mathbf{C}\\ \mathbf{C}' & \mathbf{0}\end{pmatrix} = r(\mathbf{C}). \tag{1.21}$$

Lemma 1.3. *Let* $\mathbf{A} = \mathbf{A}' \in \mathbb{R}^{m\times m}$. *Then*

$$\mathbf{A} \leq \mathbf{0} \quad \text{if and only if } i_+(\mathbf{A}) = 0. \tag{1.22}$$

In particular,

(a) $\mathbf{A} < \mathbf{0}$ *if and only if* $i_-(\mathbf{A}) = m$.
(b) $\mathbf{A} = \mathbf{0}$ *if and only if* $i_\pm(\mathbf{A}) = 0$.

Lemma 1.4. *Let* $\mathbf{A} = \mathbf{A}' \in \mathbb{R}^{m\times m}$, $\mathbf{B} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{C} = \mathbf{C}' \in \mathbb{R}^{n\times n}$. *Then*

$$i_\pm\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}' & \mathbf{0}\end{pmatrix} = r(\mathbf{B}) + i_\pm(\mathbf{E_B}\mathbf{A}\mathbf{E_B}), \tag{1.23}$$

$$i_\pm\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}' & \mathbf{C}\end{pmatrix} \geq i_\pm(\mathbf{A}). \tag{1.24}$$

In particular,

(a) $i_\pm\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}' & \mathbf{0}\end{pmatrix} = r(\mathbf{B})$, *if* $\mathscr{C}(\mathbf{A}) \subseteq \mathscr{C}(\mathbf{B})$.
(b) $i_\pm\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}' & \mathbf{C}\end{pmatrix} = i_\pm(\mathbf{A})$, *if* $\mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{A})$ *and* $\mathbf{C} = \mathbf{B}'\mathbf{A}^\dagger\mathbf{B}$.

Lemma 1.5. *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{n_1\times n_1}$, $\mathbf{A}_2 = \mathbf{A}_2' \in \mathbb{R}^{n_2\times n_2}$, $\mathbf{B}_1 \in \mathbb{R}^{q_1\times m}$, $\mathbf{B}_2 \in \mathbb{R}^{q_2\times m}$, $\mathbf{G}_1 \in \mathbb{R}^{q_1\times n_1}$ *and* $\mathbf{G}_2 \in \mathbb{R}^{q_2\times n_2}$. *If* $\mathscr{C}(\mathbf{A}_i) \subseteq \mathscr{C}(\mathbf{G}_i')$, $\mathscr{C}(\mathbf{B}_i) \subseteq \mathscr{C}(\mathbf{G}_i)$, $i = 1, 2$. *Then*

$$i_\pm\{\mathbf{B}_1'(\mathbf{G}_1')^\dagger\mathbf{A}_1\mathbf{G}_1^\dagger\mathbf{B}_1 - \mathbf{B}_2'(\mathbf{G}_2')^\dagger\mathbf{A}_2\mathbf{G}_2^\dagger\mathbf{B}_2\} = i_\pm\begin{pmatrix}\mathbf{A}_1 & \mathbf{0} & \mathbf{G}_1' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{A}_2 & \mathbf{0} & \mathbf{G}_2' & \mathbf{0}\\ \mathbf{G}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1\\ \mathbf{0} & \mathbf{G}_2 & \mathbf{0} & \mathbf{0} & \mathbf{B}_2\\ \mathbf{0} & \mathbf{0} & \mathbf{B}_1' & \mathbf{B}_2' & \mathbf{0}\end{pmatrix} - r(\mathbf{G}_1) - r(\mathbf{G}_2). \tag{1.25}$$

Hence,

$$r\{\mathbf{B}_1'(\mathbf{G}_1')^\dagger\mathbf{A}_1\mathbf{G}_1^\dagger\mathbf{B}_1 - \mathbf{B}_2'(\mathbf{G}_2')^\dagger\mathbf{A}_2\mathbf{G}_2^\dagger\mathbf{B}_2\} = r\begin{pmatrix}\mathbf{A}_1 & \mathbf{0} & \mathbf{G}_1' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\mathbf{A}_2 & \mathbf{0} & \mathbf{G}_2' & \mathbf{0}\\ \mathbf{G}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_1\\ \mathbf{0} & \mathbf{G}_2 & \mathbf{0} & \mathbf{0} & \mathbf{B}_2\\ \mathbf{0} & \mathbf{0} & \mathbf{B}_1' & \mathbf{B}_2' & \mathbf{0}\end{pmatrix} - 2r(\mathbf{G}_1) - 2r(\mathbf{G}_2). \tag{1.26}$$

The following lemma can be found in G. Marsaglia and G.P.H. Styan [6].

Lemma 1.6. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ *and* $\mathbf{C} \in \mathbb{R}^{l\times n}$. *Then:*

$$r\,(\mathbf{A}, \mathbf{B}) = r\,(\mathbf{A}) + r(\mathbf{E_A B}) = r\,(\mathbf{B}) + r(\mathbf{E_B A}). \tag{1.27}$$

$$r\begin{pmatrix}\mathbf{A}\\ \mathbf{C}\end{pmatrix} = r\,(\mathbf{A}) + r(\mathbf{C F_A}) = r\,(\mathbf{C}) + r(\mathbf{A F_C}). \tag{1.28}$$

$$r\left(\mathbf{C A^{+} B}\right) = r\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{pmatrix} - r\,(\mathbf{A})\,, \quad \text{if } \mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{A}) \quad \text{and} \quad \mathscr{C}(\mathbf{C}') \subseteq \mathscr{C}(\mathbf{A}'). \tag{1.29}$$

2. The expressions and equalities of the BLUEs

It is well known that a general solution to $\mathbf{A}_1\boldsymbol{\beta} = \mathbf{r}_1$ is given by

$$\boldsymbol{\beta} = \mathbf{A}_1^{\dagger}\mathbf{r}_1 + \mathbf{F}_{\mathbf{A}_1}\boldsymbol{\gamma}, \tag{2.1}$$

where $\boldsymbol{\gamma}$ is an arbitrary vector. Therefore, under the model (1.5), we have

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} = \mathbf{X}(\mathbf{A}_1^{\dagger}\mathbf{r}_1 + \mathbf{F}_{\mathbf{A}_1}\boldsymbol{\gamma}), \quad \text{or equivalently,}$$

$$\mathrm{E}(\mathbf{y} - \mathbf{X}\mathbf{A}_1^{\dagger}\mathbf{r}_1) = \mathbf{X}\mathbf{F}_{\mathbf{A}_1}\boldsymbol{\gamma}.$$

Thus, the model (1.5) can be equivalently written as

$$\{\mathbf{y} - \mathbf{X}\mathbf{A}_1^{\dagger}\mathbf{r}_1, \mathbf{X}\mathbf{F}_{\mathbf{A}_1}\boldsymbol{\gamma}, \boldsymbol{\Sigma}\}, \tag{2.2}$$

which leads to the fact that

$$\mathbf{y} - \mathbf{X}\mathbf{A}_1^{\dagger}\mathbf{r}_1 \in \mathscr{C}(\mathbf{X}\mathbf{F}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) \tag{2.3}$$

holds with probability 1. Y. Tian [18] points out that (2.3) is equivalent to

$$\begin{pmatrix}\mathbf{y}\\ \mathbf{r}_1\end{pmatrix} \in \mathscr{C}\begin{pmatrix}\mathbf{X} & \boldsymbol{\Sigma}\\ \mathbf{A}_1 & \mathbf{0}\end{pmatrix}. \tag{2.4}$$

Thus (2.3) implies $\mathbf{y} \in \mathscr{C}(\mathbf{X}, \boldsymbol{\Sigma})$, i.e., the consistency of the model (1.5) implies the consistency of the model (1.1). Similarly, the consistency of the model (1.6) leads to the consistency of the model (1.5). In what follows, we assume that the model in (1.6) is consistent, i.e.,

$$\begin{pmatrix}\mathbf{y}\\ \mathbf{r}\end{pmatrix} \in \mathscr{C}\begin{pmatrix}\mathbf{X} & \boldsymbol{\Sigma}\\ \mathbf{A} & \mathbf{0}\end{pmatrix} \tag{2.5}$$

holds with probability 1.

Lemma 2.1. *Consider the models* $\mathscr{M}_{r_1}$ *and* $\mathscr{M}_{r_{12}}$, *and define* $\widehat{\mathbf{X}}_1 = (\mathbf{X}', \mathbf{A}_1')'$. *Then:*

(a) *The BLUEs of* $\mathbf{X}\boldsymbol{\beta}$ *and* $\widehat{\mathbf{X}}_1\boldsymbol{\beta}$ *under* $\mathscr{M}_{r_1}$ *can be written as, respectively,*

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathbf{X}\mathbf{A}_1^{\dagger}\mathbf{r}_1 + \mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}}(\mathbf{y} - \mathbf{X}\mathbf{A}_1^{\dagger}\mathbf{r}_1), \tag{2.6}$$

$$\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \widehat{\mathbf{X}}_1\mathbf{A}_1^{\dagger}\mathbf{r}_1 + \begin{pmatrix}\mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}}(\mathbf{y} - \mathbf{X}\mathbf{A}_1^{\dagger}\mathbf{r}_1)\\ \mathbf{0}\end{pmatrix}, \tag{2.7}$$

where $\mathbf{X}_{\mathbf{A}_1} = \mathbf{X}\mathbf{F}_{\mathbf{A}_1}$ *and* $\mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}} = (\mathbf{X}_{\mathbf{A}_1}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})^{\dagger} + \mathbf{U}_1\mathbf{E}_{(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})}$, *in which* $\mathbf{U}_1$ *is arbitrary. Moreover,*

(i) $\mathscr{C}(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) = \mathscr{C}(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma})$ *and* $r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) = r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma})$.

(ii) $\mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}}\boldsymbol{\Sigma} = (\mathbf{X}_{\mathbf{A}_1}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})^{\dagger}\boldsymbol{\Sigma}$.

(b) *The BLUE of* $\mathbf{X}\boldsymbol{\beta}$ *and* $\widehat{\mathbf{X}}_1\boldsymbol{\beta}$ *under* $\mathscr{M}_{r_{12}}$ *can be written as, respectively,*

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}}) = \mathbf{X}\mathbf{A}^{\dagger}\mathbf{r} + \mathbf{P}_{\mathbf{X}_{\mathbf{A}_{12}};\boldsymbol{\Sigma}}(\mathbf{y} - \mathbf{X}\mathbf{A}^{\dagger}\mathbf{r}), \tag{2.8}$$

$$\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_{12}}) = \widehat{\mathbf{X}}_1\mathbf{A}^{\dagger}\mathbf{r} + \begin{pmatrix}\mathbf{P}_{\mathbf{X}_{\mathbf{A}_{12}};\boldsymbol{\Sigma}}(\mathbf{y} - \mathbf{X}\mathbf{A}^{\dagger}\mathbf{r})\\ \mathbf{0}\end{pmatrix}, \tag{2.9}$$

where $\mathbf{X}_{\mathbf{A}_{12}} = \mathbf{X}\mathbf{F}_{\mathbf{A}}$ *and* $\mathbf{P}_{\mathbf{X}_{\mathbf{A}_{12}};\boldsymbol{\Sigma}} = (\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_{12}}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}})^{\dagger} + \mathbf{U}_2\mathbf{E}_{(\mathbf{X}_{\mathbf{A}_{12}}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}})}$, *in which* $\mathbf{U}_2$ *is arbitrary. Moreover,*

(i) $\mathscr{C}(\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}}) = \mathscr{C}(\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{\Sigma})$ *and* $r(\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}}) = r(\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{\Sigma})$.

(ii) $\mathbf{P}_{\mathbf{X}_{\mathbf{A}_{12}};\mathbf{\Sigma}}\mathbf{\Sigma} = (\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}})^{\dagger}\mathbf{\Sigma}$.

Proof. From Lemma 1.1, it is easy to see that the BLUE of $\mathbf{X}\mathbf{F}_{\mathbf{A}_1}\boldsymbol{\gamma}$ under (2.2) is given by

$$\mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\mathbf{\Sigma}}(\mathbf{y} - \mathbf{X}\mathbf{A}_1^{\dagger}\mathbf{r}_1). \tag{2.10}$$

Substituting (2.10) into (2.1) yields (2.6). Similarly, we can get (2.8). Using (2.1), we observe that

$$\begin{aligned} \widehat{\mathbf{X}}_1\boldsymbol{\beta} &= \begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix} (\mathbf{A}_1^{\dagger}\mathbf{r}_1 + \mathbf{F}_{\mathbf{A}_1}\boldsymbol{\gamma}) \\ &= \widehat{\mathbf{X}}_1\mathbf{A}_1^{\dagger}\mathbf{r}_1 + \begin{pmatrix} \mathbf{X}_{\mathbf{A}_1}\boldsymbol{\gamma} \\ \mathbf{0} \end{pmatrix}, \end{aligned} \tag{2.11}$$

and thus, in view of (2.10),

$$\begin{aligned} \mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_1}) &= \widehat{\mathbf{X}}_1\mathbf{A}_1^{\dagger}\mathbf{r}_1 + \begin{pmatrix} \mathrm{BLUE}(\mathbf{X}_{\mathbf{A}_1}\boldsymbol{\gamma}|\mathscr{M}_{r_1}) \\ \mathbf{0} \end{pmatrix} \\ &= \widehat{\mathbf{X}}_1\mathbf{A}_1^{\dagger}\mathbf{r}_1 + \begin{pmatrix} \mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\mathbf{\Sigma}}(\mathbf{y} - \mathbf{X}\mathbf{A}_1^{\dagger}\mathbf{r}_1) \\ \mathbf{0} \end{pmatrix}, \end{aligned} \tag{2.12}$$

i.e., (2.7) is true. Similarly, we can get (2.9). □

From Lemma 1.1, we see that the BLUEs of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}_{c_1}$ and $\mathscr{M}_{c_{12}}$ can be expressed as

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1}) = (\mathbf{I}_n, \mathbf{0})\mathbf{P}_{\widehat{\mathbf{X}}_1;\widehat{\mathbf{\Sigma}}_1}\begin{pmatrix} \mathbf{y} \\ \mathbf{r}_1 \end{pmatrix}, \tag{2.13}$$

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}}) = (\mathbf{I}_n, \mathbf{0})\mathbf{P}_{\widehat{\mathbf{X}}_{12};\widehat{\mathbf{\Sigma}}_{12}}\begin{pmatrix} \mathbf{y} \\ \mathbf{r} \end{pmatrix}, \tag{2.14}$$

respectively, where

$$\mathbf{P}_{\widehat{\mathbf{X}}_1;\widehat{\mathbf{\Sigma}}_1} = (\widehat{\mathbf{X}}_1, \mathbf{0})(\widehat{\mathbf{X}}_1, \widehat{\mathbf{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})^{\dagger} + \mathbf{U}_1\mathbf{E}_{(\widehat{\mathbf{X}}_1, \widehat{\mathbf{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})}, \tag{2.15}$$

$$\mathbf{P}_{\widehat{\mathbf{X}}_{12};\widehat{\mathbf{\Sigma}}_{12}} = (\widehat{\mathbf{X}}_{12}, \mathbf{0})(\widehat{\mathbf{X}}_{12}, \widehat{\mathbf{\Sigma}}_{12}\mathbf{E}_{\widehat{\mathbf{X}}_{12}})^{\dagger} + \mathbf{U}_2\mathbf{E}_{(\widehat{\mathbf{X}}_{12}, \widehat{\mathbf{\Sigma}}_{12}\mathbf{E}_{\widehat{\mathbf{X}}_{12}})}, \tag{2.16}$$

$\widehat{\mathbf{X}}_1 = \begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix}$, $\widehat{\mathbf{X}}_{12} = \begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix}$, $\widehat{\mathbf{\Sigma}}_1 = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{(n+m_1)\times(n+m_1)}$ and $\widehat{\mathbf{\Sigma}}_{12} = \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$, in which $\mathbf{U}_1 \in \mathbb{R}^{(n+m_1)\times(n+m_1)}$ and $\mathbf{U}_2 \in \mathbb{R}^{(n+m)\times(n+m)}$ are arbitrary.

The following lemma can be found in Y. Tian [18].

Lemma 2.2. *Consider the models* $\mathscr{M}_{r_1}$, $\mathscr{M}_{c_1}$, $\mathscr{M}_{r_{12}}$ *and* $\mathscr{M}_{c_{12}}$. *Then,*

(a) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})$ *holds with probability* 1.
(b) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}})$ *holds with probability* 1.
(c) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})$ *holds with probability* 1, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(d) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_{12}})$ *holds with probability* 1, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_{12}}$.

The following result was shown by Y. Tian [17].

Lemma 2.3. *Consider the models* $\mathscr{M}$ *and* $\mathscr{M}_{r_1}$. *Then the following statements are equivalent:*

(a) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})$ *holds with probability* 1.
(b) $r\begin{pmatrix} \mathbf{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix} = r\begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix} + r(\mathbf{X}, \mathbf{\Sigma}) - r(\mathbf{X})$.

By Lemmas 2.2 and 2.3, it is easy to get the following result.

Theorem 2.4. *Consider the models* $\mathscr{M}$ *and* $\mathscr{M}_{r_1}$. *If* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}$, *then the following statements are equivalent:*

(a) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})$ *holds with probability* 1.
(b) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})$ *holds with probability* 1.
(c) $r\begin{pmatrix} \mathbf{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix} = r\begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix} + r(\mathbf{X}, \mathbf{\Sigma}) - r(\mathbf{X})$, *or equivalently*, $\mathscr{C}\begin{pmatrix} \mathbf{0} \\ \mathbf{A}_1' \end{pmatrix} \subseteq \mathscr{C}\begin{pmatrix} \mathbf{\Sigma} \\ \mathbf{X}' \end{pmatrix}$.

Proof. Since $\boldsymbol{\beta}$ is estimable under $\mathscr{M}$, i.e.,

$$r(\mathbf{X}) = p, \tag{2.17}$$

we have

$$\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) - \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathbf{X}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}) - \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}. \tag{2.18}$$

Set

$$\mathbf{V} = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) - \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}). \tag{2.19}$$

Then

$$\mathrm{cov}(\mathbf{V}) = \mathbf{X}\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}) - \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}\mathbf{X}', \tag{2.20}$$

which implies that (a) holds if and only if $\mathbf{V} = \mathbf{0}$ holds with probability 1, that is, (c) holds. By Lemma 2.2, (a) is equivalent to (b). □

Combining Lemmas 2.2 and 2.3 yields the following result.

Theorem 2.5. *Consider the models $\mathscr{M}$, $\mathscr{M}_{r_1}$ and $\mathscr{M}_{c_1}$. Then the following statements are equivalent:*

(a) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})$ *holds with probability* 1.
(b) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})$ *holds with probability* 1.
(c) $r\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{X}\\ \mathbf{0} & \mathbf{A}_1\end{pmatrix} = r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}_1\end{pmatrix} + r(\mathbf{X}, \boldsymbol{\Sigma}) - r(\mathbf{X})$.

Based on the above theorems, we can establish the following result.

Theorem 2.6. *Consider the models $\mathscr{M}_{r_1}$, $\mathscr{M}_{c_1}$, $\mathscr{M}_{r_{12}}$ and $\mathscr{M}_{c_{12}}$. Then the following statements are equivalent:*

(a) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})$ *holds with probability* 1.
(b) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}})$ *holds with probability* 1.
(c) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})$ *holds with probability* 1, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(d) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_{12}})$ *holds with probability* 1, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(e) $r\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{X}\\ \mathbf{0} & \mathbf{A}\end{pmatrix} = r\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{X}\\ \mathbf{0} & \mathbf{A}_1\end{pmatrix} + r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}\end{pmatrix} - r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}_1\end{pmatrix}$.

Proof. Observe that

$$\begin{aligned}\mathrm{cov}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_1}) - \mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} &= \begin{pmatrix}(\mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}} - \mathbf{P}_{\mathbf{X}_{\mathbf{A}_{12}};\boldsymbol{\Sigma}})\boldsymbol{\Sigma}(\mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}} - \mathbf{P}_{\mathbf{X}_{\mathbf{A}_{12}};\boldsymbol{\Sigma}})' & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) - \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}.\end{aligned} \tag{2.21}$$

Since

$$\mathrm{E}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_1}) - \mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \mathbf{0} \tag{2.22}$$

and

$$\mathrm{E}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) - \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \mathbf{0}, \tag{2.23}$$

(2.21) implies that (a) is equivalent to $\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_{12}})$ holds with probability 1, which, by Lemma 2.3, is equivalent to (e). Thus we have shown that (a) and (e) are equivalent. The equivalence of (a) and (b) follows from Lemma 2.2. Similar to the proof of Theorem 2.4, we can get (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e). □

Remark 2.1. Clearly, if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}_{r_1}$, i.e., $r(\mathbf{X}', \mathbf{A}_1')' = p$, then Theorem 2.6(e) can be simplified as $\mathscr{C}\begin{pmatrix}\mathbf{0}\\ \mathbf{A}_2'\end{pmatrix} \subseteq \mathscr{C}\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{X}' & \mathbf{A}_1'\end{pmatrix}$. For conciseness, in the following sections, we omit these simplifications.

3. Comparisons among the BLUEs

In this section, we discuss the superiority of the BLUEs of $\mathbf{X}\boldsymbol{\beta}$, as well as the superiority of the BLUEs of $\boldsymbol{\beta}$ under $\mathscr{M}$, $\mathscr{M}_{r_1}$, $\mathscr{M}_{c_1}$, $\mathscr{M}_{r_{12}}$ and $\mathscr{M}_{c_{12}}$. We begin with the following lemma.

Lemma 3.1. *Consider the models $\mathscr{M}_{r_1}$, $\mathscr{M}_{c_1}$, $\mathscr{M}_{r_{12}}$ and $\mathscr{M}_{c_{12}}$. The following statements are true:*

(a) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} = \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$.
(b) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\}$.
(c) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$, *if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}_{r_1}$.*
(d) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\}$, *if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}_{r_{12}}$.*

Proof. It is easy to see that

$$\begin{aligned}
\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} &= \mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}}\boldsymbol{\Sigma}\mathbf{P}'_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}} \\
&= (\mathbf{X}_{\mathbf{A}_1}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})^{\dagger}\boldsymbol{\Sigma}\{(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})'\}^{\dagger}(\mathbf{X}_{\mathbf{A}_1}, \mathbf{0})', \\
\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\} &= (\mathbf{I}_n, \mathbf{0})\mathbf{P}_{\widehat{\mathbf{X}}_1;\widehat{\boldsymbol{\Sigma}}_1}\widehat{\boldsymbol{\Sigma}}_1\mathbf{P}'_{\widehat{\mathbf{X}}_1;\widehat{\boldsymbol{\Sigma}}_1}(\mathbf{I}_n, \mathbf{0})' \\
&= (\mathbf{X}, \mathbf{0})(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})^{\dagger}\widehat{\boldsymbol{\Sigma}}_1\{(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})'\}^{\dagger}(\mathbf{X}, \mathbf{0})'.
\end{aligned}$$

By (1.26) and direct calculations, we have

$$\begin{aligned}
& r[\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\}] \\
&= r[(\mathbf{X}_{\mathbf{A}_1}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})^{\dagger}\boldsymbol{\Sigma}\{(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})'\}^{\dagger}(\mathbf{X}_{\mathbf{A}_1}, \mathbf{0})' - (\mathbf{X}, \mathbf{0})(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})^{\dagger}\widehat{\boldsymbol{\Sigma}}_1\{(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})'\}^{\dagger}(\mathbf{X}, \mathbf{0})'] \\
&= r\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\widehat{\boldsymbol{\Sigma}}_1 & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1 & \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'_{\mathbf{A}_1} \\
\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \widehat{\mathbf{X}}'_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{E}_{\widehat{\mathbf{X}}_1}\widehat{\boldsymbol{\Sigma}}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{0}
\end{pmatrix} - 2r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) - 2r(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1}) \\
&= r\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\widehat{\boldsymbol{\Sigma}}_1 & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{X}}_1 & \mathbf{0} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'_{\mathbf{A}_1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \widehat{\mathbf{X}}'_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{\widehat{\mathbf{X}}_1}\widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{0}
\end{pmatrix} - 2r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) - 2r(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1) \\
&= r\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\widehat{\boldsymbol{\Sigma}}_1 & \mathbf{0} & \widehat{\mathbf{X}}_1 & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'_{\mathbf{A}_1} \\
\mathbf{0} & \widehat{\mathbf{X}}'_1 & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{X} & \mathbf{0}
\end{pmatrix} - 2r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) - 2r(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1) + r(\mathbf{E}_{\widehat{\mathbf{X}}_1}\widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1}) + r(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) \\
&= r\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\widehat{\boldsymbol{\Sigma}}_1 & -\widehat{\mathbf{X}}_1\mathbf{F}_{\mathbf{A}} & \widehat{\mathbf{X}}_1 & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & -\mathbf{F}_{\mathbf{A}}\widehat{\mathbf{X}}'_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \widehat{\mathbf{X}}'_1 & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X} & \mathbf{0}
\end{pmatrix} - 2r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) - 2r(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1) + r(\mathbf{E}_{\widehat{\mathbf{X}}_1}\widehat{\boldsymbol{\Sigma}}_1) + r(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}) \\
&= r\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} \\
\mathbf{0} & -\boldsymbol{\Sigma} & -\mathbf{X}_{\mathbf{A}_1} \\
\mathbf{X}'_{\mathbf{A}_1} & -\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0}
\end{pmatrix} - r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) - r(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1) - r(\mathbf{X}_{\mathbf{A}_1}) - r(\widehat{\mathbf{X}}_1) + 2r(\widehat{\mathbf{X}}_1) \\
&= r\begin{pmatrix}
\mathbf{0} & \boldsymbol{\Sigma} & \mathbf{X}_{\mathbf{A}_1} \\
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0}
\end{pmatrix} - r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) - r(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1) - r(\mathbf{X}_{\mathbf{A}_1}) + r(\widehat{\mathbf{X}}_1) = \mathbf{0}. \qquad (3.1)
\end{aligned}$$

Hence, (a) holds. Similarly, we can also show that (b) holds. Now we show that (c) is true. Note that

$$r\{(\mathbf{A}_1, \mathbf{0})(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})^{\dagger}\widehat{\boldsymbol{\Sigma}}_1\} = r\begin{pmatrix}\widehat{\mathbf{X}}_1 & \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1} & \widehat{\boldsymbol{\Sigma}}_1\\ \mathbf{A}_1 & \mathbf{0} & \mathbf{0}\end{pmatrix} - r(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1}) = \mathbf{0}, \tag{3.2}$$

i.e.,

$$(\mathbf{A}_1, \mathbf{0})(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})^{\dagger}\widehat{\boldsymbol{\Sigma}}_1 = \mathbf{0}. \tag{3.3}$$

So we have

$$\begin{aligned}\mathrm{cov}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_1})\} &= \begin{pmatrix}\mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}}\boldsymbol{\Sigma}\mathbf{P}'_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathrm{cov}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{c_1})\} &= \mathbf{P}_{\widehat{\mathbf{X}}_1;\widehat{\boldsymbol{\Sigma}}_1}\widehat{\boldsymbol{\Sigma}}_1\mathbf{P}'_{\widehat{\mathbf{X}}_1;\widehat{\boldsymbol{\Sigma}}_1}\\ &= (\widehat{\mathbf{X}}_1, \mathbf{0})(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})^{\dagger}\widehat{\boldsymbol{\Sigma}}_1\{(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})'\}^{\dagger}(\widehat{\mathbf{X}}_1, \mathbf{0})'\\ &= \begin{pmatrix}(\mathbf{X}, \mathbf{0})(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})^{\dagger}\widehat{\boldsymbol{\Sigma}}_1\{(\widehat{\mathbf{X}}_1, \widehat{\boldsymbol{\Sigma}}_1\mathbf{E}_{\widehat{\mathbf{X}}_1})'\}^{\dagger}(\mathbf{X}, \mathbf{0})' & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\end{aligned} \tag{3.5}$$

which combined with (a) leads to

$$\mathrm{cov}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_1})\} = \mathrm{cov}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{c_1})\}, \tag{3.6}$$

i.e.,

$$\widehat{\mathbf{X}}_1[\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\}]\widehat{\mathbf{X}}'_1 = \mathbf{0}. \tag{3.7}$$

Noting that

$$r(\widehat{\mathbf{X}}_1) = r(\mathbf{X}', \mathbf{A}'_1) = p, \tag{3.8}$$

(3.7) leads to (c). Similarly, we can get (d). □

Remark 3.1. It follows from the result that the covariances of the estimations derived from the explicitly and implicitly restricted models are superficially different features, but can really be viewed as the same thing.

Theorem 3.2. *Consider the models $\mathscr{M}$, $\mathscr{M}_{r_1}$ and $\mathscr{M}_{c_1}$. Then*

(a) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} \le \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}$.
(b) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\} \le \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}$.
(c) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\} \le \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\}$, *if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}$.*
(d) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\} \le \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\}$, *if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}$.*
In particular,
(e) *The following statements are equivalent:*
  (i) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} = \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}$.
  (ii) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\} = \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}$.
  (iii) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\}$, *if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}$.*
  (iv) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\}$, *if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}$.*
  (v) $r\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{X}\\ \mathbf{0} & \mathbf{A}_1\end{pmatrix} = r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}_1\end{pmatrix} + r(\boldsymbol{\Sigma}, \mathbf{X}) - r(\mathbf{X})$.
(f) *The following statements are equivalent:*
  (i) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} < \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}$.
  (ii) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\} < \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}$.
  (iii) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\} < \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\}$, *if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}$.*
  (iv) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\} < \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\}$, *if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}$.*
  (v) $r\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{X}\\ \mathbf{0} & \mathbf{A}_1\end{pmatrix} = r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}_1\end{pmatrix} + r(\boldsymbol{\Sigma}, \mathbf{X}) - r(\mathbf{X}) + n$.

Proof. From Lemmas 1.1, 1.2, 1.4, 1.5 and direct calculations, we get

$$
\begin{aligned}
& i_{\pm}[\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}] \\
&= i_{\pm}(\mathbf{P}_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}}\boldsymbol{\Sigma}\mathbf{P}'_{\mathbf{X}_{\mathbf{A}_1};\boldsymbol{\Sigma}} - \mathbf{P}_{\mathbf{X};\boldsymbol{\Sigma}}\boldsymbol{\Sigma}\mathbf{P}'_{\mathbf{X};\boldsymbol{\Sigma}}) \\
&= i_{\pm}[(\mathbf{X}_{\mathbf{A}_1}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})^{\dagger}\boldsymbol{\Sigma}\{(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})'\}^{\dagger}(\mathbf{X}_{\mathbf{A}_1}, \mathbf{0})' - (\mathbf{X}, \mathbf{0})(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})^{\dagger}\boldsymbol{\Sigma}\{(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})'\}^{\dagger}(\mathbf{X}, \mathbf{0})'] \\
&= i_{\pm}\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{X} & \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'_{\mathbf{A}_1} \\
\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{0}
\end{pmatrix} - r\,(\mathbf{X}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}) - r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})(\text{by } (1.25)) \\
&= i_{\pm}\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'_{\mathbf{A}_1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{X} & \mathbf{0} & \mathbf{0}
\end{pmatrix} - r\,(\mathbf{X}, \boldsymbol{\Sigma}) - r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma})(\text{by } (1.19)) \\
&= i_{\pm}\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}'_{\mathbf{A}_1} \\
\mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{X} & \mathbf{0}
\end{pmatrix} + i_{\mp}(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) + i_{\pm}(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}) - r\,(\mathbf{X}, \boldsymbol{\Sigma}) - r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma})(\text{by } (1.20)) \\
&= i_{\pm}\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\boldsymbol{\Sigma} & -\mathbf{X}_{\mathbf{A}_1} & \mathbf{X} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & -\mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X} & \mathbf{0}
\end{pmatrix} + i_{\mp}(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) + i_{\pm}(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}) - r\,(\mathbf{X}, \boldsymbol{\Sigma}) - r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) \\
&= i_{\pm}\begin{pmatrix}
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\boldsymbol{\Sigma} & -\mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & -\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X} & \mathbf{0}
\end{pmatrix} + i_{\mp}(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) + i_{\pm}(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}) - r\,(\mathbf{X}, \boldsymbol{\Sigma}) - r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) \\
&= i_{\pm}\begin{pmatrix}
\boldsymbol{\Sigma} & \boldsymbol{\Sigma} & \mathbf{X}_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} \\
\boldsymbol{\Sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{X}'_{\mathbf{A}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}' \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X} & \mathbf{0}
\end{pmatrix} + i_{\mp}(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) + i_{\pm}(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}) - r\,(\mathbf{X}, \boldsymbol{\Sigma}) - r(\mathbf{X}_{\mathbf{A}_1}, \boldsymbol{\Sigma}) \\
&= i_{\mp}(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}) + i_{\pm}(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}) - r(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}).(\text{by Lemma 1.4(a)})
\end{aligned} \tag{3.9}
$$

Hence,

$$i_{+}[\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}] = \mathbf{0}, \tag{3.10}$$

$$i_{-}[\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}] = r\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix} - r\begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix} - r(\boldsymbol{\Sigma}, \mathbf{X}) + r(\mathbf{X}), \tag{3.11}$$

which results in (a). Combining Lemma 3.1 and (a) yields (b). Now we show that (c) is true. Clearly, if $\boldsymbol{\beta}$ is estimable under $\mathscr{M}$, i.e., $r(\mathbf{X}) = p$, there exists a permutation matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{PX} = (\mathbf{X}'_1, \mathbf{X}'_2)', \tag{3.12}$$

where $\mathbf{X}_1 \in \mathbb{R}^{p\times p}$ has full rank. Let

$$\mathbf{V} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\}. \tag{3.13}$$

Then

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} = \mathbf{XVX}'. \tag{3.14}$$

It follows from (1.18), (1.19), (1.24) and (3.12) that

$$\begin{aligned} i_{\pm}(\mathbf{V}) \geq i_{\pm}(\mathbf{XVX}') &= i_{\pm}[\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}] \\ &= i_{\pm}(\mathbf{PXVX}'\mathbf{P}') \\ &= i_{\pm}\begin{pmatrix} \mathbf{X}_1\mathbf{VX}_1' & \mathbf{X}_1\mathbf{VX}_2' \\ \mathbf{X}_2\mathbf{VX}_1' & \mathbf{X}_2\mathbf{VX}_2' \end{pmatrix} \\ &\geq i_{\pm}(\mathbf{X}_1\mathbf{VX}_1') = i_{\pm}(\mathbf{V}), \end{aligned} \tag{3.15}$$

i.e.,

$$i_{\pm}(\mathbf{V}) = i_{\pm}[\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}] \tag{3.16}$$

which results in (c). Combining Lemma 3.1 and (c) yields (d). Applying Lemma 1.3 to (3.11) and (3.16) gives (e) and (f). □

One of the referees pointed out that Theorem 3.2(a) can be easily derived by the united theory of least squares of C.R. Rao. In fact, by (1.13), (2.1) and (2.2), it follows that

$$\begin{aligned} \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) &= \mathbf{XA}_1^{\dagger}\mathbf{r}_1 + (\mathbf{y} - \mathbf{XA}_1^{\dagger}\mathbf{r}_1) - \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})^{-}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}(\mathbf{y} - \mathbf{XA}_1^{\dagger}\mathbf{r}_1) \\ &= \mathbf{y} - \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})^{-}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}(\mathbf{y} - \mathbf{XA}_1^{\dagger}\mathbf{r}_1). \end{aligned} \tag{3.17}$$

Now we, in view of (1.13) and (3.17), have

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}(\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}})^{-}\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma}, \tag{3.18}$$

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}(\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}})^{-}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}\boldsymbol{\Sigma}. \tag{3.19}$$

Therefore

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} = \boldsymbol{\Sigma}^{1/2}(\mathbf{P}_{\boldsymbol{\Sigma}^{1/2}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}} - \mathbf{P}_{\boldsymbol{\Sigma}^{1/2}\mathbf{E}_{\mathbf{X}}})\boldsymbol{\Sigma}^{1/2}. \tag{3.20}$$

Noting that

$$\mathscr{C}(\boldsymbol{\Sigma}^{1/2}\mathbf{E}_{\mathbf{X}}) \subset \mathscr{C}(\boldsymbol{\Sigma}^{1/2}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}), \tag{3.21}$$

we, from Proposition 7.1 in [9], get

$$\mathbf{P}_{\boldsymbol{\Sigma}^{1/2}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_1}}} - \mathbf{P}_{\boldsymbol{\Sigma}^{1/2}\mathbf{E}_{\mathbf{X}}} \geq \mathbf{0}, \tag{3.22}$$

which leads to

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} - \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} \geq \mathbf{0}, \tag{3.23}$$

i.e., Theorem 3.2(a) holds.

Remark 3.2. In comparison, part (a) of the above theorem was also investigated using linear zero function and the principle of covariance adjustment; see Section 7.9 in [14].

Remark 3.3. In the situation where $r(\mathbf{A}_1) = m_1$ and $r(\boldsymbol{\Sigma}) = n$, part (c) of the above theorem was proved in Section 5.10 in [11] using a different technique. Of course, their results need to make a slight modification.

Unifying Theorems 2.4, 2.5, 3.2(e), we obtain the following conclusion.

Corollary 3.3. *Consider the models $\mathscr{M}$, $\mathscr{M}_{r_1}$ and $\mathscr{M}_{c_1}$. Then the following statements are equivalent:*

(a) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})$ *holds with probability* 1.
(b) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})$ *holds with probability* 1.
(c) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} = \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$.
(d) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} = \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$.
(e) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})$ *holds with probability* 1, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}$.
(f) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})$ *holds with probability* 1, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}$.
(g) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}$.
(h) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}$.
(i) $r\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix} = r\begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix} + r(\mathbf{X}, \boldsymbol{\Sigma}) - r(\mathbf{X})$.

Theorem 3.4. *Consider the models $\mathscr{M}_{r_1}$, $\mathscr{M}_{c_1}$, $\mathscr{M}_{r_{12}}$ and $\mathscr{M}_{c_{12}}$. Then the following statements are true:*

(a) $\operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} \leq \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$.
(b) $\operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\} \leq \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$.
(c) $\operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} \leq \operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(d) $\operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\} \leq \operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
In particular,
(e) *The following statements are equivalent:*
(i) $\operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$.
(ii) $\operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\} = \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$.
(iii) $\operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(iv) $\operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\} = \operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(v) $r\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} = r\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix} + r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} - r\begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix}$.
(f) *The following statements are equivalent:*
(i) $\operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} < \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$.
(ii) $\operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\} < \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$.
(iii) $\operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} < \operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(iv) $\operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\} < \operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(v) $r\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} = r\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix} + r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} - r\begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix} + n$.

Proof. Note that

$$\begin{aligned}
\operatorname{cov}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} &= \mathbf{P}_{\widehat{\mathbf{X}}_{\mathbf{A}_{12}};\mathbf{X}_{\mathbf{A}_{12}};\boldsymbol{\Sigma}}\boldsymbol{\Sigma}\mathbf{P}'_{\widehat{\mathbf{X}}_{\mathbf{A}_{12}};\mathbf{X}_{\mathbf{A}_{12}};\boldsymbol{\Sigma}} \\
&= (\widehat{\mathbf{X}}_{\mathbf{A}_{12}}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_{12}}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}})^{\dagger}\boldsymbol{\Sigma}\{(\mathbf{X}_{\mathbf{A}_{12}}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}})'\}^{\dagger}(\widehat{\mathbf{X}}_{\mathbf{A}_{12}}, \mathbf{0})' \\
&= \begin{pmatrix} (\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{0})(\mathbf{X}_{\mathbf{A}_{12}}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}})^{\dagger}\boldsymbol{\Sigma}\{(\mathbf{X}_{\mathbf{A}_{12}}, \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_{\mathbf{A}_{12}}})'\}^{\dagger}(\mathbf{X}_{\mathbf{A}_{12}}, \mathbf{0})' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.
\end{aligned} \tag{3.24}$$

we set

$$\mathbf{V} = \operatorname{cov}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} - \operatorname{cov}\{\mathrm{BLUE}(\widehat{\mathbf{X}}_1\boldsymbol{\beta}|\mathscr{M}_{r_1})\}. \tag{3.25}$$

Then, by (3.4) and (3.24), we get

$$\mathbf{V} = \begin{pmatrix} \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} - \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{3.26}$$

i.e.,

$$i_{\pm}(\mathbf{V}) = i_{\pm}[\operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} - \operatorname{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\}]. \tag{3.27}$$

From (3.10) and (3.11), we obtain

$$i_{-}(\mathbf{V}) = r\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} - r\begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix} - r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} + r\begin{pmatrix} \mathbf{X} \\ \mathbf{A}_1 \end{pmatrix} \tag{3.28}$$

and

$$i_{+}(\mathbf{V}) = \mathbf{0}. \tag{3.29}$$

Combining (3.27) and (3.29) yields (a). By (a) and Lemma 3.1, we have (b). As to (c). Noticing $\boldsymbol{\beta}$ is estimable under $\mathscr{M}_{r_1}$, i.e.,

$$r(\widehat{\mathbf{X}}_1) = p, \tag{3.30}$$

then

$$\mathbf{V} = \widehat{\mathbf{X}}_1[\operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} - \operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}]\widehat{\mathbf{X}}_1'. \tag{3.31}$$

Similar to (3.12)–(3.16) of Theorem 3.2, we derive

$$i_{\pm}(\mathbf{V}) = i_{\pm}\left[\operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} - \operatorname{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}\right], \tag{3.32}$$

which combined with (3.29) results in (c). Obviously, by Lemma 3.1, (c) is equivalent to (d). Applying Lemma 1.3 to (3.28) and (3.32) gives (e) and (f). □

Combining Theorem 2.6 with 3.4 (e), we can get the following result.

Corollary 3.5. *Consider the models $\mathscr{M}_{r_1}$, $\mathscr{M}_{c_1}$, $\mathscr{M}_{r_{12}}$ and $\mathscr{M}_{c_{12}}$. The following statements are equivalent:*

(a) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})$ *holds with probability* 1.
(b) $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}})$ *holds with probability* 1.
(c) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$.
(d) $\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\} = \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$.
(e) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})$ *holds with probability* 1, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(f) $\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1}) = \mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_{12}})$ *holds with probability* 1, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(g) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(h) $\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_{12}})\} = \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{c_1})\}$, *if* $\boldsymbol{\beta}$ *is estimable under* $\mathscr{M}_{r_1}$.
(i) $r\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{X}\\ \mathbf{0} & \mathbf{A}\end{pmatrix} = r\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{X}\\ \mathbf{0} & \mathbf{A}_1\end{pmatrix} + r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}\end{pmatrix} - r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}_1\end{pmatrix}$.

4. Conclusion and example

In this article, we derived that, when we gradually add linear restrictions into the general linear model $\mathscr{M}$, the BLUEs become more and more superior, i.e.,

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} \le \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} \le \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}, \tag{4.1}$$
$$\mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} \le \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M}_{r_1})\} \le \mathrm{cov}\{\mathrm{BLUE}(\boldsymbol{\beta}|\mathscr{M})\}. \tag{4.2}$$

This result is actually easy to understand. Because when appending successively linear restrictions, we get more information about the unknown regression parameters. Consequently, we hope that its estimator BLUE should have a higher estimation accuracy. As is shown in this article, we first derived (4.1), by which we then established (4.2). Alternatively, Firstly, we can also compare the superiority of the BLUEs of estimable parametric functions $\mathbf{K}\boldsymbol{\beta}$ under $\mathscr{M}$, $\mathscr{M}_{r_1}$ and $\mathscr{M}_{r_{12}}$, by which we then get (4.1) and (4.2) with $\mathbf{K} = \mathbf{X}$ and $\mathbf{K} = \mathbf{I}_p$. We also gave the necessary and sufficient conditions for the statement $\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1}) = \mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})$ to hold.

Motivated by the referees' suggestion, we now take an example discussed in Section 4.1 of [14] to illustrate our theoretical results. Consider the linear model

$$\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}\}, \tag{4.3}$$

where

$$\mathbf{X} = \begin{pmatrix} \mathbf{1}_{10\times 1} & \mathbf{1}_{10\times 1} & \mathbf{0}_{10\times 1} & \mathbf{1}_{10\times 1} & \mathbf{0}_{10\times 1}\\ \mathbf{1}_{10\times 1} & \mathbf{0}_{10\times 1} & \mathbf{1}_{10\times 1} & \mathbf{1}_{10\times 1} & \mathbf{0}_{10\times 1}\\ \mathbf{1}_{10\times 1} & \mathbf{1}_{10\times 1} & \mathbf{0}_{10\times 1} & \mathbf{0}_{10\times 1} & \mathbf{1}_{10\times 1}\\ \mathbf{1}_{10\times 1} & \mathbf{0}_{10\times 1} & \mathbf{1}_{10\times 1} & \mathbf{0}_{10\times 1} & \mathbf{1}_{10\times 1} \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix}\boldsymbol{\mu}\\ \boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\\ \boldsymbol{\tau}_1\\ \boldsymbol{\tau}_2\end{pmatrix}. \tag{4.4}$$

This designed set-up is typical in agricultural experiments where several treatments are applied to various blocks of land. The experiment is often conducted to assess the differential impact of the treatments. Here, the parameter $\boldsymbol{\mu}$ represents a general effect which is present in all the observations, the parameters $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ represent the respective effects of two blocks and the parameters $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ represent the respective effects of two treatments. Let

$$\boldsymbol{\Sigma} = \begin{pmatrix} 3\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}'\\ 2\mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}'\\ 2\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}'\\ \mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' \end{pmatrix}, \tag{4.5}$$

where $\mathbf{1}$ denotes $\mathbf{1}_{10\times 1}$, which will be used in the following part. Also suppose that

$$\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 = 0, \quad \text{i.e., } \mathbf{A}_1\boldsymbol{\beta} = 0 \tag{4.6}$$

and

$$\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2 = 0, \quad \text{i.e., } \mathbf{A}_2\boldsymbol{\beta} = 0, \tag{4.7}$$

where $\mathbf{A}_1 = (0, 0, 0, 1, -1)$ and $\mathbf{A}_2 = (0, 1, -1, 0, 0)$. These two restrictions may be a known fact from the theory or experiment view.

Direct calculations will show that

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\} = \begin{pmatrix} 3\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ \mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' \end{pmatrix}, \tag{4.8}$$

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} = \begin{pmatrix} 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' \\ 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' \\ 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' \\ 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' \end{pmatrix}, \tag{4.9}$$

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} = \begin{pmatrix} 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \end{pmatrix}. \tag{4.10}$$

Now it is easy to see that

$$\begin{pmatrix} 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \end{pmatrix} \leq \begin{pmatrix} 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' \\ 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' \\ 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' \\ 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' & 3/2\mathbf{1}\mathbf{1}' & 5/2\mathbf{1}\mathbf{1}' \end{pmatrix} \leq \begin{pmatrix} 3\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ \mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' \end{pmatrix}, \tag{4.11}$$

i.e.,

$$\mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_{12}})\} \leq \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M}_{r_1})\} \leq \mathrm{cov}\{\mathrm{BLUE}(\mathbf{X}\boldsymbol{\beta}|\mathscr{M})\}. \tag{4.12}$$

This is in accordance with our result (4.1). It should be noted that if $\mathscr{C}(\mathbf{A}_1') \cap \mathscr{C}(\mathbf{X}') = \{\mathbf{0}\}$, then the restriction (1.2) has no effect on the BLUE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$, see Section 4.9 in [14]. Furthermore, Theorem 3.2(e) gives the conditions under which the restriction (1.2) has no effect on the covariance of the BLUE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$. Of course, if the restriction (1.2) does not meet the conditions in Theorem 3.2(e), then it improves the estimation accuracy. For example, the restriction in (4.6) raises the accuracy of the BLUE of $\mathbf{X}\boldsymbol{\beta}$ under $\mathscr{M}$, but the restriction $(0, 0, 0, 1, 1)\boldsymbol{\beta} = 0$ does nothing.

References

[1] J.K. Baksalary, R. Kala, Best linear unbiased estimation in the restricted general linear model, Statistics 10 (1979) 27–35.
[2] J.K. Baksalary, P.R. Pordzik, A note on comparing the unrestricted and restricted least-squares estimators, Linear Algebra Appl. 127 (1990) 371–378.
[3] W.T. Dent, On restricted estimation in linear models, J. Econom. 12 (1980) 45–58.
[4] B. Dong, W. Guo, Y. Tian, On relations between BLUEs under two transformed linear models, J. Multivariate Anal. 131 (2014) 279–292.
[5] H. Haupt, W. Oberhofer, Fully restricted linear regression: a pedagogical note, Econom. Bull. 3 (2002) 1–7.
[6] G. Marsaglia, G.P.H. Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Algebra 2 (1974) 269–292.
[7] T. Mathew, A note on best linear unbiased estimation in the restricted general linear model, Statistics 14 (1983) 3–6.
[8] S. Puntanen, G.P.H. Styan, The equality of the ordinary least squares estimator and the best linear unbiased estimator. With comments by O. Kempthorne, S.R. Searle, and a reply by the authors, Amer. Statist. 43 (1989) 153–164.
[9] S. Puntanen, G.P.H. Styan, J. Isotala, Matrix Tricks for Linear Statistical Models: Our Personal Top Twenty, Springer, Heidelberg, 2011.
[10] C.R. Rao, Representations of best linear unbiased estimators in the Gauss–Markoff model with a singular dispersion matrix, J. Multivariate Anal. 3 (1973) 276–292.
[11] C.R. Rao, H. Toutenburg, Shalabh, C. Heumann, Linear Models and Generalizations: Least Squares and Alternatives, third ed., Springer, 2008.
[12] B. Ravikumar, S. Ray, N.E. Savin, Robust wald tests in SUR systems with adding up restrictions, Econometrica 68 (2000) 715–719.
[13] X. Ren, On the equivalence of the BLUEs under a general linear model and its restricted and stochastically restricted models, Statist. Probab. Lett. 90 (2014) 1–10.
[14] D. Sengupta, S.R. Jammalamadaka, Linear Models: An Integrated Approach, World Scientific, River Edge, NJ, 2003.
[15] H. Theil, A.S. Goldberger, On pure and mixed statistical estimation in economics, Internat. Econom. Rev. 2 (1961) 65–78.
[16] Y. Tian, Equalities and inequalities for inertias of Hermitian matrices with applications, Linear Algebra Appl. 433 (2010) 263–296.
[17] Y. Tian, On equalities of estimations of parametric functions under a general linear model and its restricted models, Metrika 72 (2010) 313–330.
[18] Y. Tian, Characterizing relationships between estimations under a general linear model with explicit and implicit restrictions by rank of matrix, Comm. Statist. Theory Methods 41 (2012) 2588–2601.