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The Effects of Nonnormality on Tests for Dimensionality in Canonical Correlation and MANOVA Models

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In this paper we consider the usual test statistics for dimensionality in canonical correlation and MANOVA models for nonnormal populations. In order to know the effects of the null distributions of the test statistics when the populations depart from normality, perturbation expansions for test statistics are derived. The asymptotic expansions of the expectations of the test statistics are given under the class of elliptical populations. Further, modified test statistics with a better chi-squared approximation are proposed. Finally, numerical results by Monte Carlo simulations are presented.

1. INTRODUCTION

The tests for dimensionality are an important problem in multivariate statistical analysis and have been mainly studied under the assumption of multivariate normal populations. The distributions of test statistics have been discussed in terms of asymptotic expansions. As for the canonical correlation and the MANOVA models, Lawley [8] and Fujikoshi [2] have proposed modified test statistics for the likelihood ratio statistic, etc., which yield better chi-squared approximations. In this paper we consider

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asymptotic behaviors of the usual test statistics for dimensionality in canonical correlation and MANOVA models under nonnormality. In order to know the effects of the null distributions of the test statistics when the populations depart from normality, asymptotic chi-squared approximations for test statistics are examined by using perturbation expansions of sample roots with multiple population roots. The asymptotic results, which extend the work of Muirhead and Waternaux [11], are given under the class of elliptical populations, and consequently modified test statistics with better chi-squared approximations are proposed. In Section 2, the usual three test statistics in the canonical correlation model are discussed. The asymptotic results show that these tests are nonrobust. On the other hand, it is shown in Section 3 that the test statistics in the MANOVA model are asymptotically robust. Finally, the numerical results by Monte Carlo simulations are presented.

2. Tests for Dimensionality in the Canonical Correlation Model

Let $\rho_1 \geqslant \cdots \geqslant \rho_p$ and $r_1 \geqslant \cdots \geqslant r_p$ be the population and the sample canonical correlation coefficients based on a sample size $n+1$, respectively, which are canonical correlations between $\mathbf{x}_1$ and $\mathbf{x}_2$ based on p and q components ($p \leqslant q$), respectively. The test of dimensionality is to test the null hypothesis that the smallest $p-k$ population canonical correlation coefficients are zero; that is,

$$H_k: \rho_k > \rho_{k+1} = \cdots = \rho_p = 0.$$

We now consider the following three test statistics which have been proposed under the normal population: that is, (i) $Q_1 = -\log \prod_{j=k+1}^{p} (1-r_j^2)$, (ii) $Q_2 = \sum_{j=k+1}^{p} r_j^2/(1-r_j^2)$, and (iii) $Q_3 = \sum_{j=k+1}^{p} r_j^2$. With the assumption of normal populations, Lawley [8] has shown

$$E\left[\left\{n-k-\frac{1}{2}(p+q+1)+\sum_{j=1}^{k} \rho_j^{\,-2}\right\} Q_1\right] = f_c + o(n^{-1}), \tag{2.1}$$

where $f_c = (p-k)(q-k)$. Further, Fujikoshi [2] has given the modified test statistics for the other two statistics. Under the elliptical population it is known (see Muirhead and Waternaux [11]) that the limiting distribution as $n \to +\infty$ of $nQ_1/(1+\kappa)$ is chi-squared with f_c degrees of freedom when H_k is true.

We consider perturbation expansions of test statistics in order to extend the results as in (2.1) to the nonnormal case. For this, it is fundamental to obtain a perturbation expansion of sample roots with multiple population roots.

2.1. *A Perturbation Expansion for Sample Roots*

In this subsection, a perturbation expansion for the last $p-k$ sample roots of the canonical correlation is discussed under H_k. Considering an appropriate nonsingular transformation, without loss of generality we may assume

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{I}_p & \mathbf{P} \\ \mathbf{P}' & \mathbf{I}_q \end{bmatrix}, \tag{2.2}$$

where $\mathbf{P} = [\text{diag}(\rho_1, ..., \rho_p) : \mathbf{0}]$ with $\rho_{k+1} = \cdots = \rho_p = 0$. Similarly, let $\mathbf{S}$ be partitioned as

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}.$$

Then $r_1^2 \geqslant \cdots \geqslant r_p^2$ are the latent roots of the matrix $\mathbf{S}_{11}^{-1}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$, or $\mathbf{R} = \mathbf{S}_{11}^{-1/2}\mathbf{M}\mathbf{S}_{11}^{-1/2}$, where $\mathbf{M} = \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$. Let $\mathbf{U} = [u_{ij}] = \sqrt{n}(\mathbf{S} - \mathbf{\Sigma})$ and

$$\mathbf{U} = \begin{bmatrix} \mathbf{V} & \mathbf{Z} \\ \mathbf{Z}' & \mathbf{W} \end{bmatrix}.$$

It is easily seen that we can expand $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{\Delta} + n^{-1/2}\mathbf{M}^{(1)} + n^{-1}\mathbf{M}^{(2)} + n^{-3/2}\mathbf{M}^{(3)} + n^{-2}\mathbf{M}^{(4)} + \cdots,$$

where

$$\begin{aligned}
\mathbf{\Delta} &= \text{diag}(\rho_1^2, ..., \rho_p^2), \\
\mathbf{M}^{(1)} &= \mathbf{ZP}' + \mathbf{PZ}' - \mathbf{PWP}', \\
\mathbf{M}^{(2)} &= \mathbf{PW}^2\mathbf{P}' - \mathbf{PWZ}' - \mathbf{ZWP}' + \mathbf{ZZ}', \\
\mathbf{M}^{(3)} &= -\mathbf{PW}^3\mathbf{P}' + \mathbf{PW}^2\mathbf{Z}' + \mathbf{ZW}^2\mathbf{P}' - \mathbf{ZWZ}', \\
\mathbf{M}^{(4)} &= \mathbf{PW}^4\mathbf{P}' - \mathbf{PW}^3\mathbf{Z}' - \mathbf{ZW}^3\mathbf{P}' + \mathbf{ZW}^2\mathbf{Z}'.
\end{aligned}$$

Further, we can expand $\mathbf{R}$ as follows:

$$\mathbf{R} = \mathbf{R}^{(0)} + n^{-1/2}\mathbf{R}^{(1)} + n^{-1}\mathbf{R}^{(2)} + n^{-3/2}\mathbf{R}^{(3)} + n^{-2}\mathbf{R}^{(4)} + \cdots, \tag{2.3}$$

where

$$\begin{aligned}
\mathbf{R}^{(0)} &= \mathbf{\Delta}, \\
\mathbf{R}^{(1)} &= \mathbf{M}^{(1)} - \tfrac{1}{2}\mathbf{\Delta V} - \tfrac{1}{2}\mathbf{V\Delta}, \\
\mathbf{R}^{(2)} &= \mathbf{M}^{(2)} - \tfrac{1}{2}(\mathbf{M}^{(1)}\mathbf{V} + \mathbf{VM}^{(1)}) + \tfrac{3}{8}(\mathbf{\Delta V}^2 + \mathbf{V}^2\mathbf{\Delta}) + \tfrac{1}{4}\mathbf{V\Delta V},
\end{aligned}$$

$$\mathbf{R}^{(3)}=\mathbf{M}^{(3)}-\tfrac{1}{2}(\mathbf{M}^{(2)}\mathbf{V}+\mathbf{V}\mathbf{M}^{(2)})+\tfrac{3}{8}(\mathbf{M}^{(1)}\mathbf{V}^2+\mathbf{V}^2\mathbf{M}^{(1)})+\tfrac{1}{4}\mathbf{V}\mathbf{M}^{(1)}\mathbf{V}$$
$$-\tfrac{3}{16}(\mathbf{V}\boldsymbol{\Delta}\mathbf{V}^2+\mathbf{V}^2\boldsymbol{\Delta}\mathbf{V})-\tfrac{5}{16}(\boldsymbol{\Delta}\mathbf{V}^3+\mathbf{V}^3\boldsymbol{\Delta}),$$
$$\mathbf{R}^{(4)}=\mathbf{M}^{(4)}-\tfrac{1}{2}(\mathbf{M}^{(3)}\mathbf{V}+\mathbf{V}\mathbf{M}^{(3)})+\tfrac{3}{8}(\mathbf{M}^{(2)}\mathbf{V}^2+\mathbf{V}^2\mathbf{M}^{(2)})+\tfrac{1}{4}\mathbf{V}\mathbf{M}^{(2)}\mathbf{V}$$
$$-\tfrac{3}{16}(\mathbf{V}\mathbf{M}^{(1)}\mathbf{V}^2+\mathbf{V}^2\mathbf{M}^{(1)}\mathbf{V})-\tfrac{5}{16}(\mathbf{M}^{(1)}\mathbf{V}^3+\mathbf{V}^3\mathbf{M}^{(1)})$$
$$+\tfrac{5}{32}(\mathbf{V}\boldsymbol{\Delta}\mathbf{V}^3+\mathbf{V}^3\boldsymbol{\Delta}\mathbf{V})+\tfrac{35}{128}(\boldsymbol{\Delta}\mathbf{V}^4+\mathbf{V}^4\boldsymbol{\Delta})+\tfrac{9}{64}\mathbf{V}^2\boldsymbol{\Delta}\mathbf{V}^2.$$

Then, by lines similar to those given by Lawley [7] and Fujikoshi [2], we have that $r_{k+1}^2, ..., r_p^2$ are equal to the latent roots of $\mathbf{L}$ such that

$$\mathbf{L}=n^{-1}\mathbf{L}^{(2)}+n^{-3/2}\mathbf{L}^{(3)}+n^{-2}\mathbf{L}^{(4)}+\cdots, \tag{2.4}$$

where

$$\mathbf{L}^{(2)}=\mathbf{R}_{22}^{(2)}-\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(1)},$$
$$\mathbf{L}^{(3)}=\mathbf{R}_{22}^{(3)}-\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(2)}-\mathbf{R}_{21}^{(2)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(1)}+\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{11}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(1)},$$
$$\mathbf{L}^{(4)}=\mathbf{R}_{22}^{(4)}-\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(3)}-\mathbf{R}_{21}^{(3)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(1)}-\mathbf{R}_{21}^{(2)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(2)}$$
$$-\tfrac{1}{2}\mathbf{L}^{(2)}\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-2}\mathbf{R}_{12}^{(1)}-\tfrac{1}{2}\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-2}\mathbf{R}_{12}^{(1)}\mathbf{L}^{(2)}$$
$$+\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{11}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(2)}+\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{11}^{(2)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(1)}$$
$$+\mathbf{R}_{21}^{(2)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{11}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(1)}-\mathbf{R}_{21}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{11}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{11}^{(1)}\boldsymbol{\Theta}^{-1}\mathbf{R}_{12}^{(1)},$$

and

$$\mathbf{R}^{(0)}=\begin{bmatrix}\boldsymbol{\Theta} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \qquad \boldsymbol{\Theta}=\mathrm{diag}(\rho_1^2, ..., \rho_k^2),$$
$$\mathbf{R}^{(j)}=\begin{bmatrix}\mathbf{R}_{11}^{(j)} & \mathbf{R}_{12}^{(j)}\\ \mathbf{R}_{21}^{(j)} & \mathbf{R}_{22}^{(j)}\end{bmatrix} \quad (j=1, 2, 3, 4).$$

Here we note that $\mathbf{R}_{22}^{(1)}=\mathbf{0}$. Using the results, we derive asymptotic expansions for the expectations of test statistics.

2.2. *Test Statistics with the Correction Factor*

Since $r_{k+1}^2, ..., r_p^2$ are equal to the latent roots of $\mathbf{L}$, we can write

$$nQ_1=\mathrm{tr}\,\mathbf{L}^{(2)}+n^{-1/2}\,\mathrm{tr}\,\mathbf{L}^{(3)}+n^{-1}\{\mathrm{tr}\,\mathbf{L}^{(4)}+\tfrac{1}{2}\mathrm{tr}(\mathbf{L}^{(2)})^2\}+O_p(n^{-3/2}),$$
$$nQ_2=\mathrm{tr}\,\mathbf{L}^{(2)}+n^{-1/2}\,\mathrm{tr}\,\mathbf{L}^{(3)}+n^{-1}\{\mathrm{tr}\,\mathbf{L}^{(4)}+\mathrm{tr}(\mathbf{L}^{(2)})^2\}+O_p(n^{-3/2}),$$
$$nQ_3=\mathrm{tr}\,\mathbf{L}^{(2)}+n^{-1/2}\,\mathrm{tr}\,\mathbf{L}^{(3)}+n^{-1}\,\mathrm{tr}\,\mathbf{L}^{(4)}+O_p(n^{-3/2}),$$

where $\mathbf{L}^{(2)}$, $\mathbf{L}^{(3)}$, and $\mathbf{L}^{(4)}$ are given by (2.4). In order to obtain $E[nQ_i]$, $i=1, 2, 3$, in terms of the cumulants of $\mathbf{x}$, the following formulas are useful (see, e.g., Kaplan [5] and Kendall and Stuart [6]):

$$E(u_{ab}) = 0,$$
$$E(u_{ab}u_{cd}) = m(ab, cd) - n^{-1}\kappa_{abcd} + O(n^{-2}),$$
$$E(u_{ab}u_{cd}u_{ef}) = n^{-1/2}m(ab, cd, ef) + O(n^{-3/2}),$$
$$E(u_{ab}u_{cd}u_{ef}u_{gh}) = \sum_{(3)} m(ab, cd)\, m(ef, gh) + O(n^{-1}),$$

with $m(ab, cd)$ and $m(ab, cd, ef)$ given by

$$m(ab, cd) = \kappa_{abcd} + \sigma_{ac}\sigma_{bd} + \sigma_{ad}\sigma_{bc},$$
$$m(ab, cd, ef) = \kappa_{abcdef} + \sum_{(12)} \kappa_{abce}\kappa_{df} + \sum_{(4)} \kappa_{ace}\kappa_{bdf} + \sum_{(8)} \kappa_{ac}\kappa_{be}\kappa_{df},$$

where $\kappa_{i \cdots i \cdots j \cdots j}$ $(= \kappa_{r \cdots s}^{i \cdots j} \equiv \kappa_{r \cdots s}(\mathbf{x}_i, ..., \mathbf{x}_j))$ are the cumulants of $\mathbf{x}$ and the summations occur over all ways of grouping the subscripts, which are shown in the numbers in parentheses after the summation signs.

However, it is difficult to obtain a simple and united form of the asymptotic expansion of $E[nQ_i]$, $i = 1, 2, 3$, in terms of the cumulants of $\mathbf{x}$ for a general nonnormal distribution. In the following we consider the case when $\mathbf{x}$ is distributed as an elliptical distribution $EL_{p+q}(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ (see, e.g., Muirhead [10]). Then its denisty function and characteristic function are of the form $f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Lambda}) = c_{p+q}|\boldsymbol{\Lambda}|^{-1/2} g((\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Lambda}^{-1}(\mathbf{x}-\boldsymbol{\mu}))$ for some function g, where c_{p+q} is positive constant, and $\phi(\mathbf{t}) = \exp\{i\mathbf{t}'\boldsymbol{\mu}\}\, \psi\{\mathbf{t}'\boldsymbol{\Lambda}\mathbf{t})$ for some function ψ, respectively. Provided they exist, $E[\mathbf{x}] = \boldsymbol{\mu}$, $\boldsymbol{\Sigma} \equiv \mathrm{Cov}[\mathbf{x}] = -2\psi'(0)\, \boldsymbol{\Lambda}$. Let the kurtosis parameter be $\kappa \equiv \{\psi^{(2)}(0)/(\psi'(0))^2\} - 1$, and let $\varphi = \{\psi^{(3)}(0)/(\psi'(0))^3\} - 1$. Under the elliptical population, note that $\kappa_{abcd} = \kappa \sum_{(3)} \sigma_{ab}\sigma_{cd}$ and $\kappa_{abcdef} = (\varphi - 3\kappa) \sum_{(15)} \sigma_{ab}\sigma_{cd}\sigma_{ef}$, where $\sum_{(3)} \sigma_{ab}\sigma_{cd} = \sigma_{ab}\sigma_{cd} + \sigma_{ac}\sigma_{bd} + \sigma_{ad}\sigma_{bc}$, and so on. After a good deal of calculation, we may obtain the following theorem.

THEOREM 2.1. *Suppose that* $\mathbf{x}$ *is distributed as an elliptical distribution* $EL_{p+q}(\boldsymbol{\mu}, \boldsymbol{\Lambda})$. *Then it holds that under* H_k,

$$E[c_i Q_i] = f_c + o(n^{-1}), \qquad i = 1, 2, 3,$$

with the correction factors given by

$$c_1 = (n+2)(1+\kappa)^{-1} - \tfrac{1}{2}(3p + 3q + 11) - k + \sum_{j=1}^{k} \rho_j^{-2} + (h-1)(1+\kappa)^{-2},$$

$$c_2 = (n+2)(1+\kappa)^{-1} - 2(p+q+3) + \sum_{j=1}^{k} \rho_j^{-2} + (h-1)(1+\kappa)^{-2},$$

$$c_3 = (n+2)(1+\kappa)^{-1} - (p+q+5) - 2k + \sum_{j=1}^{k} \rho_j^{-2} + (h-1)(1+\kappa)^{-2},$$

where $f_c = (p-k)(q-k)$ *and* $h = (p+q+4)(1+\varphi)$.

When $\kappa = 0$ and $\varphi = 0$, we can see that the correction factors c_i ($i = 1, 2, 3$) coincide with the results in the normal case (see, e.g., Fujikoshi [2]). Thus it is shown that the three test statistics are nonrobust when the populations depart from normality. It may be noted that the kurtosis parameters κ and φ are invariant under the nonsingular linear transformation. Therefore, when the population has an elliptical distribution, we may use κ and φ of the original vector $\mathbf{x}$ instead of the transformed vector with the covariance matrix (2.2).

3. Tests for Dimensionality in the MANOVA Model

Suppose that we have q groups with N_j ($j = 1, ..., q$) observations drawn from p-variate distribution with mean vector $\boldsymbol{\mu}^{(j)}$ and the same covariance matrix $\boldsymbol{\Sigma}$, respectively. Let $\mathbf{S}_{\mathrm{W}}$ and $\mathbf{S}_{\mathrm{B}}$ be the usual "within groups" and "between groups" matrix sums of squares and products in the MANOVA, respectively: that is,

$$\mathbf{S}_{\mathrm{W}} = \sum_{j=1}^{q} \mathbf{S}_j, \qquad \mathbf{S}_j = \sum_{\alpha=1}^{N_j} (\mathbf{x}_\alpha^{(j)} - \bar{\mathbf{x}}^{(j)})(\mathbf{x}_\alpha^{(j)} - \bar{\mathbf{x}}^{(j)})',$$

$$\mathbf{S}_{\mathrm{B}} = \sum_{j=1}^{q} N_j(\bar{\mathbf{x}}^{(j)} - \bar{\mathbf{x}})(\bar{\mathbf{x}}^{(j)} - \bar{\mathbf{x}})',$$

where $\mathbf{x}_\alpha^{(j)}$ is the αth observation from the jth population, $\bar{\mathbf{x}}^{(j)} = (1/N_j)\sum_{\alpha=1}^{N_j} \mathbf{x}_\alpha^{(j)}$, $\bar{\mathbf{x}} = (1/N)\sum_{j=1}^{q} N_j \bar{\mathbf{x}}^{(j)}$, and $N = \sum_{j=1}^{q} N_j$. Further, let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ be the latent roots of $\mathbf{S}_{\mathrm{B}}\mathbf{S}_{\mathrm{W}}^{-1}$ and let $\delta_1 \geqslant \cdots \geqslant \delta_p \geqslant 0$ be the corresponding population roots, i.e., the latent roots of

$$\boldsymbol{\Omega} \equiv \boldsymbol{\Sigma}^{-1/2} \sum_{j=1}^{q} v_j(\boldsymbol{\mu}^{(j)} - \bar{\boldsymbol{\mu}})(\boldsymbol{\mu}^{(j)} - \bar{\boldsymbol{\mu}})' \boldsymbol{\Sigma}^{-1/2},$$

where $v_j = N_j/N$ and $\bar{\boldsymbol{\mu}} = \sum_{j=1}^{q} v_j \boldsymbol{\mu}^{(j)}$. Then the hypothesis on dimensionality is defined by $H_k: \delta_k > \delta_{k+1} = \cdots = \delta_p = 0$. The hypothesis means that the number of significant discriminant functions in multiple discriminant analysis is k. When the observations are normal, the following statistics have been proposed: that is, (i) $T_1 = \log \prod_{j=k+1}^{p} (1 + d_j)$, (ii) $T_2 = \sum_{j=k+1}^{p} d_j$, and (iii) $T_3 = \sum_{j=k+1}^{p} d_j/(1 + d_j)$.

Theoretical discussions related to these test statistics under normality are to be found in Siotani *et al.* [13] and others. Especially, the null distributions of these test statistics are asymptotically distributed as a chi-squared variate with $f_m = (p-k)(q-1-k)$ degrees of freedom, and also their refinements to the chi-squared approximation have been obtained. In this section, we study the effects of nonnormality of their refinements.

Without loss of generality, we can assume $\mathbf{\Sigma}=\mathbf{I}_p$, and let $\mathbf{S}_j/N_j=\mathbf{I}_p+\mathbf{V}_j/\sqrt{N_j}$ and $\mathbf{y}^{(j)}=\sqrt{N_j}\,(\bar{\mathbf{x}}^{(j)}-\boldsymbol{\mu}^{(j)})$. Let $\mathbf{H}$ be an orthogonal matrix such that $\mathbf{\Omega}=\mathbf{H}\mathbf{\Delta}\mathbf{H}'$, where $\mathbf{\Delta}=\mathrm{diag}(\delta_1, ..., \delta_p)$. Then $d_1\geqslant\cdots\geqslant d_p$ are latent roots of $\mathbf{S}_\mathbf{B}\mathbf{S}_\mathbf{W}^{-1}$ or $\mathbf{R}=(\mathbf{S}_e/N)^{-1/2}(\mathbf{S}_h/N)(\mathbf{S}_e/N)^{-1/2}$, where $\mathbf{S}_h=\mathbf{H}\mathbf{S}_\mathbf{B}\mathbf{H}'$ and $\mathbf{S}_e=\mathbf{H}\mathbf{S}_\mathbf{W}\mathbf{H}'$. Further, we can write $\mathbf{S}_e/N=\mathbf{I}+\mathbf{V}/\sqrt{N}$ and $\mathbf{S}_h/N=\mathbf{\Delta}+\mathbf{D}/\sqrt{N}+\mathbf{G}/N$, where

$$\mathbf{V}=\mathbf{H}\left(\sum_j\sqrt{v_j}\,\mathbf{V}_j\right)\mathbf{H}',$$

$$\mathbf{D}=\mathbf{H}\left(\sum_{j=1}^{q}\sqrt{v_j}\,\mathbf{y}^{(j)}(\boldsymbol{\mu}^{(j)}-\bar{\boldsymbol{\mu}})'+\sum_{j=1}^{q}\sqrt{v_j}\,(\boldsymbol{\mu}^{(j)}-\bar{\boldsymbol{\mu}})\,\mathbf{y}^{(j)'}\right)\mathbf{H}',$$

$$\mathbf{G}=\mathbf{H}\left(\sum_{j=1}^{q}(\mathbf{y}^{(j)}-\sqrt{v_j}\,\bar{\mathbf{y}})(\mathbf{y}^{(j)}-\sqrt{v_j}\,\bar{\mathbf{y}})'\right)\mathbf{H}',$$

and $\bar{\mathbf{y}}=\sum_{j=1}^{q}\sqrt{v_j}\,\mathbf{y}^{(j)}$. Therefore, a perturbation expansin for $d_{k+1}, ..., d_p$ under H_k can be obtained as a special case of (2.3). In fact, we may only replace $\mathbf{M}^{(j)}$ as $\mathbf{M}^{(1)}=\mathbf{D}$, $\mathbf{M}^{(2)}=\mathbf{G}$, and $\mathbf{M}^{(3)}=\mathbf{M}^{(4)}=\mathbf{0}$. Thus the test statistics in the MANOVA model can be expanded as

$$NT_1=\mathrm{tr}\,\mathbf{L}^{(2)}+N^{-1/2}\,\mathrm{tr}\,\mathbf{L}^{(3)}+N^{-1}\{\mathrm{tr}\,\mathbf{L}^{(4)}-\tfrac{1}{2}\mathrm{tr}(\mathbf{L}^{(2)})^2\}+O_p(N^{-3/2}),$$

$$NT_2=\mathrm{tr}\,\mathbf{L}^{(2)}+N^{-1/2}\,\mathrm{tr}\,\mathbf{L}^{(3)}+N^{-1}\,\mathrm{tr}\,\mathbf{L}^{(4)}+O_p(N^{-3/2}),$$

$$NT_3=\mathrm{tr}\,\mathbf{L}^{(2)}+N^{-1/2}\,\mathrm{tr}\,\mathbf{L}^{(3)}+N^{-1}\{\mathrm{tr}\,\mathbf{L}^{(4)}-\mathrm{tr}(\mathbf{L}^{(2)})^2\}+O_p(N^{-3/2}),$$

where the latent roots of $\mathbf{L}$ in the form of (2.4) are $d_{k+1}, ..., d_p$.

In the following we assume that $\mathbf{x}_\alpha^{(j)}$ is distributed as the elliptical distribution $EL_p(\boldsymbol{\mu}^{(j)}, \mathbf{I}_p)$. Then the joint density function of $\mathbf{V}_j$ and $\mathbf{y}^{(j)}$ is given by

$$\begin{aligned}f(\mathbf{V}_j, \mathbf{y}^{(j)})&=(2\pi)^{-p(p+3)/4}|\mathbf{\Xi}|^{-1/2}(1+\kappa)^{-p(p-1)/4}\\&\times\exp\left[-\frac{1}{2}\left\{\mathbf{v}_1^{(j)'}\mathbf{\Xi}^{-1}\mathbf{v}_1^{(j)}+\frac{1}{1+\kappa}\mathbf{v}_2^{(j)'}\mathbf{v}_2^{(j)}+\mathbf{y}^{(j)'}\mathbf{y}^{(j)}\right\}\right]\\&\times[1+\{g_j(\mathbf{V}_j, \mathbf{y}^{(j)})-\mathbf{1}_p'\mathbf{\Xi}^{-1}\mathbf{v}_1^{(j)}\}\,N_j^{-1/2}+O(N_j^{-1})],\end{aligned}$$

where

$$\begin{aligned}g_j(\mathbf{V}_j, \mathbf{y}^{(j)})&=w_1\,\mathrm{tr}\,\mathbf{V}_j+w_2(\mathrm{tr}\,\mathbf{V}_j)^3+w_3\,\mathrm{tr}\,\mathbf{V}_j^3\\&+w_4\,\mathrm{tr}\,\mathbf{V}_j\,\mathrm{tr}\,\mathbf{V}_j^2+w_5\,\mathbf{y}^{(j)'}\mathbf{y}^{(j)}\mathrm{tr}\,\mathbf{V}_j+w_6\,\mathbf{y}^{(j)'}\mathbf{V}_j\mathbf{y}^{(j)},\end{aligned}$$

and

$$\mathbf{V}_j = [v_{\alpha\beta}^{(j)}], \qquad \boldsymbol{\Xi} = 2(1+\kappa)\,\mathbf{I}_p + \kappa \mathbf{1}_p \mathbf{1}_p',$$
$$\mathbf{v}_1^{(j)} = (v_{11}^{(j)}, ..., v_{pp}^{(j)})', \qquad \mathbf{v}_2^{(j)} = (v_{12}^{(j)}, v_{13}^{(j)}, ..., v_{p-1,p}^{(j)})'.$$

The coefficients w_i $(i = 1, ..., 6)$ are given by

$$\begin{aligned}
w_1 &= -\varphi\{st(4p+1-4p^{-1}) + t^2(\tfrac{1}{2}p+3+4p^{-1})\} \\
&\quad + \kappa\{st(2p-1) + t^2(\tfrac{3}{2}p+3) - t(\tfrac{1}{2}p+1)\} - 2st(p+1-2p^{-1}) - 4t^2p^{-1}, \\
w_2 &= \varphi\{\tfrac{8}{3}s^3p^{-2} - s^2t(p^{-1}+4p^{-2}) + t^3(\tfrac{1}{6}+p^{-1}+\tfrac{4}{3}p^{-2})\} \\
&\quad + \kappa\{s^2tp^{-1} - t^3(\tfrac{1}{2}+p^{-1})\} + \tfrac{8}{3}s^3p^{-2} - 4s^2tp^{-2} + \tfrac{4}{3}t^3p^{-2}, \\
w_3 &= \tfrac{4}{3}(1+\varphi)\,s^3, \\
w_4 &= \varphi\{-4s^3p^{-1} + s^2t(4p^{-1}+1) - \kappa s^2 t - 4s^3p^{-1} + 4s^2tp^{-1}\}, \\
w_5 &= \kappa\{-sp^{-1} + t(\tfrac{1}{2}+p^{-1})\}, \\
w_6 &= \kappa s,
\end{aligned}$$

where $s = \{2(1+\kappa)\}^{-1}$ and $t = \{(p+2)\kappa + 2\}^{-1}$. The joint density can be obtained by slightly modifying the result of Wakaki [14]. Using the joint density, the expectations of the expanded test statistics can be calculated in a straightforward method. The following theorem may be obtained after a good deal of calculation.

THEOREM 3.1. *Suppose that* $\mathbf{x}^{(j)}$ *are distributed as the elliptical distribution* $EL_p(\boldsymbol{\mu}^{(j)}, \boldsymbol{\Lambda})$. *Then it holds that under* H_k

$$E[m_i T_i] = f_m + o(N^{-1}), \qquad i = 1, 2, 3,$$

with the correction factors given by

$$\begin{aligned}
m_1 &= N - \left\{\tfrac{1}{2}(p+q+2) - \sum_{i=1}^{k} \delta_i^{-1} + (p-k+2)\,\kappa - e\right\}, \\
m_2 &= N - \left\{p+q-k+1 - \sum_{i=1}^{k} \delta_i^{-1} + (p-k+2)\,\kappa - e\right\}, \\
m_3 &= N - \left\{k+1 - \sum_{i=1}^{k} \delta_i^{-1} + (p-k+2)\,\kappa - e\right\},
\end{aligned}$$

where

$$
\begin{aligned}
f_m &= (p-k)(q-k-1),\\
e &= (p\kappa+2\kappa+2)[w_1 q + 3w_2 pq(p\kappa+2\kappa+2) + 3w_3 q(p\kappa+2\kappa+p+1)\\
&\quad + w_4 q\{(p+2)^2\kappa + p^2+p+4\} + w_5(pq+2) + w_6 q]\\
&\quad + 2w_6\{(p-k+2)\kappa + p-k+1\}.
\end{aligned}
$$

When $\kappa=0$ and $\varphi=0$, we can see that the m_i $(i=1, 2, 3)$ coincide with the results in the normal case (see, e.g., Fujikoshi [2]). As for the result above, it may be noted that the test statistics in the MANOVA model are asymptotically robust, since the effect of nonnormality appears in the terms of $O(N^{-1})$. In the case of the MANOVA model, we may also use the kurtosis parameter κ and the φ based on the original vector $\mathbf{x}$ when the population has an elliptical distribution.

4. Monte Carlo Simulations

In this section, we investigate the accuracy of chi-squared approximations of the test statistics, including the modified test statistics for the canonical correlation and MANOVA models by Monte Carlo simulations. The results also show how much these test statistics are affected by departures from normality. A simulation study of the likelihood ratio test statistic for the canonical correlation model under elliptical populations has been reported by Muirhead and Waternaux [11]. On the other hand, for the MANOVA model, for example, Ito [4] and Olson [12] have also examined the robustness of the test statistics for testing a linear hypothesis by Monte Carlo simulations. A simulation study related to several tests of dimensionality under normal populations has been reported by Backhouse and McKay [1].

4.1. *The Canonical Correlation Model*

We consider the Monte Carlo simulation in the same setup as that used in the Muirhead and Waternaux [11]. The simulation study was done for several elliptical populations: multivariate normal, multivariate t (d.f. $\nu=8$), and contaminated normal ($\varepsilon=0.1$, $\sigma=3$). For each population we consider the following cases: $p=3$; $q=4$; $k=0, 1$; $N=50$; $\alpha=0.25$, 0.10, 0.05, 0.01; and two covariance matrices (i) $\mathbf{\Sigma}=\mathbf{I}_7$ when $k=0$ and (ii) $\mathbf{\Sigma}_{11}=\mathbf{I}_3$, $\mathbf{\Sigma}_{22}=\mathbf{I}_4$, $\mathbf{P}=[\mathrm{diag}(0.8, 0, 0){:}\mathbf{0}]$ when $k=1$. Note that $\kappa=\varphi=0$ for a normal population, and for the multivariate t population (d.f. $\nu=8$), $\kappa=2/(\nu-4)=0.5$, and $\varphi=(\nu-2)^2/\{(\nu-4)(\nu-6)\}-1=3.5$, for the contaminated normal distribution ($\varepsilon = 0.1$, $\sigma = 3$), $\kappa = \{1 + \varepsilon(\sigma^4 - 1)\}/\{1 + \varepsilon(\sigma^2 - 1)\}^2 - 1 = 1.78$ and $\varphi = \{1 + \varepsilon(\sigma^6 - 1)\}/\{1 + \varepsilon(\sigma^2 - 1)\}^3 - 1 = 11.65$. In practical use, we must estimate κ and φ, since κ and φ are unknown. Note here that an estimate of κ, which is a proper consistent estimate, has been use in Muirhead and Waternaux [11]. Extending the estimation method, we can estimate κ and φ as follows. Let y_{i1} and y_{i2} be the ith canonical variables ($i = 1, ..., p$), and let $R^2 = \sum_{i=k+1}^{p} (y_{i1}^2 + y_{i2}^2)$. Then, we have $E(R^2) = r\lambda$, $E(R^4) = r(r+2)(1+\kappa)\lambda^2$, $E(R^6) = r(r+2)(r+4) \times (1+\varphi)\lambda^3$, where $r = 2(p-k)$, and λ and 3κ are variance and kurtosis parameters, respectively. Thus κ and φ satisfy

$$(1+\kappa) = \frac{r}{r+2} \frac{E(R^4)}{\{E(R^2)\}^2}, \qquad (1+\varphi) = \frac{r^2}{(r+2)(r+4)} \frac{E(R^6)}{\{E(R^2)\}^3}.$$

Then the consistent estimates by the moment method are given by

$$\hat{\kappa} = \frac{r}{r+2} \frac{M_4}{M_2^2} - 1, \qquad \hat{\varphi} = \frac{r^2}{(r+2)(r+4)} \frac{M_6}{M_2^3} - 1, \tag{4.1}$$

where

$$M_{2j} = N^{-1} \sum_{\alpha=1}^{N} \left(\sum_{i=k+1}^{p} y_{i1\alpha}^2 + y_{i2\alpha}^2 \right)^j, \qquad j = 1, 2, 3.$$

Further, we may use the estimate $\sum_{j=1}^{k} r_j^{-2}$ instead of $\sum_{j=1}^{k} \rho_j^{-2}$. It may be noted that more efficient estimates could be found. For the present, however, the simulation results based on the estimates (4.1) are given in Table I. In the Monte Carlo simulation, the number of times which are over the α percentile of the $\chi^2_{(p-k)(q-k)}$ out of 200 simulations is calculated. This process is repeated 100 times to obtain 100 estimates, and the averages for 100 estimates are given. Their values for $n_i Q_i$, $n_i Q_i/(1+\hat{\kappa})$, and $\hat{c}_i Q_i$ are summarized in Table I for the selected parameters. Further, the standard deviations for 100 estimates are given. It can be seen from simulation results that the test statistics are nonrobust when the populations depart from normality and that the modified test statistics yield better chi-squared approximations under the class of elliptical populations. In addition, it may be noted that the modified test statistics with the estimates $\hat{\kappa}$ and $\hat{\varphi}$ give considerably better chi-squared approximations. Thus the use of test statistics needs special care under nonnormality, and we recommend the use of modified tests statistics if the population has the class of elliptical distributions including normal distributions.

4.2. *The MANOVA Model*

In the Monte Carlo simulation, there is no loss of generality in assuming that the covariance matrix $\mathbf{\Sigma} = \mathbf{I}$. The simulation study was done for the

TABLE I

Simulation Results for the Canonical Correlation Model ($N = 50$)

	α:	0.25		0.10		0.05		0.01	
	Expected:	50		20		10		2	
		χ_6^2	χ_{12}^2	χ_6^2	χ_{12}^2	χ_6^2	χ_{12}^2	χ_6^2	χ_{12}^2
M.N.[a]	nQ_1	59	65	26	30	14	16	3	4
	$nQ_1/(1+\hat{\kappa})$	57	66	25	31	13	17	3	4
	$\hat{c}_1 Q_1$	45	51	18	20	9	10	2	2
	nQ_2	67	81	33	44	20	29	6	10
	$nQ_2/(1+\hat{\kappa})$	65	82	31	45	19	29	6	10
	$\hat{c}_2 Q_2$	46	52	19	23	10	13	3	4
	nQ_3	51	48	20	16	9	7	1	1
	$nQ_3/(1+\hat{\kappa})$	49	50	18	17	8	7	1	1
	$\hat{c}_3 Q_3$	45	49	16	17	7	7	1	1
M.T.	nQ_1	96	118	57	76	38	54	14	23
	$nQ_1/(1+\hat{\kappa})$	51	62	21	27	11	14	2	3
	$\hat{c}_1 Q_1$	45	55	17	22	8	11	2	2
	nQ_2	104	132	67	96	48	75	22	42
	$nQ_2/(1+\hat{\kappa})$	61	83	30	46	18	30	6	11
	$\hat{c}_2 Q_2$	45	56	19	26	10	14	3	5
	nQ_3	88	100	46	54	27	33	7	10
	$nQ_3/(1+\hat{\kappa})$	41	39	13	11	6	4	1	0
	$\hat{c}_3 Q_3$	45	52	15	17	6	7	1	1
C.N.	nQ_1	152	180	121	159	100	143	61	107
	$nQ_1/(1+\hat{\kappa})$	52	47	20	15	10	7	2	1
	$\hat{c}_1 Q_1$	53	53	20	18	10	8	2	1
	nQ_2	157	185	130	170	112	158	77	131
	$nQ_2/(1+\hat{\kappa})$	69	85	35	46	21	29	7	11
	$\hat{c}_2 Q_2$	52	54	24	23	13	13	4	4
	nQ_3	146	172	109	143	85	121	43	71
	$nQ_3/(1+\hat{\kappa})$	34	16	10	2	3	1	0	0
	$\hat{c}_3 Q_3$	52	48	17	11	7	4	1	0

[a] M.N., multivariate normal; M.T.: multivariate t (d.f. 8); C.N., contaminated normal ($\varepsilon = 0.1$, $\sigma = 3$).

same elliptical populations as in the canonical correlation model case: multivariate normal, multivariate t (d.f. $= 5, 8$), and contaminated normal ($\varepsilon = 0.1$, $\sigma = 3$). For parameters, we set them as follows: $p = 3$; $q = 4$; $N_i = 10$ ($N = 40$); $k = 0, 1$; and $\alpha = 0.25, 0.10, 0.05, 0.01$, where $\boldsymbol{\mu}^{(j)} = \mathbf{0}$; that is, $N\boldsymbol{\Delta} = \operatorname{diag}(0, 0, 0)$ for $k = 0$ and $\boldsymbol{\mu}^{(j)} = (\sqrt{3/5}\, j, 0, 0)$, $j = 1, ..., 4$, that is, $N\boldsymbol{\Delta} = \operatorname{diag}(30, 0, 0)$ for $k = 1$, which corresponds to the matrices of non-

TABLE II

Simulation Results for the MANOVA Model ($N = 40$)

	$k = 0$											
α:	0.25			0.10			0.05			0.01		
Expected:	50			20			10			2		
	M.N.[a]	M.T.	C.N.	M.N.	M.T.	C.N.	M.N.	M.T.	C.N.	M.N.	M.T.	C.N.
NT_1	68	67	67	31	30	31	18	16	17	4	4	4
	(6.44)	(7.08)	(7.61)	(4.78)	(4.55)	(5.42)	(3.91)	(4.00)	(4.28)	(1.86)	(1.76)	(2.13)
$m_1 T_1$	49	48	49	20	18	19	10	8	9	2	1	2
	(5.67)	(6.74)	(6.26)	(4.21)	(3.98)	(4.39)	(3.13)	(2.83)	(3.08)	(1.32)	(1.13)	(1.22)
NT_2	82	83	82	46	45	46	30	28	30	12	10	11
	(6.63)	(6.88)	(7.69)	(5.34)	(6.16)	(6.33)	(5.05)	(4.58)	(5.21)	(3.36)	(3.30)	(3.27)
$m_2 T_2$	50	49	50	23	21	22	13	12	12	4	3	3
	(5.68)	(6.79)	(6.56)	(4.56)	(4.11)	(4.79)	(3.58)	(3.61)	(3.47)	(1.80)	(1.66)	(1.96)
NT_3	51	49	51	19	16	17	8	6	7	1	1	1
	(5.47)	(6.80)	(6.33)	(4.29)	(4.05)	(4.29)	(2.84)	(2.42)	(2.95)	(0.99)	(0.85)	(0.94)
$m_3 T_3$	47	45	47	16	14	15	7	5	6	1	0	1
	(5.64)	(6.44)	(5.94)	(3.84)	(3.82)	(4.13)	(2.67)	(2.26)	(2.50)	(0.87)	(0.66)	(0.79)

[a] M.N., multivariate normal; M.T., multivariate t (d.f. 5); C.N., contaminated normal ($\varepsilon = 0.1$, $\sigma = 3$).

centrality parameters. The simulation results which calculated the number of times that T_i and $m_i T_i$ $(i = 1, 2, 3)$ exceed $\chi^2_{(p-k)(q-1-k)}(\alpha)$ out of 200 are given in Table II for the selected parameters. It can be seen from simulation results that the test statistics in the MANOVA model are asymptotically robust, and the results for three elliptical populations are much the same. It may be noted that the tests for dimensionality in the MANOVA model do not have much effect on nonnormality. Simulation results also show that the modified test statistics yield better chi-squared approximations under the class of elliptical populations, though the magnitude of improvement is small.

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