

Asymptotic Expansions and Bootstrap Approximations in Factor Analysis

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We derive asymptotic expansions for the distributions of the normal theory maximum likelihood estimators of unique variances and uniquenesses (standardized unique variances) in the factor analysis model. Asymptotic expansions are given for the distributions of non-Studentized and also Studentized statistics to construct accurate confidence intervals. In the case of Studentized statistics, we investigate the accuracy of the asymptotic approximations to the exact distributions that are determined by Monte Carlo simulations. The results show that, compared with normal approximations, the asymptotic expansions generally improve the accuracy of the approximations in the tail area except for the cases of the uniqueness estimators whose true values are close to their upper bounds unity. We also compare three types of confidence intervals that are based on the distributions of the Studentized statistics; each of which employs normal approximation, asymptotic expansion of the percentile points of the Studentized statistic, and further modification using the bootstrap. The results show that while the first type of confidence intervals were far from equal-tailed, the latter two achieved better balance in both sides. © 2001 Elsevier Science (USA)

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1. INTRODUCTION

Factor analysis is one of the most popular methods of multivariate statistical analysis in social and behavioral sciences. Statistical inference in factor analysis is usually based on the asymptotic distributions of

estimators. Anderson and Rubin [2] derived asymptotic distributions of the normal theory maximum likelihood estimators of unique variances. Asymptotic standard errors of standardized unique variance estimates and unrotated loading estimates for both standardized and nonstandardized cases were given in Lawley and Maxwell [9]. An error in their formulae was corrected by Jennrich and Thayer [8]. These asymptotic results can be used to approximate the distributions of the estimators by suitable normal distributions and then to construct approximate confidence intervals or to judge the significance of factor loadings. It is noted, however, that there are many parameters in the factor analysis model and, as a consequence, the normal approximations may not be so accurate, in particular, when the sample size is not large enough compared with the number of the observed variables.

An approach to cope with this problem is to find asymptotic expansions for the distributions of the estimators. Since such expansions have not been given in factor analysis, we derive asymptotic expansions for the distributions of the normal theory maximum likelihood estimators of unique variances and uniquenesses (standardized unique variances) for both Studentized and non-Studentized statistics. By inverting the derived expansions, we may also obtain expansions of the percentile points, which can be used to construct confidence intervals. Because it is the distributions of the Studentized statistics that are of interest in practice, we investigate the accuracy of the approximations by comparing with the exact distributions of the Studentized statistics that are determined numerically by Monte Carlo simulations.

Another approach is to use the bootstrap method introduced by Efron [3]. Ichikawa and Konishi [7] investigated the application of the bootstrap methods in factor analysis. As to the construction of confidence intervals, various types of bootstrap confidence interval have been proposed (see, for example, Efron and Tibshirani [4]). In contrast to the asymptotic expansions mentioned above, the bootstrap- t method approximates the exact distributions of the Studentized statistics by their bootstrap distributions. By combining the asymptotic expansion and the bootstrap, further modification may be possible. Abramovitch and Singh [1] have given a bootstrap method of improving confidence intervals that are based on asymptotic expansions.

We carry out a Monte Carlo simulation to compare three types of confidence intervals, each of which employs normal approximation, asymptotic expansion of the percentile points of the Studentized statistic, and further modification using the bootstrap.

2. MAXIMUM LIKELIHOOD FACTOR ANALYSIS

Let $\mathbf{x} = (x_1, \dots, x_p)'$ be an observable random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Omega} = (\omega_{ij})$. Then under the factor analysis model \mathbf{x} can be written in the form

$$\mathbf{x} = \boldsymbol{\mu} + \Lambda \mathbf{f} + \mathbf{e},$$

where $\Lambda = (\lambda_{ik})$ is a $p \times m$ matrix of factor loadings and $\mathbf{f} = (f_1, \dots, f_m)'$ and $\mathbf{e} = (e_1, \dots, e_p)'$ are unobservable random vectors. The elements of \mathbf{f} and \mathbf{e} are called the common factors and the unique factors, respectively. We assume that the means of the elements of \mathbf{f} and \mathbf{e} are zero and that $E(\mathbf{f}\mathbf{f}') = \mathbf{I}_m$ and $E(\mathbf{e}\mathbf{e}') = \boldsymbol{\Psi}$, where \mathbf{I}_m is the identity matrix of order m and $\boldsymbol{\Psi}$ is a diagonal matrix, of which the diagonal elements $\psi_j (> 0)$ are called the *unique variances*. We also assume that $E(\mathbf{f}\mathbf{e}') = \mathbf{0}$, then $\boldsymbol{\Omega}$ is decomposed as

$$\boldsymbol{\Omega} = \Lambda \Lambda' + \boldsymbol{\Psi}. \quad (2.1)$$

In many cases the unit of measurement of each component of \mathbf{x} is arbitrary. If we standardize the observed variable then the corresponding standardized unique variance is given by $\zeta_i = \psi_i / \omega_{ii}$, which is the proportion of the variance of x_i not "explained" by the common factors. We shall call ζ_i the *uniqueness* throughout the paper.

For the existence of consistent estimators we henceforth assume that the solution $\boldsymbol{\Psi}$ of the decomposition (2.1) is unique. Anderson and Rubin [2, Theorem 5.6] have proved that a necessary condition for the uniqueness of the decomposition (2.1) except for multiplication on the right of Λ by an orthogonal matrix is that each column of ΛT has at least three nonzero elements for every nonsingular matrix T , which condition implies that $\text{rank}(\Lambda) = m$. In order to remove this rotational indeterminacy from Λ , we assume that $\Gamma = \Lambda' \boldsymbol{\Psi}^{-1} \Lambda$ is diagonal and that the diagonal elements $\gamma_{kk} (> 0)$ are distinct and ordered.

Let $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ be a random sample of size $N = n + 1$ from the multivariate normal population $N_p(\boldsymbol{\mu}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = \Lambda \Lambda' + \boldsymbol{\Psi}$ and let $\mathbf{S} = (s_{ij})$ be the usual unbiased estimator of $\boldsymbol{\Omega}$. Then the maximum likelihood estimators $\hat{\Lambda}$ and $\hat{\boldsymbol{\Psi}}$ based on the likelihood of \mathbf{S} are defined by the following equations:

$$(\mathbf{S} - \hat{\boldsymbol{\Omega}}) \hat{\boldsymbol{\Psi}}^{-1} \hat{\Lambda} = \mathbf{0}, \quad \text{diag}(\mathbf{S} - \hat{\boldsymbol{\Omega}}) = \mathbf{0}, \quad \text{nondiag } \hat{\Gamma} = \mathbf{0}, \quad (2.2)$$

where $\hat{\boldsymbol{\Omega}} = \hat{\Lambda} \hat{\Lambda}' + \hat{\boldsymbol{\Psi}}$ and $\hat{\Gamma} = \hat{\Lambda}' \hat{\boldsymbol{\Psi}}^{-1} \hat{\Lambda}$ with $\hat{\gamma}_{11} > \dots > \hat{\gamma}_{mm}$.

3. ASYMPTOTIC EXPANSIONS

The maximum likelihood estimators $\hat{\psi}_i$ of the unique variances given as the solutions of the equations (2.2) and $\hat{\zeta}_i$ of the uniquenesses given by $\hat{\psi}_i/\hat{\omega}_{ii}$ can be essentially expressed as a function of the sample covariance matrix \mathbf{S} . Hence, we first derive an asymptotic expansion for the distribution of a function of \mathbf{S} . Asymptotic expansions are given for the distributions of Studentized and non-Studentized versions of estimators.

3.1. Functions of a Sample Covariance Matrix

Let $h(\mathbf{S})$ be a real valued function, not depending on n , of the sample covariance matrix \mathbf{S} and assume that the partial derivatives of $h(\mathbf{S})$ of second order are continuous in a neighborhood of $\mathbf{S}=\mathbf{\Omega}$. It is known that the limiting distribution of

$$y = n^{1/2}\{h(\mathbf{S}) - h(\mathbf{\Omega})\}$$

is normal with zero mean as n tends to infinity. Let $\mathbf{U} = n^{1/2}(\mathbf{S} - \mathbf{\Omega})$. By expanding $h(\mathbf{S})$ in a Taylor series around $\mathbf{\Omega}$ and substituting $\mathbf{U} = (u_{ij})$ in the resulting expansion, we have the stochastic expansion in the form

$$\begin{aligned} y &= \sum_{a,b} \frac{\partial h(\mathbf{S})}{\partial s_{ab}} \bigg|_{\mathbf{S}=\mathbf{\Omega}} u_{ab} + \frac{1}{2} n^{-1/2} \sum_{a,b,c,d} \frac{\partial^2 h(\mathbf{S})}{\partial s_{ab} \partial s_{cd}} \bigg|_{\mathbf{S}=\mathbf{\Omega}} u_{ab} u_{cd} + O_p(n^{-1}) \\ &= \text{tr } \mathbf{A}\mathbf{U} + \frac{1}{2} n^{-1/2} \sum_{a,b,c,d} \tau(ab, cd) u_{ab} u_{cd} + O_p(n^{-1}). \end{aligned} \quad (3.1)$$

Then the asymptotic bias b , variance σ^2 , and skewness κ of y are given by

$$\begin{aligned} E(y) &= n^{-1/2}b + O(n^{-3/2}), \\ E\{y - E(y)\}^2 &= \sigma^2 + O(n^{-1}), \\ E\{y - E(y)\}^3 &= n^{-1/2}\kappa + O(n^{-3/2}), \end{aligned}$$

where

$$\begin{aligned} b &= \frac{1}{2} \sum_{a,b,c,d} \tau(ab, cd)(\omega_{ac}\omega_{bd} + \omega_{ad}\omega_{bc}), \\ \sigma^2 &= 2 \text{tr}(\mathbf{A}\mathbf{\Omega})^2, \\ \kappa &= 12T + 8 \text{tr}(\mathbf{A}\mathbf{\Omega})^3, \end{aligned} \quad (3.2)$$

with

$$T = \sum_{a, b, c, d} \tau(ab, cd) [\mathbf{\Omega A \Omega}]_{ab} [\mathbf{\Omega A \Omega}]_{cd}.$$

Hence, we have an asymptotic expansion for the distribution of $y = n^{1/2}\{h(\mathbf{S}) - h(\mathbf{\Omega})\}$

$$P\left(\frac{y}{\sigma} \leq x\right) = \Phi(x) - n^{-1/2} \left\{ \frac{b}{\sigma} + \frac{\kappa}{6\sigma^3} (x^2 - 1) \right\} \phi(x) + O(n^{-1}), \quad (3.3)$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the distribution and probability density function of the standard normal distribution, respectively (Siotani, Hayakawa and Fujikoshi [10]).

In practice the asymptotic variance σ^2 would be unknown, so we consider the Studentized statistic

$$t = \frac{y}{\hat{\sigma}},$$

where $\hat{\sigma}^2 = \sigma^2(\mathbf{S})$ is a consistent estimator of σ^2 . Suppose $\sigma^2(\cdot)$ is differentiable at $\mathbf{S} = \mathbf{\Omega}$ and can be written as

$$\begin{aligned} \hat{\sigma}^2 - \sigma^2 &= n^{-1/2} \sum_{a, b} \frac{\partial \sigma^2(\mathbf{S})}{\partial s_{ab}} \bigg|_{\mathbf{S}=\mathbf{\Omega}} u_{ab} + O_p(n^{-1}) \\ &= n^{-1/2} \text{tr } \mathbf{H} \mathbf{U} + O_p(n^{-1}). \end{aligned} \quad (3.4)$$

It follows from (3.1) and (3.4) that the Studentized statistic $t = y/\hat{\sigma}$ can be expanded in a power series of order $n^{-1/2}$ as

$$\begin{aligned} t &= \sigma^{-1} \text{tr } \mathbf{A} \mathbf{U} + \frac{1}{2} n^{-1/2} \sigma^{-1} \sum_{a, b, c, d} \tau(ab, cd) u_{ab} u_{cd} \\ &\quad - \frac{1}{2} n^{-1/2} \sigma^{-3} \text{tr } \mathbf{A} \mathbf{U} \cdot \text{tr } \mathbf{H} \mathbf{U} + O_p(n^{-1}). \end{aligned}$$

Then the asymptotic bias b' and skewness κ' of t are of the form

$$\begin{aligned} b' &= \sigma^{-1} b - \sigma^{-3} \text{tr } \mathbf{A} \mathbf{\Omega} \mathbf{H} \mathbf{\Omega}, \\ \kappa' &= \sigma^{-3} \kappa - 6\sigma^{-3} \text{tr } \mathbf{A} \mathbf{\Omega} \mathbf{H} \mathbf{\Omega}, \end{aligned} \quad (3.5)$$

where b , σ^2 , and κ are given by (3.2). Hence, we have an asymptotic expansion of the distribution of the Studentized statistic $t = n^{1/2}\{h(\mathbf{S}) - h(\mathbf{\Omega})\}/\hat{\sigma}$

$$P(t \leq x) = \Phi(x) - n^{-1/2} \left\{ b' + \frac{\kappa'}{6} (x^2 - 1) \right\} \phi(x) + O(n^{-1}). \quad (3.6)$$

Under the normality assumption the expansions (3.3) and (3.6) are valid if y and t , respectively, have continuous derivatives to order two in a neighborhood of $\mathbf{S} = \mathbf{\Omega}$. As to the validity of such expansions, we refer to Hall [6].

3.2. Main Results

In this subsection we derive asymptotic expansions for the distributions of the maximum likelihood estimators of unique variances and uniquenesses. For the purpose, we obtain stochastic expansions for $\hat{\psi}_i$ and $\hat{\xi}_i$ by using the perturbation method. Suppose that $\hat{\Psi}$ and $\hat{\Lambda}$ can be expanded for small ϵ as

$$\begin{aligned} \hat{\Psi} &= \Psi + \epsilon \mathbf{P}^{(1)} + \epsilon^2 \mathbf{P}^{(2)} + \dots, \\ \hat{\Lambda} &= \Lambda + \epsilon \mathbf{L}^{(1)} + \epsilon^2 \mathbf{L}^{(2)} + \dots, \end{aligned} \quad (3.7)$$

where $\mathbf{P}^{(k)}$ and $\mathbf{L}^{(k)}$ ($k = 1, 2, \dots$) are $p \times p$ diagonal and $p \times m$ matrices, respectively. Substituting these expansions into (2.2) with $\mathbf{S} = \mathbf{\Omega} + \epsilon \mathbf{U}$ and equating the coefficients of ϵ^k ($k = 1, 2$) on both sides, we obtain expressions of $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ in terms of \mathbf{U} . To this end we define $\Xi = (\xi_{ij})$ by

$$\Xi = \Phi \odot \Phi,$$

where

$$\Phi = \Psi^{-1} - \Psi^{-1} \Lambda (\Lambda' \Psi^{-1} \Lambda)^{-1} \Lambda' \Psi^{-1}$$

and the symbol \odot indicates Hadamard product of matrices. We assume that Ξ is positive definite and define a diagonal matrix Θ_i whose diagonal elements are the i -th row (column) of $\Xi^{-1} = (\xi^{ij})$. Then it can be shown that the stochastic expansion of $\hat{\psi}_i$ is of the form

$$\hat{\psi}_i = \psi_i + n^{-1/2} p_i^{(1)} + n^{-1} p_i^{(2)} + O_p(n^{-3/2}),$$

where

$$p_i^{(1)} = \text{tr } \Phi \Theta_i \Phi U, \quad p_i^{(2)} = \text{tr } \Theta_i Q,$$

and

$$\begin{aligned} Q = & -\Phi(U - P^{(1)}) B(U - P^{(1)}) \Phi \\ & - 2\Phi(U - P^{(1)}) \Phi(U - P^{(1)}) B + 2\Phi(U - P^{(1)}) \Phi P^{(1)} \Pi \end{aligned} \quad (3.8)$$

with $B = (\beta_{ij}) = \Psi^{-1} \Lambda \Gamma^{-2} \Lambda' \Psi^{-1}$ and $\Pi = \Psi^{-1} \Lambda \Gamma^{-1} \Lambda' \Psi^{-1}$.

Asymptotic expansions for the distributions of the estimator $\hat{\psi}_i$ can be now derived by an argument similar to that discussed in subsection 3.1. The results are summarized in the following theorem.

THEOREM 1. *Let $\hat{\psi}_i$ be the maximum likelihood estimator of the unique variance defined as a solution of (2.2). Then the asymptotic variance of $\hat{\psi}_i$ is given by $2\xi^{ii}$.*

1. *An asymptotic expansion for the distribution of $\hat{\psi}_i$ is given by*

$$\begin{aligned} P \left\{ \frac{n^{1/2}(\hat{\psi}_i - \psi_i)}{(2\xi^{ii})^{1/2}} \leq x \right\} \\ = \Phi(x) - n^{-1/2} \left\{ \frac{b_i}{(2\xi^{ii})^{1/2}} + \frac{\kappa_i}{6(2\xi^{ii})^{3/2}} (x^2 - 1) \right\} \phi(x) + O(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} b_i &= -\psi_i \text{tr}(\mathbf{I} + \Gamma^{-1}) - 2 \text{tr } \Phi \Theta_i \Phi (\mathbf{B} \odot \Xi^{-1}), \\ \kappa_i &= 8 \text{tr}(\Phi \Theta_i)^3 - 24 \text{tr } \Phi \Theta_i \Phi \Theta_i \mathbf{B} \Theta_i. \end{aligned} \quad (3.9)$$

2. *The Studentized statistic $n^{1/2}(\hat{\psi}_i - \psi_i)/(2\hat{\xi}^{ii})^{1/2}$ has the asymptotic expansion*

$$P \left\{ \frac{n^{1/2}(\hat{\psi}_i - \psi_i)}{(2\hat{\xi}^{ii})^{1/2}} \leq x \right\} = \Phi(x) - n^{-1/2} \left\{ b'_i + \frac{\kappa'_i}{6} (x^2 - 1) \right\} \phi(x) + O(n^{-1}),$$

where

$$\begin{aligned} b'_i &= (2\xi^{ii})^{-1/2} b_i - (2\xi^{ii})^{-3/2} \{4 \text{tr}(\Phi \Theta_i)^3 - 8 \text{tr } \Phi \Theta_i \Phi \Theta_i \mathbf{B} \Theta_i\}, \\ \kappa'_i &= (2\xi^{ii})^{-3/2} \kappa_i - 6(2\xi^{ii})^{-3/2} \{4 \text{tr}(\Phi \Theta_i)^3 - 8 \text{tr } \Phi \Theta_i \Phi \Theta_i \mathbf{B} \Theta_i\}, \end{aligned}$$

with b_i and κ_i given by (3.9).

Noting that $\hat{\omega}_{ii} = s_{ii}$ from (2.2) and $u_{ii} = n^{1/2}(s_{ii} - \omega_{ii})$, the stochastic expansion of the estimator $\hat{\zeta}_i = \hat{\psi}_i / \hat{\omega}_{ii}$ of the uniqueness ζ_i is

$$\begin{aligned}\hat{\zeta}_i &= \hat{\psi}_i / s_{ii} \\ &= \zeta_i + n^{-1/2} \omega_{ii}^{-1} (p_i^{(1)} - \zeta_i u_{ii}) \\ &\quad + n^{-1} (\omega_{ii}^{-1} p_i^{(2)} - \omega_{ii}^{-2} p_i^{(1)} u_{ii} + \omega_{ii}^{-2} \zeta_i u_{ii}^2) + O_p(n^{-3/2}).\end{aligned}\quad (3.10)$$

Then using the results (3.3) and (3.6), we have the following theorem.

THEOREM 2. *Let $\hat{\zeta}_i$ be the maximum likelihood estimator of the uniqueness ζ_i . Then the asymptotic variance of $\hat{\zeta}_i$ is*

$$\sigma_i^2 = 2\omega_{ii}^{-2} \xi^{ii} + 2\zeta_i^2(1 - 2\zeta_i). \quad (3.11)$$

1. *An asymptotic expansion for the distribution of $\hat{\zeta}_i$ is given by*

$$\begin{aligned}\mathbf{P} \left\{ \frac{n^{1/2}(\hat{\zeta}_i - \zeta_i)}{\sigma_i} \leq x \right\} \\ = \Phi(x) - n^{-1/2} \left\{ \frac{b_i}{\sigma_i} + \frac{\kappa_i}{6\sigma_i^3} (x^2 - 1) \right\} \phi(x) + O(n^{-1}),\end{aligned}$$

where

$$b_i = -\zeta_i \operatorname{tr} (\mathbf{I} + \mathbf{\Gamma}^{-1}) - 2\omega_{ii}^{-1} \operatorname{tr} \mathbf{\Phi} \mathbf{\Theta}_i \mathbf{\Phi} (\mathbf{B} \odot \mathbf{\Xi}^{-1}) + 2\zeta_i(1 - \zeta_i), \quad (3.12)$$

and κ_i is given by (3.2) with

$$\begin{aligned}T &= -2\omega_{ii}^{-3} \operatorname{tr} \mathbf{\Phi} \mathbf{\Theta}_i \mathbf{\Phi} \mathbf{\Theta}_i \mathbf{B} \mathbf{\Theta}_i + 4\omega_{ii}^{-1} \zeta_i^3 [\mathbf{\Phi} \mathbf{\Theta}_i \mathbf{\Phi} \mathbf{\Theta}_i \mathbf{\Phi}]_{ii} \\ &\quad + 2\zeta_i(1 - 3\zeta_i) \{ \omega_{ii}^{-2} \xi^{ii} + \zeta_i^2(1 - \zeta_i) \}, \\ \operatorname{tr}(\mathbf{A} \mathbf{\Omega})^3 &= \omega_{ii}^{-3} \operatorname{tr}(\mathbf{\Phi} \mathbf{\Theta}_i)^3 - 3\omega_{ii}^{-1} \zeta_i^3 [\mathbf{\Phi} \mathbf{\Theta}_i \mathbf{\Phi} \mathbf{\Theta}_i \mathbf{\Phi}]_{ii} - \zeta_i^3(1 - 3\zeta_i).\end{aligned}\quad (3.13)$$

2. *The Studentized statistic $n^{1/2}(\hat{\zeta}_i - \zeta_i) / \hat{\sigma}_i$ has the asymptotic expansion*

$$\mathbf{P} \left\{ \frac{n^{1/2}(\hat{\zeta}_i - \zeta_i)}{\hat{\sigma}_i} \leq x \right\} = \Phi(x) - n^{-1/2} \left\{ b'_i + \frac{\kappa'_i}{6} (x^2 - 1) \right\} \phi(x) + O(n^{-1}),$$

where

$$\begin{aligned}b'_i &= \sigma_i^{-1} b_i - \sigma_i^{-3} \operatorname{tr} \mathbf{A} \mathbf{\Omega} \mathbf{H} \mathbf{\Omega}, \\ \kappa'_i &= \sigma_i^{-3} \kappa_i - 6\sigma_i^{-3} \operatorname{tr} \mathbf{A} \mathbf{\Omega} \mathbf{H} \mathbf{\Omega},\end{aligned}$$

with

$$\begin{aligned} \text{tr } \mathbf{A}\mathbf{\Omega}\mathbf{H}\mathbf{\Omega} &= 4\omega_{ii}^{-3} \text{tr}(\mathbf{\Phi}\mathbf{\Theta}_i)^3 - 8\omega_{ii}^{-3} \text{tr } \mathbf{\Phi}\mathbf{\Theta}_i\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{B}\mathbf{\Theta}_i \\ &\quad + 4\omega_{ii}^{-1}\zeta_i^3 [\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{\Phi}]_{ii} \\ &\quad + 4\zeta_i(1-3\zeta_i)\{2\omega_{ii}^{-2}\zeta_i^2 + \zeta_i^2(1-2\zeta_i)\} \end{aligned}$$

and b_i given by (3.12); the asymptotic skewness κ_i is of the form (3.2) with T and $\text{tr}(\mathbf{A}\mathbf{\Omega})^3$ given by (3.13).

4. CONSTRUCTION OF CONFIDENCE INTERVALS

In this section, we introduce three types of confidence intervals that are based on the distribution of the Studentized statistic. Each of the three intervals employs the normal approximation, the asymptotic expansion of percentile points, and further modification by using the bootstrap.

Let t_α be the 100α percentile point of the Studentized statistic $t = n^{1/2}\{h(\mathbf{S}) - h(\mathbf{\Omega})\}/\hat{\sigma}$. Then the exact confidence interval CI0 of $h(\mathbf{\Omega})$ with confidence coefficient $1 - 2\alpha$ based on the distribution of t is given by

$$(h(\mathbf{S}) - t_{1-\alpha}n^{-1/2}\hat{\sigma}, h(\mathbf{S}) - t_\alpha n^{-1/2}\hat{\sigma}). \quad (4.1)$$

By approximating the distribution of t by the standard normal distribution we obtain a confidence interval, which we denote CI1,

$$(h(\mathbf{S}) - z_{1-\alpha}n^{-1/2}\hat{\sigma}, h(\mathbf{S}) - z_\alpha n^{-1/2}\hat{\sigma}), \quad (4.2)$$

where z_α is the 100α percentile point of the standard normal distribution.

By the inverse Cornish-Fisher expansion (see, for example, Hall [6]), the 100α percentile point of the distribution function $P(t \leq x)$ can be written as

$$z_\alpha - n^{-1/2}p(z_\alpha) + O(n^{-1}),$$

where $p(x) = p(x; b', \kappa')$ is defined by

$$p(x) = -\left\{b' + \frac{\kappa'}{6}(x^2 - 1)\right\}.$$

Then we obtain a confidence interval, which we denote CI2,

$$(h(\mathbf{S}) - \{z_{1-\alpha} - n^{-1/2}\hat{p}(z_{1-\alpha})\}n^{-1/2}\hat{\sigma}, h(\mathbf{S}) - \{z_\alpha - n^{-1/2}\hat{p}(z_\alpha)\}n^{-1/2}\hat{\sigma}), \quad (4.3)$$

where $\hat{p}(x) = p(x; \hat{b}', \hat{\kappa}')$ and \hat{b}' and $\hat{\kappa}'$ are consistent estimators of b' and κ' , respectively. We note here that the confidence interval CI2 has the same length as CI1 because $p(x)$ is an even function.

It may be possible, at least in principle, to derive the term of order n^{-1} in the asymptotic expansion (3.6) of the distribution of t . But the result would be so complicated and may not be practical. Abramovitch and Singh [1] have given a method of improving asymptotic expansion based confidence intervals by using the bootstrap. That is, let v_α^* be the 100α percentile point of the bootstrap distribution of the statistic

$$v = t + n^{-1/2} \hat{p}(t),$$

and let c_α be defined by

$$c_\alpha = v_\alpha^* - n^{-1/2} \hat{p}(v_\alpha^*) + n^{-1} \hat{p}(v_\alpha^*) \hat{p}'(v_\alpha^*).$$

Abramovitch and Singh [1] showed that the confidence interval CI3

$$(h(\mathbf{S}) - c_{1-\alpha} n^{-1/2} \hat{\sigma}, h(\mathbf{S}) - c_\alpha n^{-1/2} \hat{\sigma}) \quad (4.4)$$

is more accurate than CI2. In practice v_α^* is approximated by a Monte Carlo simulation as follows:

1. Generate a bootstrap version of the sample covariance matrix \mathbf{S}^* from a Wishart population $W_p(n-1, (n-1)^{-1} (\hat{\Lambda} \hat{\Lambda}' + \hat{\Psi}))$.
2. Compute bootstrap replication v^* from \mathbf{S}^* .
3. Repeat the above steps B times and take 100α percentile point of these B values as v_α^* .

5. NUMERICAL EXPERIMENTS

In this section we give numerical examples to demonstrate how approximations to the exact distributions of the Studentized statistics are improved by using the asymptotic expansion (3.6). We also compare by a Monte Carlo simulation the performance of the three types of confidence intervals CI1, CI2, and CI3 described in the previous section. The values of the population factor loadings λ_{ik} and unique variances ψ_i were

$$\Lambda = \begin{pmatrix} .668 & .692 & .500 & .839 & .700 & .800 & .670 & .442 & .775 \\ .304 & .236 & .287 & -.321 & -.319 & -.372 & .385 & .245 & .424 \end{pmatrix}',$$

$$\Psi = (.462 \ .465 \ .667 \ .192 \ .408 \ .221 \ .403 \ .745 \ .219)',$$

which were determined by fitting with the maximum likelihood method the two factor model to the correlation matrix reported in Emmett [5] as if it were a covariance matrix.

TABLE I
Asymptotic Biases, Standard Errors, and Skewnesses

$\hat{\psi}_i$					
i	b'	κ'	b/σ	κ/σ^3	σ
8	-3.007	-5.554	-1.625	2.732	1.102
3	-2.954	-5.485	-1.595	2.671	1.012
2	-2.848	-5.385	-1.517	2.595	.743
1	-2.780	-5.237	-1.495	2.469	.758
5	-2.613	-5.006	-1.396	2.298	.699
7	-2.565	-4.772	-1.422	2.087	.724
6	-1.523	-2.017	-1.185	.012	.608
9	-1.847	-3.239	-1.190	.705	.589
4	-1.459	-1.905	-1.148	-.041	.553
$\hat{\zeta}_i$					
i	b'	κ'	b/σ	κ/σ^3	σ
8	-1.118	1.680	-1.720	-1.931	.820
3	-1.239	.239	-1.373	-.567	.852
2	-1.589	-2.518	-.825	2.067	.763
1	-1.554	-2.479	-.816	1.948	.779
5	-1.544	-2.949	-.664	2.331	.742
7	-1.557	-2.897	-.715	2.152	.766
6	-1.344	-2.873	-.578	1.727	.651
9	-1.607	-3.884	-.568	2.354	.633
4	-1.362	-3.029	-.547	1.866	.592

Table I shows the asymptotic biases and skewnesses of both Studentized and non-Studentized statistics. It also shows the asymptotic variances of the non-Studentized statistics. These values were calculated by using the formulae given in Section 3. The variables are reordered in descending order of the values of $\psi_i (= \zeta_i)$. It can be seen that the values of b' and κ' are negative for $\hat{\psi}_i$ and their absolute values increase with ψ_i . In the case of $\hat{\zeta}_i$, the values of b' are negative but there is no clear relation between the values of ζ_i and those of b' and κ' .

Table II shows the exact values of the biases, variances, skewnesses, and kurtoses of the Studentized statistics. These values were determined by Monte Carlo simulations, that is, by generating \mathbf{S} from a Wishart population $W_p(n, n^{-1}(\Lambda\Lambda' + \Psi))$, calculating $\hat{\psi}_i$, $\hat{\zeta}_i$, and their standard error estimates to obtain Studentized statistics, and repeating these steps 10^6 times for each sample size. The maximum likelihood estimates $\hat{\psi}_i$ and $\hat{\zeta}_i$ were calculated by using the program coded in C by one of the present authors, which program employs a Newton-Raphson algorithm with line minimization. It is noted that the exact values of the variances were greater than the

TABLE II

Exact Values of the Bias (B), Variance (V), Skewness (S), and Kurtosis (K)
of the Studentized Statistics

$N = 150$								
$\hat{\psi}_i$					$\hat{\zeta}_i$			
i	B	V	S	K	B	V	S	K
8	-.255	1.117	-.466	.402	-.094	1.112	.178	.546
3	-.248	1.117	-.465	.412	-.104	1.101	.035	.353
2	-.244	1.121	-.456	.372	-.137	1.102	-.225	.260
1	-.236	1.118	-.445	.375	-.132	1.103	-.222	.272
5	-.215	1.100	-.412	.283	-.126	1.096	-.252	.265
7	-.219	1.118	-.407	.282	-.134	1.118	-.259	.272
6	-.128	1.066	-.159	-.115	-.112	1.091	-.216	.022
9	-.158	1.112	-.262	.042	-.138	1.135	-.316	.206
4	-.126	1.061	-.155	-.120	-.117	1.083	-.230	.006
$N = 300$								
$\hat{\psi}_i$					$\hat{\zeta}_i$			
i	B	V	S	K	B	V	S	K
8	-.177	1.055	-.330	.209	-.065	1.048	.108	.216
3	-.174	1.056	-.323	.197	-.074	1.051	.019	.157
2	-.167	1.057	-.317	.182	-.094	1.045	-.154	.129
1	-.163	1.057	-.307	.183	-.092	1.050	-.153	.122
5	-.151	1.048	-.295	.156	-.088	1.042	-.175	.131
7	-.151	1.055	-.278	.138	-.091	1.055	-.170	.118
6	-.088	1.029	-.115	-.062	-.078	1.043	-.161	.008
9	-.109	1.052	-.191	.017	-.095	1.064	-.231	.102
4	-.089	1.025	-.106	-.071	-.083	1.037	-.166	.010

asymptotic variance unity. It is also noted that the exact values of the kurtoses increased with ψ_i and ζ_i , in particular, when ζ_i was close to its upper bound unity. It may be seen that $n^{-1/2}b'$ and $n^{-1/2}\kappa'$ as shown in Table 5 approximated the exact values fairly well.

Figure 1 shows the graphs of errors in approximating the distributions of the Studentized statistics. It can be seen that, in general, the asymptotic expansion improved the accuracy of the approximation in the tail area except for the case of $\hat{\zeta}_8$ whose true value was close to its upper bound unity. It can be also seen that, except for $\hat{\zeta}_8$,

$$t_\alpha < z_\alpha - n^{-1/2}p(z_\alpha) < z_\alpha,$$

$$z_{1-\alpha} - n^{-1/2}p(z_{1-\alpha}) < t_{1-\alpha} < z_{1-\alpha},$$

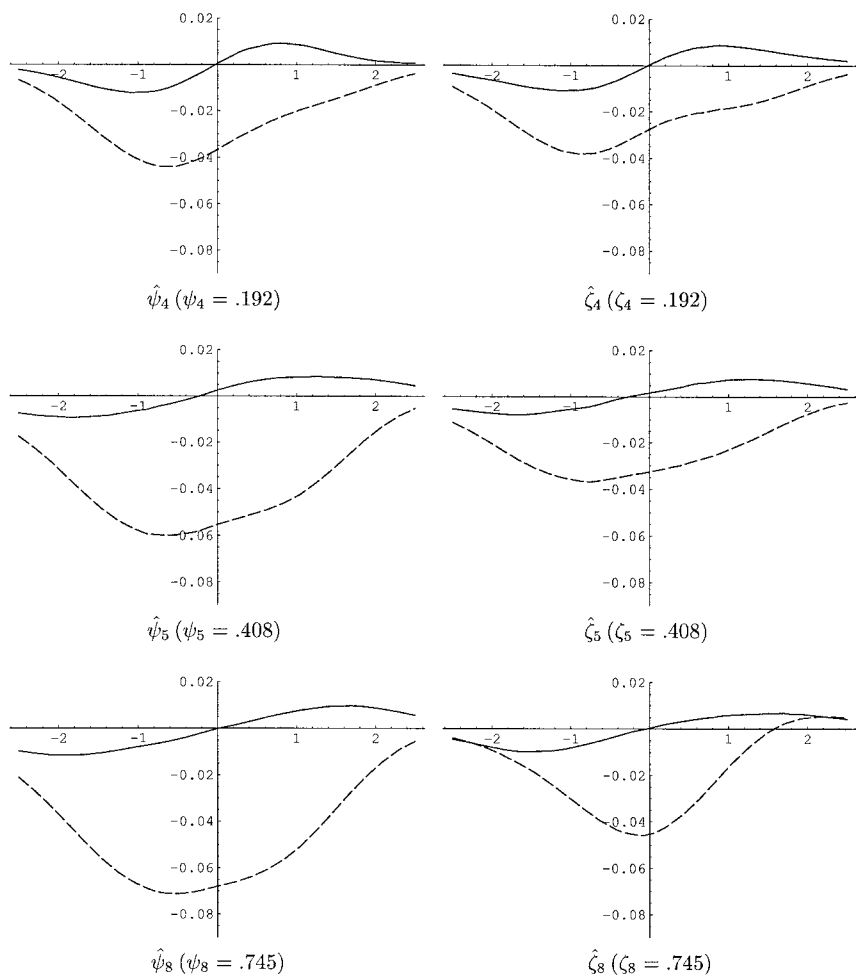


FIG. 1. Errors in approximating the distribution $P(t \leq x)$ of the Studentized statistics with $N=150$: Error=Approximate value—Exact value. Solid line indicates the error by asymptotic expansion (3.6) and dashed line by normal approximation.

with $\alpha < .5$. This implies that, neglecting errors in estimating b' and κ' in CI2,

$$CI1_{lo} < CI0_{lo} < CI2_{lo},$$

$$CI1_{up} < CI2_{up} < CI0_{up},$$

where $CI1_{lo}$ and $CI1_{up}$, for example, indicate the lower and upper confidence limits of CI1, respectively.

Table III compares the performance over 5,000 trials of the three types of confidence intervals CI1, CI2, and CI3 with $N = 150$ and $N = 300$. The number of the bootstrap samples in CI3 was chosen as $B = 1,000$. The symbol L (R) indicates the percentage of trials in which the true value was less (greater) than the lower (upper) confidence limit. The symbol T indicates the sum of L and R. Except for ζ_8 and ζ_3 CI1 underestimated L and overestimated R. The intervals CI2 and CI3 achieved better balance in the left and right sides, but they tended to overestimate both L and R. For the case of CI2, this may be due to the fact that CI2 did not take into account of the fact that the exact variances of u were greater than the asymptotic variance unity. The variations of T across ψ_i s and ζ_i s in CI2 and CI3 were

TABLE III
Results of 5,000 Confidence Intervals

$(N = 150)$									
$\hat{\psi}$									
	CI1			CI2			CI3		
i	L	R	T	L	R	T	L	R	T
8	.80	6.16	6.96	3.28	3.04	6.32	2.70	2.70	5.40
3	.68	6.28	6.96	3.10	2.98	6.08	2.56	2.60	5.16
2	.64	6.98	7.62	3.66	3.70	7.36	2.68	3.06	5.74
1	.76	5.92	6.68	3.54	2.98	6.52	2.70	2.34	5.04
5	.88	5.86	6.74	3.22	2.88	6.10	2.48	2.12	4.60
7	.92	5.98	6.90	3.06	3.32	6.38	2.30	2.26	4.56
6	1.46	4.30	5.76	3.98	4.08	8.06	3.82	3.86	7.68
9	1.68	4.96	6.64	3.94	3.62	7.56	3.16	3.10	6.26
4	1.64	4.20	5.84	4.00	4.12	8.12	3.66	4.30	7.96
Range	1.04	2.78	1.86	.94	1.24	2.02	1.52	2.18	3.40
ζ_i									
	CI1			CI2			CI3		
i	L	R	T	L	R	T	L	R	T
8	3.18	3.70	6.88	2.56	2.70	5.26	5.08	2.70	7.78
3	2.78	3.78	6.56	2.54	2.74	5.28	3.70	2.54	6.24
2	1.68	4.54	6.22	2.42	2.80	5.22	2.54	2.68	5.22
1	1.50	4.84	6.34	2.32	2.76	5.08	2.46	2.52	4.98
5	1.64	4.50	6.14	2.56	2.62	5.18	2.88	2.36	5.24
7	1.48	4.84	6.32	2.54	2.74	5.28	2.64	2.36	5.00
6	1.56	4.66	6.22	3.28	3.60	6.88	3.40	3.30	6.70
9	1.60	5.30	6.90	3.28	3.70	6.98	2.80	2.94	5.74
4	1.54	4.44	5.98	3.06	3.78	6.84	3.06	4.00	7.06
Range	1.70	1.60	.92	.96	1.16	1.90	2.62	1.64	2.80

TABLE III—Continued

(N = 300)										
ψ_i										
	CI1			CI2			CI3			
<i>i</i>	L	R	T	L	R	T	L	R	T	
8	1.00	5.02	6.02	2.82	2.82	5.64	2.56	2.70	5.26	
3	1.10	4.60	5.70	2.86	2.72	5.58	2.68	2.62	5.30	
2	1.00	4.86	5.86	2.70	2.70	5.40	2.56	2.46	5.02	
1	1.08	5.06	6.14	3.08	3.16	6.24	2.72	2.80	5.52	
5	1.18	4.80	5.98	2.44	3.22	5.66	2.36	2.96	5.32	
7	1.42	4.48	5.90	3.32	2.78	6.10	2.96	2.30	5.26	
6	1.76	3.32	5.08	2.88	2.86	5.74	2.72	2.10	4.82	
9	1.48	3.78	5.26	3.10	2.80	5.90	2.82	1.98	4.80	
4	1.90	3.42	5.32	3.08	3.02	6.10	2.86	1.94	4.80	
Range	.90	1.74	1.06	.88	.52	.84	.60	1.02	.72	
ζ_i										
	CI1			CI2			CI3			
<i>i</i>	L	R	T	L	R	T	L	R	T	
8	2.84	3.08	5.92	2.56	2.58	5.14	3.10	2.38	5.48	
3	2.36	3.02	5.38	2.40	2.22	4.62	2.80	2.16	4.96	
2	1.42	3.72	5.14	2.20	2.52	4.72	2.36	2.50	4.86	
1	2.06	4.22	6.28	2.78	3.00	5.78	2.86	2.72	5.58	
5	1.72	4.04	5.76	2.50	2.84	5.34	2.72	2.68	5.40	
7	1.92	3.66	5.58	2.92	2.72	5.64	3.04	2.56	5.60	
6	1.64	3.34	4.98	2.96	2.68	5.64	2.76	2.12	4.88	
9	1.36	4.00	5.36	2.38	2.70	5.08	2.28	2.08	4.36	
4	1.84	3.10	4.94	3.16	2.60	5.76	3.12	1.90	5.02	
Range	1.48	1.20	1.34	.96	.78	1.16	.84	.82	1.24	

Note. This table shows the percentage of trials in which the indicated interval missed the true value on the left (L) or right (R) side. For example, the symbol L means that the left end point was greater than the true value. The symbol T indicates the sum of L and R. The desired coverage is 95%, so that the ideal values of L and R are both 2.5%.

greater than those of CI1 when $N = 150$; but the variations of L, R, and T across ψ_i s and ζ_i s in CI2 and CI3 were smaller than those of CI1 when $N = 300$. It may be seen from Table 1 that both CI2 and CI3 performed somewhat poorly for ψ_i s and ζ_i s whose true values were small.

It is known that the occurrence of improper solution or nonconvergence in maximum likelihood factor analysis is by no means an exception (van Driel [11]). The frequencies of these problems before we obtained $5,000 \times 1,000$ estimates from bootstrap samples were 68,859 ($N = 150$) and 1,025 ($N = 300$), respectively.

6. CONCLUDING REMARKS

We have derived asymptotic expansions for the distributions of the normal theory maximum likelihood estimators of the unique variances $\hat{\psi}_i$ and uniquenesses $\hat{\zeta}_i$. In particular the asymptotic results for the Studentized statistics are useful to construct confidence intervals. Our experiment showed that the confidence interval CI1 that employs the normal approximation was far from equal-tailed even when the sample size was moderate. For a small sample there was no clear evidence that the bootstrap modified confidence interval CI3 performed over the confidence interval CI2 that employs the asymptotic expansion of percentile points. It seems that the performance of CI3 depends on the accuracy of the bootstrap simulations, which is reflected by the frequency of improper solutions from the bootstrap samples. Hence, we would recommend CI2 for a small sample. The use of CI3 would be recommended only when the sample size is moderate to large and would be better avoided when the improper solutions are frequent in bootstrap simulation.

Under nonnormality, $\hat{\psi}_i$ and $\hat{\zeta}_i$ are no longer maximum likelihood estimators; The formulae (3.2) and (3.5) for the asymptotic biases, variances and skewnesses of the distributions of functions of a sample covariance matrix \mathbf{S} have to be modified because the distributions of the elements of \mathbf{S} and hence those of $\hat{\psi}_i$ and $\hat{\zeta}_i$ depend on the fourth order moment of the observed random vector $\mathbf{x} = (x_1, \dots, x_p)'$. It may be possible to obtain asymptotic expansions for the distributions of $\hat{\psi}_i$ and $\hat{\zeta}_i$ under nonnormality, but the resulting coefficients would be so complicated and it may not be easy even to observe the effect of nonnormality.

APPENDIX

This appendix outlines the derivation of the stochastic expansions of the maximum likelihood estimators of unique variances and uniquenesses in Section 3.2, using the perturbation method.

By substituting the expansions (3.7) into (2.2) with $\mathbf{S} = \mathbf{\Omega} + \epsilon \mathbf{U}$ and equating the coefficients of $\epsilon^k (k = 1, 2)$ on both sides, we obtain the following equations.

$$(\mathbf{U} - \mathbf{\Lambda} \mathbf{L}^{(1)'} - \mathbf{L}^{(1)} \mathbf{\Lambda}' - \mathbf{P}^{(1)}) \mathbf{\Psi}^{-1} \mathbf{\Lambda} = \mathbf{0}, \quad (\text{A1})$$

$$\begin{aligned} & (\mathbf{U} - \mathbf{\Lambda} \mathbf{L}^{(1)'} - \mathbf{L}^{(1)} \mathbf{\Lambda}' - \mathbf{P}^{(1)}) \mathbf{\Psi}^{-1} (\mathbf{L}^{(1)} - \mathbf{P}^{(1)} \mathbf{\Psi}^{-1} \mathbf{\Lambda}) \\ & - (\mathbf{P}^{(2)} + \mathbf{L}^{(1)} \mathbf{L}^{(1)'} + \mathbf{L}^{(2)} \mathbf{\Lambda}' + \mathbf{\Lambda} \mathbf{L}^{(2)'}) \mathbf{\Psi}^{-1} \mathbf{\Lambda} = \mathbf{0}, \end{aligned} \quad (\text{A2})$$

$$\text{diag}(\mathbf{U} - \mathbf{\Lambda} \mathbf{L}^{(1)'} - \mathbf{L}^{(1)} \mathbf{\Lambda}' - \mathbf{P}^{(1)}) = \mathbf{0}, \quad (\text{A3})$$

$$\text{diag}(\mathbf{P}^{(2)} + \mathbf{L}^{(1)} \mathbf{L}^{(1)'} + \mathbf{L}^{(2)} \mathbf{\Lambda}' + \mathbf{\Lambda} \mathbf{L}^{(2)'}) = \mathbf{0}. \quad (\text{A4})$$

Anderson and Rubin [2] showed that

$$\mathbf{U} - (\mathbf{L}^{(1)}\mathbf{\Lambda}' + \mathbf{\Lambda}\mathbf{L}^{(1)'} + \mathbf{P}^{(1)}) = \mathbf{\Psi}\mathbf{\Phi}(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{\Phi}\mathbf{\Psi} \quad (\text{A5})$$

and that $p_i^{(1)}$ can be written as

$$p_i^{(1)} = \sum_{j, a, b} \xi^{ij} \phi_{ja} \phi_{jb} u_{ab}.$$

Hence, for the maximum likelihood estimator, $\hat{\psi}_i$, of the unique variance, the matrix \mathbf{A} in (3.1) is given by

$$\mathbf{A}_{\hat{\psi}_i} = \mathbf{\Phi}\mathbf{\Theta}_i\mathbf{\Phi}.$$

Next, we express $\mathbf{P}^{(2)}$ in terms of \mathbf{U} and $\mathbf{P}^{(1)}$. By substituting (A5) into (A2), we have

$$\begin{aligned} & \mathbf{\Psi}\mathbf{\Phi}(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{\Phi}(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\mathbf{\Psi}^{-1}\mathbf{\Lambda}) \\ & - (\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'} + \mathbf{L}^{(2)}\mathbf{\Lambda}' + \mathbf{\Lambda}\mathbf{L}^{(2)'})\mathbf{\Psi}^{-1}\mathbf{\Lambda} = \mathbf{0}, \end{aligned}$$

from which we obtain

$$\begin{aligned} \mathbf{L}^{(2)}\mathbf{\Lambda}' &= \mathbf{\Psi}\mathbf{\Phi}(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{\Phi}(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\mathbf{\Psi}^{-1}\mathbf{\Lambda})\mathbf{\Gamma}^{-1}\mathbf{\Lambda}' \\ & - (\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'})\mathbf{\Pi}\mathbf{\Psi} - \mathbf{\Lambda}\mathbf{L}^{(2)'}\mathbf{\Pi}\mathbf{\Psi}, \end{aligned}$$

and

$$\begin{aligned} & \mathbf{L}^{(2)}\mathbf{\Lambda}' + \mathbf{\Lambda}\mathbf{L}^{(2)'} \\ &= \mathbf{\Psi}\mathbf{\Phi}(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{\Phi}(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\mathbf{\Psi}^{-1}\mathbf{\Lambda})\mathbf{\Gamma}^{-1}\mathbf{\Lambda}' \\ & - (\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'})\mathbf{\Pi}\mathbf{\Psi} - \mathbf{\Lambda}\mathbf{L}^{(2)'}\mathbf{\Pi}\mathbf{\Psi} \\ & + \mathbf{\Lambda}\mathbf{\Gamma}^{-1}(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\mathbf{\Psi}^{-1}\mathbf{\Lambda})'\mathbf{\Phi}(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{\Phi}\mathbf{\Psi} \\ & - \mathbf{\Psi}\mathbf{\Pi}(\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'}) - \mathbf{\Psi}\mathbf{\Pi}\mathbf{\Lambda}\mathbf{L}^{(2)'}\mathbf{\Lambda}' \\ &= \mathbf{\Psi}\mathbf{\Phi}(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{\Phi}(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\mathbf{\Psi}^{-1}\mathbf{\Lambda})\mathbf{\Gamma}^{-1}\mathbf{\Lambda}' \\ & - (\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'})\mathbf{\Pi}\mathbf{\Psi} - \mathbf{\Lambda}\mathbf{L}^{(2)'}\mathbf{\Pi}\mathbf{\Psi} \\ & + \mathbf{\Lambda}\mathbf{\Gamma}^{-1}(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\mathbf{\Psi}^{-1}\mathbf{\Lambda})'\mathbf{\Phi}(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{\Phi}\mathbf{\Psi} \\ & - \mathbf{\Psi}\mathbf{\Pi}(\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'}) \\ & - \mathbf{\Psi}\mathbf{\Pi}\mathbf{\Psi}\mathbf{\Phi}(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{\Phi}(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\mathbf{\Psi}^{-1}\mathbf{\Lambda})\mathbf{\Gamma}^{-1}\mathbf{\Lambda}' \\ & + \mathbf{\Psi}\mathbf{\Pi}(\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'})\mathbf{\Pi}\mathbf{\Psi} + \mathbf{\Psi}\mathbf{\Pi}\mathbf{\Lambda}\mathbf{L}^{(2)'}\mathbf{\Pi}\mathbf{\Psi}. \end{aligned}$$

Since $\Pi\Psi\Phi = \mathbf{0}$, $\Pi\Lambda = \Psi^{-1}\Lambda$ and $\mathbf{I} - \Psi\Pi = \Psi\Phi$, we have

$$\begin{aligned} & \mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'} + \mathbf{L}^{(2)}\Lambda' + \Lambda\mathbf{L}^{(2)'} \\ &= \Psi\Phi(\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'})\Phi\Psi \\ & \quad + \Psi\Phi(\mathbf{U} - \mathbf{P}^{(1)})\Phi(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\Psi^{-1}\Lambda)\Gamma^{-1}\Lambda' \\ & \quad + \Lambda\Gamma^{-1}(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\Psi^{-1}\Lambda)'\Phi(\mathbf{U} - \mathbf{P}^{(1)})\Phi\Psi. \end{aligned} \quad (\text{A6})$$

By using (A1), $\mathbf{L}^{(1)}$ can be written as

$$\mathbf{L}^{(1)} = (\mathbf{U} - \mathbf{P}^{(1)})\Psi^{-1}\Lambda\Gamma^{-1} - \Lambda\mathbf{L}^{(1)'}\Psi^{-1}\Lambda\Gamma^{-1},$$

and since $\Phi\Lambda = \mathbf{0}$ we have

$$\Phi\mathbf{L}^{(1)} = \Phi(\mathbf{U} - \mathbf{P}^{(1)})\Psi^{-1}\Lambda\Gamma^{-1}, \quad (\text{A7})$$

$$\Phi(\mathbf{L}^{(1)} - \mathbf{P}^{(1)}\Psi^{-1}\Lambda)\Gamma^{-1}\Lambda' = \Phi\{\mathbf{U}\mathbf{B} - \mathbf{P}^{(1)}(\mathbf{B} + \Pi)\}\Psi. \quad (\text{A8})$$

Hence, by substituting (A7) and (A8) into (A6) we have

$$\begin{aligned} & \text{diag}(\mathbf{P}^{(2)} + \mathbf{L}^{(1)}\mathbf{L}^{(1)'} + \mathbf{L}^{(2)}\Lambda' + \Lambda\mathbf{L}^{(2)'}) \\ &= \text{diag}[\Psi\Phi\mathbf{P}^{(2)}\Phi\Psi + \Psi\Phi(\mathbf{U} - \mathbf{P}^{(1)})\mathbf{B}(\mathbf{U} - \mathbf{P}^{(1)})\Phi\Psi \\ & \quad + \Psi\Phi(\mathbf{U} - \mathbf{P}^{(1)})\Phi\{\mathbf{U}\mathbf{B} - \mathbf{P}^{(1)}(\mathbf{B} + \Pi)\}\Psi \\ & \quad + \Psi\{\mathbf{U}\mathbf{B} - \mathbf{P}^{(1)}(\mathbf{B} + \Pi)\}'\Phi(\mathbf{U} - \mathbf{P}^{(1)})\Phi\Psi] \\ &= \text{diag}(\Psi\Phi\mathbf{P}^{(2)}\Phi\Psi - \Psi\mathbf{Q}\Psi), \end{aligned}$$

where \mathbf{Q} is defined in (3.8). Since Ψ is a diagonal matrix with positive diagonal elements and Ξ is assumed to be positive definite, $p_i^{(2)}$ can be written in the form

$$p_i^{(2)} = \sum_j \xi^{ij} q_{jj}. \quad (\text{A9})$$

It can be seen that $\tau(ab, cd)$ in (3.1) is 2 times the coefficient of $u_{ab}u_{cd}$ in $p_i^{(2)}$ and is given by

$$\begin{aligned} \tau_{\psi_i}(ab, cd) &= -2[\Phi\Theta_i\Phi]_{ad}\beta_{bc} - 2\sum_{e,f} [\Phi\Theta_i\Phi]_{ef}\beta_{ef}[\Phi\Theta_e\Phi]_{ab}[\Phi\Theta_f\Phi]_{cd} \\ & \quad + 4\sum_e [\Phi\Theta_i\Phi]_{ae}\beta_{be}[\Phi\Theta_e\Phi]_{cd} \end{aligned}$$

$$\begin{aligned}
& + 4 \sum_e [\mathbf{\Phi}\mathbf{\Theta}_i(\mathbf{B} + \mathbf{\Pi})]_{ae} \phi_{be} [\mathbf{\Phi}\mathbf{\Theta}_e\mathbf{\Phi}]_{cd} \\
& + 4 \sum_e [\mathbf{\Phi}\mathbf{\Theta}_e\mathbf{\Phi}]_{ab} \phi_{ce} [\mathbf{B}\mathbf{\Theta}_i\mathbf{\Phi}]_{de} - 4[\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{B}]_{ad} \phi_{bc} \\
& - 4 \sum_{e,f} \phi_{ef} [\mathbf{\Phi}\mathbf{\Theta}_i(\mathbf{B} + \mathbf{\Pi})]_{ef} [\mathbf{\Phi}\mathbf{\Theta}_e\mathbf{\Phi}]_{ab} [\mathbf{\Phi}\mathbf{\Theta}_f\mathbf{\Phi}]_{cd}. \tag{A10}
\end{aligned}$$

Next, we approximate $\hat{\Xi}^{-1} - \Xi^{-1}$ in terms of $\mathbf{P}^{(1)}$ and $\mathbf{L}^{(1)}$ and by using (A7) we have

$$\begin{aligned}
& (\hat{\Xi}^{-1} - \Xi^{-1}) \\
& = 2n^{1/2}\Xi^{-1}\{\mathbf{\Phi} \odot (\mathbf{\Phi}\mathbf{P}^{(1)}\mathbf{\Phi} - \mathbf{\Phi}\mathbf{P}^{(1)}\mathbf{B} - \mathbf{B}\mathbf{P}^{(1)}\mathbf{\Phi} + \mathbf{\Phi}\mathbf{U}\mathbf{B} + \mathbf{B}\mathbf{U}\mathbf{\Phi})\}\Xi^{-1} \\
& + O_p(n^{-1}).
\end{aligned}$$

The element of \mathbf{H} in (3.4) is the coefficient of u_{ab} in $2n^{1/2}(\hat{\xi}^{ii} - \xi^{ii})$ and is given by

$$\begin{aligned}
[\mathbf{H}_{\hat{\psi}_i}]_{ab} & = 4 \sum_c [\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{\Phi}]_{cc} [\mathbf{\Phi}\mathbf{\Theta}_c\mathbf{\Phi}]_{ab} - 8 \sum_c [\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{B}]_{cc} [\mathbf{\Phi}\mathbf{\Theta}_c\mathbf{\Phi}]_{ab} \\
& + 8[\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{\Phi}\mathbf{\Theta}_i\mathbf{B}]_{ab}. \tag{A11}
\end{aligned}$$

In the case of the maximum likelihood estimator, $\hat{\xi}_i$, of the uniqueness, we obtain from (3.10) and (3.11) the relations.

$$[\mathbf{A}_{\hat{\xi}_i}]_{ab} = \omega_{ii}^{-1}[\mathbf{A}_{\hat{\psi}_i}]_{ab} - \omega_{ii}^{-1}\zeta_i\delta_{ia}\delta_{ib}, \tag{A12}$$

$$\begin{aligned}
\tau_{\hat{\xi}_i}(ab, cd) & = \omega_{ii}^{-1}\tau_{\hat{\psi}_i}(ab, cd) - \omega_{ii}^{-2}\delta_{ia}\delta_{ib}[\mathbf{A}_{\hat{\psi}_i}]_{cd} \\
& + \omega_{ii}^{-2}\zeta_i\delta_{ia}\delta_{ib}\delta_{ic}\delta_{id}, \tag{A13}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{H}_{\hat{\xi}_i}]_{ab} & = \omega_{ii}^{-2}[\mathbf{H}_{\hat{\psi}_i}]_{ab} + 4\omega_{ii}^{-1}\zeta_i(1 - 3\zeta_i)[\mathbf{A}_{\hat{\psi}_i}]_{ab} \\
& - 4\omega_{ii}^{-1}\{\omega_{ii}^{-2}\xi^{ii} + \zeta_i^2(1 - 3\zeta_i)\}\delta_{ia}\delta_{ib}, \tag{A14}
\end{aligned}$$

where δ_{ij} is the Kronecker delta.

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