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On sample ranges in multiple-outlier models[☆]

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ABSTRACT

In this paper, we investigate the ordering properties of sample ranges arising from multiple-outlier models in terms of the reversed hazard rate order and the usual stochastic order. Under the setup of an exponential model, it is shown that the weak majorization order between the two hazard rate vectors is equivalent to the reversed hazard rate order between exponential sample ranges; the *p*-larger order between two hazard rate vectors implies the usual stochastic order between exponential sample ranges. Under the setup of a proportional hazard rate (PHR) model, we prove that the majorization order between two parameter vectors implies the usual stochastic order between sample ranges. The results established here strengthen and generalize some of the results known in the literature. Some numerical examples are provided to illustrate the theoretical results.

1. Introduction

Order statistics and those statistics having a close relation to order statistics play a prominent role in many areas of probability and statistics. In particular, spacings are of great interest in many areas such as goodness-of-fit tests, reliability theory, auction theory, actuarial science, life testing, operations research, information sciences, and many other areas. One may refer to [2,3] for some goodness-of-fit tests based on functions of sample spacings. Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the order statistics arising from random variables $X_1, X_2, \ldots, X_n$. Then, the k-th order statistic $X_{k:n}$ is just the lifetime of a $(n-k+1)$-out-of-n system, which is a very popular structure of redundancy in fault-tolerant systems that have been studied extensively. In particular, $X_{n:n}$ and $X_{1:n}$ correspond to the lifetimes of parallel and series systems, respectively.

Due to the nice mathematical form and the unique memoryless property, the exponential distribution has widely been used in many fields including reliability analysis. One may refer to [4,1] for an encyclopedic treatment to developments on the exponential distribution. There are a large number of papers in the literature on stochastic comparisons of exponential sample spacings; see, for example, [13] for a review on this topic. Recently, some researchers carried out stochastic comparisons of sample ranges wherein one sample is homogeneous while another sample is heterogenous. Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate λ. Then, Kochar and Rojo [8] showed, for $\lambda \ge \bar{\lambda} = \sum_{i=1}^{n} \lambda_i/n$, that

$$X_{n:n} - X_{1:n} \ge_{\text{st}} Y_{n:n} - Y_{1:n}, \tag{1}$$

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where $\geq_{\mathrm{st}}$ denotes the usual stochastic order and the formal definitions of various stochastic orders used in this paper will be given in Section 2. Zhao and Li [20] strengthened this result and presented the following equivalent characterization:

$$\lambda \geq \lambda^* \Longleftrightarrow X_{n:n} - X_{1:n} \geq_{\mathrm{st}} Y_{n:n} - Y_{1:n},$$

where

$$\lambda^* = \left(\frac{\prod_{i=1}^{n} \lambda_i}{\bar{\lambda}} \right)^{1/(n-1)}.$$

Kochar and Xu [9] improved the result in (1) from the usual stochastic order to the reversed hazard rate order as

$$X_{n:n} - X_{1:n} \geq_{\mathrm{rh}} Y_{n:n} - Y_{1:n}.$$

Recently, Genest et al. [7] proved, for $\lambda = \bar{\lambda}$, that

$$X_{n:n} - X_{1:n} \geq_{\mathrm{lr}} Y_{n:n} - Y_{1:n}$$

and

$$X_{n:n} - X_{1:n} \geq_{\mathrm{disp}} Y_{n:n} - Y_{1:n}.$$

Mao and Hu [15] further presented the following equivalent characterizations:

$$\lambda \geq \bar{\lambda} \Longleftrightarrow X_{n:n} - X_{1:n} \geq_{\mathrm{lr}} Y_{n:n} - Y_{1:n} \Longleftrightarrow X_{n:n} - X_{1:n} \geq_{\mathrm{rh}} Y_{n:n} - Y_{1:n}.$$

If $X_1, \ldots, X_n$ are independent random variables with X_i having survival function $\overline{F}^{\lambda_i}$, $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ is a random sample with common survival distribution $\overline{F}^{\bar{\lambda}}$, where $\bar{\lambda} = \sum_1^n \lambda_i / n$, [10] then proved that

$$X_{n:n} - X_{1:n} \geq_{\mathrm{st}} Y_{n:n} - Y_{1:n}.$$

In this paper, we will investigate the ordering properties between sample ranges arising from multiple-outlier exponential and proportional hazard rate (PHR) models. *It should be mentioned here that some researchers studied order statistics from the multiple-outlier exponential model; see, for example*, [12,19]. Let $X_1, \ldots, X_n$ follow the multiple-outlier exponential model with parameters

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}),$$

where $p + q = n$, and let $Y_1, \ldots, Y_n$ be another set of random variables following the multiple-outlier exponential model with parameters

$$(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}).$$

Under the condition $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, we prove that

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{w}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longleftrightarrow X_{n:n} - X_{1:n} \geq_{\mathrm{rh}} Y_{n:n} - Y_{1:n}$$

and

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{p}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longrightarrow X_{n:n} - X_{1:n} \geq_{\mathrm{st}} Y_{n:n} - Y_{1:n}.$$

As a matter of fact, the above results reveal a correspondence between the various stochastic orders between sample ranges arising from multiple-outlier exponential models and majorization type orders of the vectors of hazard rates.

Let $X_1, X_2, \ldots, X_n$ follow a PHR model with survival functions

$$(\underbrace{[\overline{F}(x)]^{\lambda_1}, \ldots, [\overline{F}(x)]^{\lambda_1}}_{p}, \underbrace{[\overline{F}(x)]^{\lambda_2}, \ldots, [\overline{F}(x)]^{\lambda_2}}_{q}),$$

where $p + q = n$. Let $Y_1, Y_2, \ldots, Y_n$ follow another PHR model with survival functions

$$(\underbrace{[\overline{F}(x)]^{\lambda_1^*}, \ldots, [\overline{F}(x)]^{\lambda_1^*}}_{p}, \underbrace{[\overline{F}(x)]^{\lambda_2^*}, \ldots, [\overline{F}(x)]^{\lambda_2^*}}_{q}).$$

Suppose $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, we prove that

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{m}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longrightarrow X_{n:n} - X_{1:n} \geq_{\mathrm{st}} Y_{n:n} - Y_{1:n}.$$

2. Definitions

In this section, we recall some notions of stochastic orders, and majorization and related orders. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*.

Stochastic orders

Definition 2.1. For two random variables X and Y with densities f_X and f_Y, and distribution functions F_X and F_Y, respectively, let $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ be the corresponding survival functions. Then:

(i) X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{\text{lr}} Y$) if $f_Y(x)/f_X(x)$ is increasing in x;
(ii) X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{\text{hr}} Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in x;
(iii) X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \leq_{\text{rh}} Y$) if $F_Y(x)/F_X(x)$ is increasing in x;
(iv) X is said to be smaller than Y in the stochastic order (denoted by $X \leq_{\text{st}} Y$) if $\overline{F}_Y(x) \geq \overline{F}_X(x)$;
(v) X is said to be smaller than Y in the mean residual life order (denoted by $X \leq_{\text{mrl}} Y$) if $\mathsf{E}X_t \leq \mathsf{E}Y_t$, where $X_t = (X - t | X > t)$ is the residual life at age $t > 0$ of the random lifetime X;

It is known that likelihood ratio order implies both usual stochastic order and hazard rate order, but neither usual stochastic order nor hazard rate order implies the other; see [17].

Majorization and related orders

The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 2.2. (i) A vector $\mathbf{x} = (x_1, \ldots, x_n) \in \Re^n$ is said to majorize another vector $\mathbf{y} = (y_1, \ldots, y_n) \in \Re^n$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$;

(ii) A vector $\mathbf{x} \in \Re^n$ is said to weakly majorize another vector $\mathbf{y} \in \Re^n$ (written as $\mathbf{x} \overset{w}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n;$$

(iii) A vector $\mathbf{x} \in \Re_+^n$ is said to be p-larger than another vector $\mathbf{y} \in \Re_+^n$ (written as $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Obviously, $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$ implies $\mathbf{x} \overset{w}{\succeq} \mathbf{y}$, and $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$ is equivalent to $\log(\mathbf{x}) \overset{w}{\succeq} \log(\mathbf{y})$, where $\log(\mathbf{x})$ is the vector of logarithms of the coordinates of $\mathbf{x}$. Note that $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$ implies $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \Re_+^n$. The converse is, however, not true. For example, $(1, 5.5) \overset{p}{\succeq} (2, 3)$, but clearly the majorization order does not hold between these two vectors.

For more details on majorization and p-larger orders and their applications, one may refer to [14,5]. Zhao and Balakrishnan [18] recently introduced a new partial order, called reciprocal majorization order.

Definition 2.3. The vector $\mathbf{x} \in \Re_+^n$ is said to reciprocal majorize another vector $\mathbf{y} \in \Re_+^n$ (written as $\mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \geq \sum_{i=1}^{j} \frac{1}{y_{(i)}}$$

for $j = 1, \ldots, n$.

It is known from [11] that

$$\mathbf{x} \overset{w}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$$

for any two non-negative vectors $\mathbf{x}$ and $\mathbf{y}$. On the other hand, the $\overset{\text{rm}}{\succeq}$ order does not imply the $\overset{p}{\succeq}$ order. For example, from the definition of the $\overset{\text{rm}}{\succeq}$ order, it follows that $(1, 4) \overset{\text{rm}}{\succeq} (4/3, 2)$, but clearly the $\overset{p}{\succeq}$ order does not hold between these two vectors.

3. Reversed hazard rate ordering

In this section, we carry out stochastic comparisons between sample ranges in terms of the reversed hazard rate order in exponential multiple-outlier models. The following several lemmas will be used to establish the main results.

Lemma 3.1. *The functions*

$$\frac{1-e^{-x}}{x}, \qquad \frac{xe^{-x}}{1-e^{-x}} \quad \text{and} \quad \frac{x^2e^{-x}}{(1-e^{-x})^2}$$

are all decreasing in $x \in \Re_+$.

Lemma 3.2. *The function*

$$\frac{e^{-x}(1-x-e^{-x})}{(1-e^{-x})^2}$$

is increasing in $x \in \Re_+$.

Proof. Denote

$$f(x) = \frac{e^{-x}(1-x-e^{-x})}{(1-e^{-x})^2}, \quad x \ge 0.$$

Taking the derivative of $f(x)$ with respect to x gives rise to

$$\begin{aligned} f'(x) &\overset{\text{sgn}}{=} \left[-e^{-x}(1-x-e^{-x}) + e^{-x}(-1+e^{-x})\right](1-e^{-x})^2 - 2e^{-2x}(1-e^{-x})(1-x-e^{-x}) \\ &= e^{-x}(x-2+2e^{-x})(1-e^{-x})^2 - 2e^{-2x}(1-e^{-x})(1-x-e^{-x}) \\ &\overset{\text{sgn}}{=} (x-2+2e^{-x})(1-e^{-x}) - 2e^{-x}(1-x-e^{-x}) \\ &= xe^{-x} + 2e^{-x} + x - 2 \\ &= g(x). \end{aligned}$$

Taking the derivative of $g(x)$ with respect to x, we have

$$g'(x) = -xe^{-x} - e^{-x} + 1,$$

and we then have

$$g''(x) = xe^{-x} \ge 0.$$

It is easy to see that, for any $x \in \Re_+$,

$$g''(x) \ge 0 \Longrightarrow g'(x) \ge g'(0) = 0 \Longrightarrow f'(x) \overset{\text{sgn}}{=} g(x) \ge g(0) = 0,$$

which implies that $f(x)$ is increasing in $x \in \Re_+$. □

Lemma 3.3 ([14])**.** *Let* $I \subset \Re$ *be an open interval and let* $\phi : I^n \to \Re$ *be continuously differentiable. Then,* ϕ *is Schur-convex [Schur-concave] on* I^n *if and only if* ϕ *is symmetric on* I^n *and for all* $i \ne j$,

$$(z_i - z_j)\left[\frac{\partial}{\partial z_i}\phi(\boldsymbol{z}) - \frac{\partial}{\partial z_j}\phi(\boldsymbol{z})\right] \ge [\le] 0 \quad \text{for all } \boldsymbol{z} \in I^n,$$

where $\frac{\partial}{\partial z_i}\phi(\boldsymbol{z})$ *denotes the partial derivative of* $\phi(\boldsymbol{z})$ *with respect to its i-th argument.*

We are now ready to present our main results.

Theorem 3.4. *Let* $X_1, X_2, \ldots, X_n$ *be independent exponential random variables such that* X_i *has failure rate* λ_1 *for* $i = 1, \ldots, p$ *and,* X_j *has failure rate* λ *for* $j = p+1, \ldots, n$*, where* $p \ge 1$ *and* $q = n - p \ge 1$*. Let* $Y_1, Y_2, \ldots, Y_n$ *be independent exponential random variables such that* Y_i *has failure rate* λ_2 *for* $i = 1, \ldots, p$ *and,* Y_j *has failure rate* λ *for* $j = p+1, \ldots, n$*. Suppose* $\lambda \ge \max(\lambda_1, \lambda_2)$*. Then, the sufficient and necessary condition for* $X_{n:n} - X_{1:n} \ge_{\text{rh}} Y_{n:n} - Y_{1:n}$ *is* $\lambda_1 \le \lambda_2$.

Proof. According to [6, p. 26], the distribution function of $R(X) = X_{n:n} - X_{1:n}$ can be written as

$$F_{R(X)}(t) = \left(\frac{p\lambda_1}{1-e^{-\lambda_1 t}} + \frac{q\lambda}{1-e^{-\lambda t}}\right) \cdot \frac{\left(1-e^{-\lambda_1 t}\right)^p \left(1-e^{-\lambda t}\right)^q}{p\lambda_1 + q\lambda}, \quad t \ge 0. \tag{2}$$

Actually, the formula in (2) can also be deduced from Theorem 4.1 in [16]. Thus, its reversed hazard rate function is given by

$$\begin{aligned}\tilde{r}_{R(X)}(t) &= \frac{d[\ln F_{R(X)}(t)]}{dt}\\ &= \frac{p\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1 t}} + \frac{q\lambda e^{-\lambda t}}{1-e^{-\lambda t}} - \frac{\frac{p\lambda_1^2 e^{-\lambda_1 t}}{(1-e^{-\lambda_1 t})^2} + \frac{q\lambda^2 e^{-\lambda t}}{(1-e^{-\lambda t})^2}}{\frac{p\lambda_1}{1-e^{-\lambda_1 t}} + \frac{q\lambda}{1-e^{-\lambda t}}}, \quad t \geq 0.\end{aligned}$$

Necessity. Suppose that $X_{n:n} - X_{1:n} \geq_{\rm rh} Y_{n:n} - Y_{1:n}$ and then it holds that $X_{n:n} - X_{1:n} \geq_{\rm st} Y_{n:n} - Y_{1:n}$. From (2), it follows that

$$\left(\frac{p\lambda_1}{1-e^{-\lambda_1 t}} + \frac{q\lambda}{1-e^{-\lambda t}}\right)\cdot\frac{\left(1-e^{-\lambda_1 t}\right)^p\left(1-e^{-\lambda t}\right)^q}{p\lambda_1+q\lambda} \leq \left(\frac{p\lambda_2}{1-e^{-\lambda_2 t}} + \frac{q\lambda}{1-e^{-\lambda t}}\right)\cdot\frac{\left(1-e^{-\lambda_2 t}\right)^p\left(1-e^{-\lambda t}\right)^q}{p\lambda_2+q\lambda},$$

which is equivalent to

$$\left(\frac{px_1}{1-e^{-x_1}} + \frac{qx}{1-e^{-x}}\right)\cdot\frac{\left(1-e^{-x_1}\right)^p}{px_1+qx} \leq \left(\frac{px_2}{1-e^{-x_2}} + \frac{qx}{1-e^{-x}}\right)\cdot\frac{\left(1-e^{-x_2}\right)^p}{px_2+qx},$$

where $x_1 \geq 0$, $x_2 \geq 0$ and $x \geq \max(x_1, x_2)$. Denote

$$F(x_1) = \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx}{1-e^{-x}}\right)\cdot\frac{\left(1-e^{-x_1}\right)^p}{px_1+qx}.$$

Taking the derivative with respect to x_1, we have

$$\begin{aligned}F'(x_1) &= \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx}{1-e^{-x}}\right)\cdot\frac{pe^{-x_1}(1-e^{-x_1})^{p-1}(px_1+qx) - p(1-e^{-x_1})^p}{(px_1+qx)^2}\\ &\quad + \frac{p(1-e^{-x_1}-x_1e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{(1-e^{-x_1})^p}{px_1+qx}\\ &\overset{\rm sgn}{=} \frac{1-e^{-x_1}-x_1e^{-x_1}}{1-e^{-x_1}} + \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx}{1-e^{-x}}\right)\cdot\frac{e^{-x_1}(px_1+qx)-(1-e^{-x_1})}{px_1+qx}\\ &= \frac{1-e^{-x_1}-x_1e^{-x_1}}{1-e^{-x_1}} + \frac{px_1e^{-x_1}}{1-e^{-x_1}} - \frac{px_1}{px_1+qx} + \frac{qxe^{-x_1}}{1-e^{-x}} - \frac{qx}{px_1+qx}\cdot\frac{1-e^{-x_1}}{1-e^{-x}}\\ &= \frac{qx}{px_1+qx} + \frac{(p-1)x_1e^{-x_1}}{1-e^{-x_1}} + \frac{qxe^{-x_1}}{1-e^{-x}} - \frac{qx}{px_1+qx}\cdot\frac{1-e^{-x_1}}{1-e^{-x}}\\ &= \frac{qx}{px_1+qx}\cdot\frac{e^{-x_1}-e^{-x}}{1-e^{-x}} + \frac{(p-1)x_1e^{-x_1}}{1-e^{-x_1}} + \frac{qxe^{-x_1}}{1-e^{-x}}.\end{aligned}$$

It can be readily seen that $F'(x_1) \geq 0$ due to $x \geq \max(x_1, x_2)$, and hence $F(x_1)$ is increasing in $x_1 \in [0, x]$. Thus, we have $x_1 \leq x_2$, or equivalently, $\lambda_1 \leq \lambda_2$.

Sufficiency. Denote by $\tilde{r}_{R(Y)}(t)$ the reversed hazard rate function of $Y_{n:n} - Y_{1:n}$. It suffices to show $\tilde{r}_{R(X)}(t) \geq \tilde{r}_{R(Y)}(t)$ for $\lambda \geq \lambda_2 \geq \lambda_1 \geq 0$, i.e.,

$$\frac{p\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1 t}} - \frac{\frac{p\lambda_1^2 e^{-\lambda_1 t}}{(1-e^{-\lambda_1 t})^2} + \frac{q\lambda^2 e^{-\lambda t}}{(1-e^{-\lambda t})^2}}{\frac{p\lambda_1}{1-e^{-\lambda_1 t}} + \frac{q\lambda}{1-e^{-\lambda t}}} \geq \frac{p\lambda_2 e^{-\lambda_2 t}}{1-e^{-\lambda_2 t}} - \frac{\frac{p\lambda_2^2 e^{-\lambda_2 t}}{(1-e^{-\lambda_2 t})^2} + \frac{q\lambda^2 e^{-\lambda t}}{(1-e^{-\lambda t})^2}}{\frac{p\lambda_2}{1-e^{-\lambda_2 t}} + \frac{q\lambda}{1-e^{-\lambda t}}},$$

which is actually equivalent to showing that the function

$$f(x_1) = \frac{px_1e^{-x_1}}{1-e^{-x_1}} - \frac{\frac{px_1^2e^{-x_1}}{(1-e^{-x_1})^2} + \frac{qx^2e^{-x}}{(1-e^{-x})^2}}{\frac{px_1}{1-e^{-x_1}} + \frac{qx}{1-e^{-x}}}$$

is decreasing in $x_1 \in [0, x]$ under the condition that $x \geq x_2 \geq x_1 \geq 0$. Taking the derivative of $f(x_1)$ with respect to x_1, we have

$$\begin{aligned}f'(x_1) &= \frac{pe^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2} + \frac{\frac{p(1-e^{-x_1}-x_1e^{-x_1})}{(1-e^{-x_1})^2}}{\left(\frac{px_1}{1-e^{-x_1}} + \frac{qx}{1-e^{-x}}\right)^2}\cdot\left[\frac{px_1^2e^{-x_1}}{(1-e^{-x_1})^2} + \frac{qx^2e^{-x}}{(1-e^{-x})^2}\right]\\ &\quad - \frac{p}{\frac{px_1}{1-e^{-x_1}} + \frac{qx}{1-e^{-x}}}\cdot\left[\frac{e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x_1}{1-e^{-x_1}} + \frac{x_1e^{-x_1}}{1-e^{-x_1}}\cdot\frac{1-e^{-x_1}-x_1e^{-x_1}}{(1-e^{-x_1})^2}\right].\end{aligned}$$

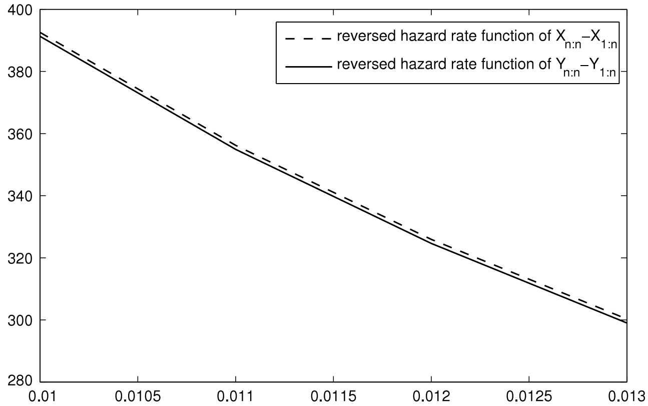

Fig. 1. Plot of $\tilde{r}_{R(X)}(t)$ and $\tilde{r}_{R(Y)}(t)$ when $p=2, q=3, \lambda_1=0.4, \lambda_2=2$ and $\lambda=6$.

After some simplification, it can be obtained that

$$\begin{aligned}
&\frac{f'(x_1)}{p}\cdot\left(\frac{px_1}{1-e^{-x_1}}+\frac{qx}{1-e^{-x}}\right)^2\\
&=\frac{e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\left(\frac{px_1}{1-e^{-x_1}}+\frac{qx}{1-e^{-x}}\right)^2-\frac{pe^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x_1^2}{(1-e^{-x_1})^2}\\
&\quad-\frac{qe^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x_1x}{(1-e^{-x_1})(1-e^{-x})}+\frac{qe^{-x}(1-e^{-x_1}-x_1e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x^2}{(1-e^{-x})^2}\\
&\quad-\frac{qe^{-x_1}(1-e^{-x_1}-x_1e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x_1x}{(1-e^{-x_1})(1-e^{-x})}\\
&=\frac{p(p-1)e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x_1^2}{(1-e^{-x_1})^2}+\frac{q^2e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x^2}{(1-e^{-x})^2}\\
&\quad+\frac{q(2p-1)e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x_1x}{(1-e^{-x_1})(1-e^{-x})}+\frac{qx(1-e^{-x_1}-x_1e^{-x_1})}{(1-e^{-x_1})^2(1-e^{-x})}\cdot\left(\frac{xe^{-x}}{1-e^{-x}}-\frac{x_1e^{-x_1}}{1-e^{-x_1}}\right)\\
&=\alpha+\beta+\gamma+\delta,\quad\text{say,}
\end{aligned}$$

where

$$\begin{aligned}
\alpha&=\frac{p(p-1)e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x_1^2}{(1-e^{-x_1})^2},\\
\beta&=\frac{q^2e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x^2}{(1-e^{-x})^2},\\
\gamma&=\frac{q(2p-1)e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}\cdot\frac{x_1x}{(1-e^{-x_1})(1-e^{-x})},\\
\delta&=\frac{qx(1-e^{-x_1}-x_1e^{-x_1})}{(1-e^{-x_1})^2(1-e^{-x})}\cdot\left(\frac{xe^{-x}}{1-e^{-x}}-\frac{x_1e^{-x_1}}{1-e^{-x_1}}\right).
\end{aligned}$$

Since $1-x_1-e^{-x_1}\le 0$, $1-e^{-x_1}-x_1e^{-x_1}\ge 0$, and from Lemma 3.1, it can be seen that

$$\alpha\le 0,\qquad \beta\le 0,\qquad \gamma\le 0,\qquad \delta\le 0\implies f'(x_1)\le 0,$$

thus, $f(x_1)\ge f(x_2)$, which implies $\tilde{r}_{R(X)}(t)\ge\tilde{r}_{R(Y)}(t)$, i.e., $X_{n:n}-X_{1:n}\ge_{\mathrm{rh}}Y_{n:n}-Y_{1:n}$. □

Example 3.5. Set $p=2$, $q=3$, $\lambda_1=0.4$, $\lambda_2=2$, $\lambda=6$ in Theorem 3.4, we then have $\lambda\ge\lambda_2\ge\lambda_1\ge 0$. Fig. 1 plots the reversed hazard rate functions of sample range in multiple-outlier exponential models. It can be seen that the reversed hazard rate of $X_{n:n}-X_{1:n}$ is larger than that of $Y_{n:n}-Y_{1:n}$ for $t\in\Re_+$, which shows the validity of the result in Theorem 3.4. □

Theorem 3.6. *Let $X_1,X_2,\ldots,X_n$ be independent exponential random variables such that X_i has failure rate λ_1 for $i=1,\ldots,p$ and, X_j has failure rate λ_2 for $j=p+1,\ldots,n$, where $p\ge 1$ and $q=n-p\ge 1$. Let $Y_1,Y_2,\ldots,Y_n$ be independent exponential* *random variables such that Y_i has failure rate λ_1^* for $i = 1, \ldots, p$ and, Y_j has failure rate λ_2^* for $j = p+1, \ldots, n$. If*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{m}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}),$$

then,

$$X_{n:n} - X_{1:n} \geq_{\text{rh}} Y_{n:n} - Y_{1:n}.$$

Proof. Without loss of generality, we assume that $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$. We need to show

$$\tilde{r}_{R(X)}(t) \ge \tilde{r}_{R(Y)}(t),$$

i.e.,

$$\frac{p\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1 t}} + \frac{q\lambda_2 e^{-\lambda_2 t}}{1-e^{-\lambda_2 t}} - \frac{\frac{p\lambda_1^2 e^{-\lambda_1 t}}{(1-e^{-\lambda_1 t})^2} + \frac{q\lambda_2^2 e^{-\lambda_2 t}}{(1-e^{-\lambda_2 t})^2}}{\frac{p\lambda_1}{1-e^{-\lambda_1 t}} + \frac{q\lambda_2}{1-e^{-\lambda_2 t}}} \ge \frac{p\lambda_1^* e^{-\lambda_1^* t}}{1-e^{-\lambda_1^* t}} + \frac{q\lambda_2^* e^{-\lambda_2^* t}}{1-e^{-\lambda_2^* t}} - \frac{\frac{p{\lambda_1^*}^2 e^{-\lambda_1^* t}}{(1-e^{-\lambda_1^* t})^2} + \frac{q{\lambda_2^*}^2 e^{-\lambda_2^* t}}{(1-e^{-\lambda_2^* t})^2}}{\frac{p\lambda_1^*}{1-e^{-\lambda_1^* t}} + \frac{q\lambda_2^*}{1-e^{-\lambda_2^* t}}}.$$

Denote

$$\varphi(\underbrace{x_1, \ldots, x_1}_{p}, \underbrace{x_2, \ldots, x_2}_{q}) = \frac{px_1 e^{-x_1}}{1-e^{-x_1}} + \frac{qx_2 e^{-x_2}}{1-e^{-x_2}} - \frac{\frac{px_1^2 e^{-x_1}}{(1-e^{-x_1})^2} + \frac{qx_2^2 e^{-x_2}}{(1-e^{-x_2})^2}}{\frac{px_1}{1-e^{-x_1}} + \frac{qx_2}{1-e^{-x_2}}}.$$

It is then sufficient to prove

$$\varphi(\underbrace{x_1, \ldots, x_1}_{p}, \underbrace{x_2, \ldots, x_2}_{q}) \ge \varphi(\underbrace{x_1^*, \ldots, x_1^*}_{p}, \underbrace{x_2^*, \ldots, x_2^*}_{q})$$

under the condition $x_1 \le x_1^* \le x_2^* \le x_2$ and

$$(\underbrace{x_1, \ldots, x_1}_{p}, \underbrace{x_2, \ldots, x_2}_{q}) \overset{m}{\succeq} (\underbrace{x_1^*, \ldots, x_1^*}_{p}, \underbrace{x_2^*, \ldots, x_2^*}_{q}).$$

In other words, we need to show $\varphi(\underbrace{x_1, \ldots, x_1}_{p}, \underbrace{x_2, \ldots, x_2}_{q})$ is Schur-convex in $(\underbrace{x_1, \ldots, x_1}_{p}, \underbrace{x_2, \ldots, x_2}_{q})$. Note that

$$\begin{aligned}
\frac{\partial\varphi}{\partial x_1} \cdot \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx_2}{1-e^{-x_2}}\right)^2 &= \frac{e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2} \cdot \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx_2}{1-e^{-x_2}}\right)^2 - \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx_2}{1-e^{-x_2}}\right) \\
&\quad \cdot \left[\frac{e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2} \cdot \frac{x_1}{1-e^{-x_1}} + \frac{x_1 e^{-x_1}}{1-e^{-x_1}} \cdot \frac{1-e^{-x_1}-x_1 e^{-x_1}}{(1-e^{-x_1})^2}\right] \\
&\quad + \frac{1-e^{-x_1}-x_1 e^{-x_1}}{(1-e^{-x_1})^2} \cdot \left(\frac{px_1 e^{-x_1}}{1-e^{-x_1}} \cdot \frac{x_1}{1-e^{-x_1}} + \frac{qx_2 e^{-x_2}}{1-e^{-x_2}} \cdot \frac{x_2}{1-e^{-x_2}}\right) \\
&= \frac{e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2} \cdot \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx_2}{1-e^{-x_2}}\right)^2 \\
&\quad - \frac{px_1^2 e^{-x_1}}{(1-e^{-x_1})^2} \cdot \frac{1-x_1-e^{-x_1}}{(1-e^{-x_1})^2} \\
&\quad + \frac{qx_2}{1-e^{-x_2}} \cdot \left[\frac{1-e^{-x_1}-x_1 e^{-x_1}}{(1-e^{-x_1})^2} \cdot \frac{x_2 e^{-x_2}}{1-e^{-x_2}} - \frac{x_1 e^{-x_1}}{1-e^{-x_1}} \cdot \frac{1-x_1-e^{-x_1}}{(1-e^{-x_1})^2}\right] \\
&\quad - \frac{qx_2}{1-e^{-x_2}} \cdot \left[\frac{1-e^{-x_1}-x_1 e^{-x_1}}{(1-e^{-x_1})^2} \cdot \frac{x_1 e^{-x_1}}{1-e^{-x_1}}\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial\varphi}{\partial x_2} \cdot \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx_2}{1-e^{-x_2}}\right)^2 &= \frac{e^{-x_2}(1-x_2-e^{-x_2})}{(1-e^{-x_2})^2} \cdot \left(\frac{px_1}{1-e^{-x_1}} + \frac{qx_2}{1-e^{-x_2}}\right)^2 \\
&\quad - \frac{qx_2^2 e^{-x_2}}{(1-e^{-x_2})^2} \cdot \frac{1-x_2-e^{-x_2}}{(1-e^{-x_2})^2}
\end{aligned}$$

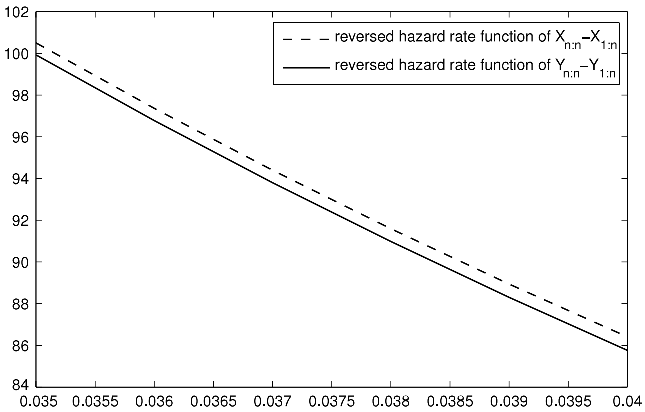

Fig. 2. Plot of $\tilde{r}_{R(X)}(t)$ and $\tilde{r}_{R(Y)}(t)$ when $p=2$, $q=3$, $\lambda_1=0.01$, $\lambda_2=12.66$, $\lambda_1^*=4$ and $\lambda_2^*=10$.

$$+\frac{px_1}{1-e^{-x_1}}\cdot\left[\frac{1-e^{-x_2}-x_2e^{-x_2}}{(1-e^{-x_2})^2}\cdot\frac{x_1e^{-x_1}}{1-e^{-x_1}}-\frac{x_2e^{-x_2}}{1-e^{-x_2}}\cdot\frac{1-x_2-e^{-x_2}}{(1-e^{-x_2})^2}\right]$$
$$-\frac{px_1}{1-e^{-x_1}}\cdot\left[\frac{1-e^{-x_2}-x_2e^{-x_2}}{(1-e^{-x_2})^2}\cdot\frac{x_2e^{-x_2}}{1-e^{-x_2}}\right].$$

Thus, we have

$$\begin{aligned}\frac{\partial\varphi}{\partial x_1}-\frac{\partial\varphi}{\partial x_2}&\overset{\text{sgn}}{=}\left[\frac{p(p-1)x_1^2}{(1-e^{-x_1})^2}+\frac{2pqx_1x_2}{(1-e^{-x_1})(1-e^{-x_2})}+\frac{q(q-1)x_2^2}{(1-e^{-x_2})^2}\right]\\&\quad\times\left[\frac{e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}-\frac{e^{-x_2}(1-x_2-e^{-x_2})}{(1-e^{-x_2})^2}\right]\\&\quad+\left[\frac{px_1e^{-x_2}(1-x_2-e^{-x_2})}{(1-e^{-x_1})(1-e^{-x_2})^2}+\frac{qx_2e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2(1-e^{-x_2})}\right]\cdot\left(\frac{x_2}{1-e^{-x_2}}-\frac{x_1}{1-e^{-x_1}}\right)\\&\quad+\left[\frac{px_1(1-e^{-x_2}-x_2e^{-x_2})}{(1-e^{-x_1})(1-e^{-x_2})^2}+\frac{qx_2(1-e^{-x_1}-x_1e^{-x_1})}{(1-e^{-x_1})^2(1-e^{-x_2})}\right]\cdot\left(\frac{x_2e^{-x_2}}{1-e^{-x_2}}-\frac{x_1e^{-x_1}}{1-e^{-x_1}}\right)\\&=\alpha+\beta+\gamma,\quad\text{say,}\end{aligned}$$

where

$$\alpha=\left[\frac{p(p-1)x_1^2}{(1-e^{-x_1})^2}+\frac{2pqx_1x_2}{(1-e^{-x_1})(1-e^{-x_2})}+\frac{q(q-1)x_2^2}{(1-e^{-x_2})^2}\right]\times\left[\frac{e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2}-\frac{e^{-x_2}(1-x_2-e^{-x_2})}{(1-e^{-x_2})^2}\right],$$
$$\beta=\left[\frac{px_1e^{-x_2}(1-x_2-e^{-x_2})}{(1-e^{-x_1})(1-e^{-x_2})^2}+\frac{qx_2e^{-x_1}(1-x_1-e^{-x_1})}{(1-e^{-x_1})^2(1-e^{-x_2})}\right]\cdot\left(\frac{x_2}{1-e^{-x_2}}-\frac{x_1}{1-e^{-x_1}}\right),$$
$$\gamma=\left[\frac{px_1(1-e^{-x_2}-x_2e^{-x_2})}{(1-e^{-x_1})(1-e^{-x_2})^2}+\frac{qx_2(1-e^{-x_1}-x_1e^{-x_1})}{(1-e^{-x_1})^2(1-e^{-x_2})}\right]\cdot\left(\frac{x_2e^{-x_2}}{1-e^{-x_2}}-\frac{x_1e^{-x_1}}{1-e^{-x_1}}\right).$$

Notice that $1-x-e^{-x}\le 0$ and $1-e^{-x}-xe^{-x}\ge 0$ for all $x\in\Re_+$. Using this fact and Lemma 3.1, we have $\beta\le 0$ and $\gamma\le 0$ for $x_2\ge x_1$. On the other hand, it can be checked that $\alpha\le 0$ for $x_2\ge x_1$ from Lemma 3.2. Therefore, it holds that

$$(x_1-x_2)\left(\frac{\partial\varphi}{\partial x_1}-\frac{\partial\varphi}{\partial x_2}\right)\ge 0.$$

Now the desired result follows by using Lemma 3.3 and the entire proof is completed. □

Example 3.7. Set $p=2$, $q=3$, $\lambda_1=0.01$, $\lambda_2=12.66$, $\lambda_1^*=4$, $\lambda_2^*=10$ in Theorem 3.6, we then have $(0.01, 0.01, 12.66, 12.66, 12.66)\overset{m}{\succeq}(4, 4, 10, 10, 10)$. Fig. 2 plots the reversed hazard rate functions $\tilde{r}_{R(X)}(t)$ and $\tilde{r}_{R(Y)}(t)$. Observe that the reversed hazard rate of $X_{n:n}-X_{1:n}$ is larger than that of $Y_{n:n}-Y_{1:n}$ for $t\in\Re_+$, which is in accordance with the theoretical result in Theorem 3.6. □

The following result extends the result of Theorem 3.6.

Theorem 3.8. *Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables such that X_i has failure rate λ_1 for $i=1,\ldots,p$ and, X_j has failure rate λ_2 for $j=p+1,\ldots,n$, where $p\ge 1$ and $q=n-p\ge 1$. Let $Y_1, Y_2, \ldots, Y_n$ be independent exponential* *random variables such that Y_i has failure rate λ_1^* for $i = 1, \ldots, p$ and, Y_j has failure rate λ_2^* for $j = p+1, \ldots, n$. Suppose $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$, Then the following two statements are equivalent:*

(i) $(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{w}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q})$;

(ii) $X_{n:n} - X_{1:n} \ge_{\mathrm{rh}} Y_{n:n} - Y_{1:n}$.

Proof. (ii)⇒(i). Assume that $\tilde{r}_{R(X)}(t) \ge \tilde{r}_{R(Y)}(t)$. Upon using Taylor's expansion at the origin, we have, for $t \in \Re_+$,

$$\begin{aligned}
\tilde{r}_{R(X)}(t) &= \frac{p\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1 t}} + \frac{q\lambda_2 e^{-\lambda_2 t}}{1-e^{-\lambda_2 t}} - \frac{\frac{p\lambda_1^2 e^{-\lambda_1 t}}{(1-e^{-\lambda_1 t})^2} + \frac{q\lambda_2^2 e^{-\lambda_2 t}}{(1-e^{-\lambda_2 t})^2}}{\frac{p\lambda_1}{1-e^{-\lambda_1 t}} + \frac{q\lambda_2}{1-e^{-\lambda_2 t}}} \\
&= \frac{p\lambda_1\left(1-\lambda_1 t + O(t^2)\right)}{1-\left(1-\lambda_1 t + O(t^2)\right)} + \frac{q\lambda_2\left(1-\lambda_2 t + O(t^2)\right)}{1-\left(1-\lambda_2 t + O(t^2)\right)} - \frac{\frac{p\lambda_1^2(1-\lambda_1 t + O(t^2))}{\left(1-(1-\lambda_1 t+O(t^2))\right)^2} + \frac{q\lambda_2^2(1-\lambda_2 t + O(t^2))}{\left(1-(1-\lambda_2 t+O(t^2))\right)^2}}{\frac{p\lambda_1}{1-(1-\lambda_1 t+O(t^2))} + \frac{q\lambda_2}{1-(1-\lambda_2 t+O(t^2))}} \\
&= \frac{n}{t} - (p\lambda_1 + q\lambda_2) - \frac{\frac{n}{t^2} - \frac{p\lambda_1 + q\lambda_2}{t}}{\frac{n}{t}} + o(1) \\
&= \frac{n-1}{t} - \frac{n-1}{n} \cdot (p\lambda_1 + q\lambda_2) + o(1).
\end{aligned}$$

Similarly,

$$\tilde{r}_{R(Y)}(t) = \frac{n-1}{t} - \frac{n-1}{n} \cdot (p\lambda_1^* + q\lambda_2^*) + o(1).$$

Thus, we have

$$\tilde{r}_{R(X)}(t) \ge \tilde{r}_{R(Y)}(t) \Longrightarrow p\lambda_1 + q\lambda_2 \le p\lambda_1^* + q\lambda_2^*.$$

(i)⇒(ii). Assume that $p\lambda_1 + q\lambda_2 \le p\lambda_1^* + q\lambda_2^*$. The result follows from Theorem 3.6 for the case when $p\lambda_1 + q\lambda_2 = p\lambda_1^* + q\lambda_2^*$. In what follows we only need to consider the case when $p\lambda_1 + q\lambda_2 < p\lambda_1^* + q\lambda_2^*$. In this case, there exists some λ satisfying $\lambda_1 < \lambda \le \lambda_1^*$ and $p\lambda + q\lambda_2 = p\lambda_1^* + q\lambda_2^*$. Let $Z_{n:n} - Z_{1:n}$ denote the sample range from the independent exponential variables $Z_1, Z_2, \ldots, Z_n$ with hazard rates $(\underbrace{\lambda, \ldots, \lambda}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q})$. Apparently, $(\underbrace{\lambda, \ldots, \lambda}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{m}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q})$. Upon using Theorem 3.6, it holds that

$$Z_{n:n} - Z_{1:n} \ge_{\mathrm{rh}} Y_{n:n} - Y_{1:n}. \tag{3}$$

On the other hand, we have

$$X_{n:n} - X_{1:n} \ge_{\mathrm{rh}} Z_{n:n} - Z_{1:n} \tag{4}$$

for $\lambda_1 < \lambda \le \lambda_1^* \le \lambda_2$ from Theorem 3.4. Then, the desired result is obtained from (3) and (4). □

Example 3.9. Set $p = 2$, $q = 3$, $\lambda_1 = 0.01$, $\lambda_2 = 6$, $\lambda_1^* = 4$, $\lambda_2^* = 10$ in Theorem 3.8, we have $(0.01, 0.01, 6, 6, 6) \overset{w}{\succeq} (4, 4, 10, 10, 10)$. Fig. 3 plots the reversed hazard rate functions $\tilde{r}_{R(X)}(t)$ and $\tilde{r}_{R(Y)}(t)$. It can be observed that the reversed hazard rate function $X_{n:n} - X_{1:n}$ is larger than that of $Y_{n:n} - Y_{1:n}$ for $t \in \Re_+$, as stated of the result in Theorem 3.8.

4. Usual stochastic ordering

In this section, we establish some comparison results in terms of the usual stochastic order in multiple-outlier exponential and PHR models.

Theorem 4.1. *Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables such that X_i has failure rate λ_1 for $i = 1, \ldots, p$ and, X_j has failure rate λ_2 for $j = p+1, \ldots, n$, where $p \ge 1$ and $q = n - p \ge 1$. Let $Y_1, Y_2, \ldots, Y_n$ be independent exponential random variables such that Y_i has failure rate λ_1^* for $i = 1, \ldots, p$ and, Y_j has failure rate λ_2^* for $j = p+1, \ldots, n$. Suppose that $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$. If*

$$\lambda_1^p \lambda_2^q = (\lambda_1^*)^p (\lambda_2^*)^q,$$

then,

$$X_{n:n} - X_{1:n} \ge_{\mathrm{st}} Y_{n:n} - Y_{1:n}.$$

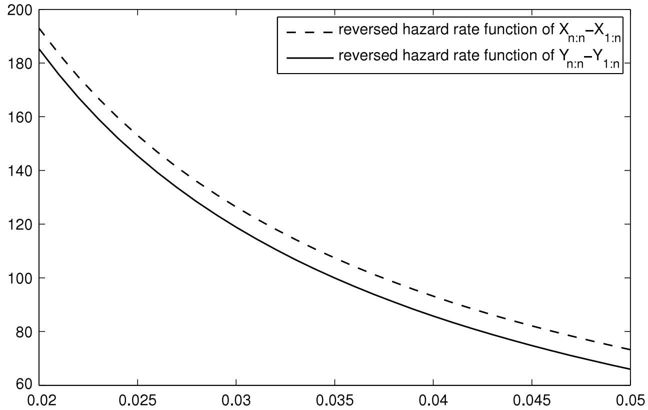

Fig. 3. Plot of $\tilde{r}_{R(X)}(t)$ and $\tilde{r}_{R(Y)}(t)$ when $p = 2, q = 3, \lambda_1 = 0.01, \lambda_2 = 6, \lambda_1^* = 4$ and $\lambda_2^* = 10$.

Proof. We need to show

$$F_{R(X)}(t) \le F_{R(Y)}(t),$$

i.e.,

$$\left[\frac{p\lambda_1}{1-e^{-\lambda_1 t}} + \frac{q\lambda_2}{1-e^{-\lambda_2 t}}\right] \frac{\left(1-e^{-\lambda_1 t}\right)^p \left(1-e^{-\lambda_2 t}\right)^q}{p\lambda_1 + q\lambda_2} \le \left[\frac{p\lambda_1^*}{1-e^{-\lambda_1^* t}} + \frac{q\lambda_2^*}{1-e^{-\lambda_2^* t}}\right] \frac{\left(1-e^{-\lambda_1^* t}\right)^p \left(1-e^{-\lambda_2^* t}\right)^q}{p\lambda_1^* + q\lambda_2^*},$$

under the condition $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$ and $\lambda_1^p \lambda_2^q = (\lambda_1^*)^p (\lambda_2^*)^q$, which is actually equivalent to showing that

$$\left(\frac{px_1}{1-e^{-x_1}} + \frac{qx_2}{1-e^{-x_2}}\right) \cdot \frac{(1-e^{-x_1})^p (1-e^{-x_2})^q}{px_1 + qx_2} \le \left(\frac{px_1^*}{1-e^{-x_1^*}} + \frac{qx_2^*}{1-e^{-x_2^*}}\right) \cdot \frac{(1-e^{-x_1^*})^p (1-e^{-x_2^*})^q}{px_1^* + qx_2^*}$$

under the conditions $x_1 \le x_1^* \le x_2^* \le x_2$ and $x_1^p x_2^q = (x_1^*)^p (x_2^*)^q$. Denote $y_1 = \log x_1, y_2 = \log x_2, y_1^* = \log x_1^*$ and $y_2^* = \log x_2^*$. We then have the following relation:

$$(\underbrace{y_1, \ldots, y_1}_{p}, \underbrace{y_2, \ldots, y_2}_{q}) \overset{m}{\succeq} (\underbrace{y_1^*, \ldots, y_1^*}_{p}, \underbrace{y_2^*, \ldots, y_2^*}_{q}).$$

Up to now it suffices to show that the symmetrical differentiable function $\Phi\colon (-\infty, \infty)^n \to (0, \infty)$ given by

$$\Phi(\underbrace{y_1, \ldots, y_1}_{p}, \underbrace{y_2, \ldots, y_2}_{q}) = \left(\frac{pe^{y_1}}{1-e^{-e^{y_1}}} + \frac{qe^{y_2}}{1-e^{-e^{y_2}}}\right) \cdot \frac{(1-e^{-e^{y_1}})^p (1-e^{-e^{y_2}})^q}{pe^{y_1} + qe^{y_2}}$$

is Schur-concave. Taking the partial derivative of $\Phi(\underbrace{y_1, \ldots, y_1}_{p}, \underbrace{y_2, \ldots, y_2}_{q})$ with respect to y_1, we have

$$\begin{aligned}
\frac{\partial \Phi}{\partial y_1} &= (p-1) \cdot (1-e^{-e^{y_1}})^{p-1} \cdot (1-e^{-e^{y_2}})^q \cdot \frac{e^{2y_1} \cdot e^{-e^{y_1}}}{(1-e^{-e^{y_1}})(pe^{y_1} + qe^{y_2})} \\
&\quad + q \cdot (1-e^{-e^{y_1}})^{p-1} \cdot (1-e^{-e^{y_2}})^q \cdot \left[\frac{e^{y_1} e^{y_2}}{(pe^{y_1} + qe^{y_2})^2} + \frac{e^{y_2}}{1-e^{-e^{y_2}}} \cdot \frac{e^{y_1} \cdot e^{-e^{y_1}}}{pe^{y_1} + qe^{y_2}}\right] \\
&\quad - q \cdot (1-e^{-e^{y_1}})^{p-1} \cdot (1-e^{-e^{y_2}})^q \cdot \left[\frac{e^{y_1} e^{y_2}}{1-e^{-e^{y_2}}} \cdot \frac{1-e^{-e^{y_1}}}{(pe^{y_1} + qe^{y_2})^2}\right] \\
&\overset{\text{sgn}}{=} q(1-e^{-e^{y_2}}) \cdot \left[\frac{e^{y_1} e^{y_2}}{(pe^{y_1} + qe^{y_2})^2} + \frac{e^{y_2}}{1-e^{-e^{y_2}}} \cdot \frac{e^{y_1} e^{-e^{y_1}}}{pe^{y_1} + qe^{y_2}} - \frac{e^{y_1} \cdot e^{y_2}}{1-e^{-e^{y_2}}} \cdot \frac{1-e^{-e^{y_1}}}{(pe^{y_1} + qe^{y_2})^2}\right] \\
&\quad + (p-1) \cdot \frac{e^{2y_1} \cdot e^{-e^{y_1}} \cdot (1-e^{-e^{y_2}})}{(1-e^{-e^{y_1}}) \cdot (pe^{y_1} + qe^{y_2})}.
\end{aligned}$$

Similarly,

$$\frac{\partial \Phi}{\partial y_2} \overset{\text{sgn}}{=} p(1-e^{-e^{y_1}}) \cdot \left[\frac{e^{y_1}e^{y_2}}{(pe^{y_1}+qe^{y_2})^2} + \frac{e^{y_1}}{1-e^{-e^{y_1}}} \cdot \frac{e^{y_2}e^{-e^{y_2}}}{pe^{y_1}+qe^{y_2}} - \frac{e^{y_1} \cdot e^{y_2}}{1-e^{-e^{y_1}}} \cdot \frac{1-e^{-e^{y_2}}}{(pe^{y_1}+qe^{y_2})^2} \right] + (q-1) \cdot \frac{e^{2y_2} \cdot e^{-e^{y_2}} \cdot (1-e^{-e^{y_1}})}{(1-e^{-e^{y_2}}) \cdot (pe^{y_1}+qe^{y_2})}.$$

Observe that

$$\begin{aligned}
\frac{\partial \Phi}{\partial y_1} - \frac{\partial \Phi}{\partial y_2} \overset{\text{sgn}}{=}& (p-1) \cdot \left[\frac{e^{2y_1} \cdot e^{-e^{y_1}} \cdot (1-e^{-e^{y_2}})}{(1-e^{-e^{y_1}}) \cdot (pe^{y_1}+qe^{y_2})} - \frac{e^{y_1} \cdot e^{y_2} \cdot e^{-e^{y_2}}}{pe^{y_1}+qe^{y_2}} + \frac{e^{y_1} \cdot e^{y_2} \cdot (e^{-e^{y_1}}-e^{-e^{y_2}})}{(pe^{y_1}+qe^{y_2})^2} \right] \\
&+ q \cdot \left[\frac{e^{y_1} \cdot e^{y_2} \cdot (e^{-e^{y_1}}-e^{-e^{y_2}})}{(pe^{y_1}+qe^{y_2})^2} + \frac{e^{y_1} \cdot e^{y_2} \cdot e^{-e^{y_1}}}{pe^{y_1}+qe^{y_2}} - \frac{e^{2y_2} \cdot e^{-e^{y_2}} \cdot (1-e^{-e^{y_1}})}{(1-e^{-e^{y_2}}) \cdot (pe^{y_1}+qe^{y_2})} \right] \\
&- \frac{e^{y_1} \cdot e^{y_2} \cdot (e^{-e^{y_2}}-e^{-e^{y_1}})}{(pe^{y_1}+qe^{y_2})^2} - \frac{e^{y_1} \cdot e^{y_2} \cdot e^{-e^{y_2}}}{pe^{y_1}+qe^{y_2}} + \frac{e^{2y_2} \cdot e^{-e^{y_2}} \cdot (1-e^{-e^{y_1}})}{(1-e^{-e^{y_2}}) \cdot (pe^{y_1}+qe^{y_2})} \\
=& \frac{(p-1)e^{y_1} \cdot e^{y_2}(e^{-e^{y_1}}-e^{-e^{y_2}})}{(pe^{y_1}+qe^{y_2})^2} + \frac{(p-1)e^{y_1}}{(1-e^{-e^{y_2}})(pe^{y_1}+qe^{y_2})} \cdot \left(\frac{e^{y_1} \cdot e^{-e^{y_1}}}{1-e^{-e^{y_1}}} - \frac{e^{y_2} \cdot e^{-e^{y_2}}}{1-e^{-e^{y_2}}} \right) \\
&+ \frac{(q+1)e^{y_1} \cdot e^{y_2}(e^{-e^{y_1}}-e^{-e^{y_2}})}{(pe^{y_1}+qe^{y_2})^2} + \frac{e^{y_1} \cdot e^{y_2}(e^{-e^{y_1}}-e^{-e^{y_2}})}{pe^{y_1}+qe^{y_2}} \\
&+ \frac{(q-1)e^{y_2}}{(1-e^{-e^{y_1}})(pe^{y_1}+qe^{y_2})} \cdot \left(\frac{e^{y_1} \cdot e^{-e^{y_1}}}{1-e^{-e^{y_1}}} - \frac{e^{y_2} \cdot e^{-e^{y_2}}}{1-e^{-e^{y_2}}} \right) \\
=& \alpha + \beta + \gamma + \delta + \xi, \quad \text{say,}
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= \frac{(p-1)e^{y_1} \cdot e^{y_2}(e^{-e^{y_1}}-e^{-e^{y_2}})}{(pe^{y_1}+qe^{y_2})^2}, \\
\beta &= \frac{(p-1)e^{y_1}}{(1-e^{-e^{y_2}})(pe^{y_1}+qe^{y_2})} \cdot \left(\frac{e^{y_1} \cdot e^{-e^{y_1}}}{1-e^{-e^{y_1}}} - \frac{e^{y_2} \cdot e^{-e^{y_2}}}{1-e^{-e^{y_2}}} \right), \\
\gamma &= \frac{(q+1)e^{y_1} \cdot e^{y_2}(e^{-e^{y_1}}-e^{-e^{y_2}})}{(pe^{y_1}+qe^{y_2})^2}, \\
\delta &= \frac{e^{y_1} \cdot e^{y_2}(e^{-e^{y_1}}-e^{-e^{y_2}})}{pe^{y_1}+qe^{y_2}}
\end{aligned}$$

and

$$\xi = \frac{(q-1)e^{y_2}}{(1-e^{-e^{y_1}})(pe^{y_1}+qe^{y_2})} \cdot \left(\frac{e^{y_1} \cdot e^{-e^{y_1}}}{1-e^{-e^{y_1}}} - \frac{e^{y_2} \cdot e^{-e^{y_2}}}{1-e^{-e^{y_2}}} \right).$$

Since $e^{-e^{y_1}} - e^{-e^{y_2}} \geq 0$ for $y_1 \leq y_2$, using this and Lemma 3.1, we have

$$\begin{aligned}
\alpha &\overset{\text{sgn}}{=} e^{-e^{y_1}} - e^{-e^{y_2}} \overset{\text{sgn}}{=} y_2 - y_1, \\
\beta &\overset{\text{sgn}}{=} \frac{e^{y_1} \cdot e^{-e^{y_1}}}{1-e^{-e^{y_1}}} - \frac{e^{y_2} \cdot e^{-e^{y_2}}}{1-e^{-e^{y_2}}} \overset{\text{sgn}}{=} y_2 - y_1, \\
\gamma &\overset{\text{sgn}}{=} e^{-e^{y_1}} - e^{-e^{y_2}} \overset{\text{sgn}}{=} y_2 - y_1, \\
\delta &\overset{\text{sgn}}{=} e^{-e^{y_1}} - e^{-e^{y_2}} \overset{\text{sgn}}{=} y_2 - y_1
\end{aligned}$$

and

$$\xi \overset{\text{sgn}}{=} \frac{e^{y_1} \cdot e^{-e^{y_1}}}{1-e^{-e^{y_1}}} - \frac{e^{y_2} \cdot e^{-e^{y_2}}}{1-e^{-e^{y_2}}} \overset{\text{sgn}}{=} y_2 - y_1,$$

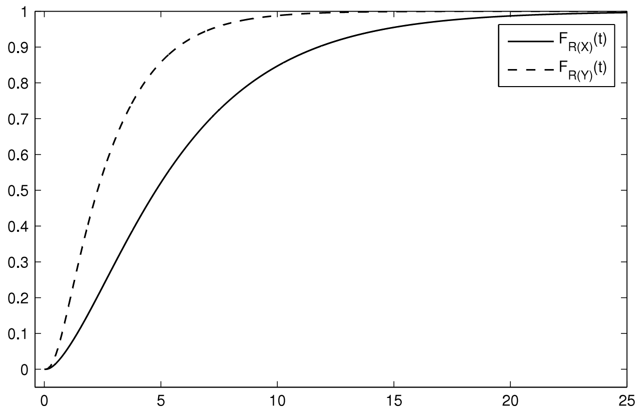

Fig. 4. Plot of $F_{R(X)}(t)$ and $F_{R(Y)}(t)$ when $p = 2, q = 2, \lambda_1 = 1/4, \lambda_2 = 4, \lambda_1^* = 1/2$ and $\lambda_2^* = 2$.

using which we have

$$(y_1 - y_2)\left(\frac{\partial \Phi}{\partial y_1} - \frac{\partial \Phi}{\partial y_2}\right) \le 0.$$

Now, upon applying Lemma 3.3, we can conclude that the function $\Phi(\underbrace{y_1, \dots, y_1}_{p}, \underbrace{y_2, \dots, y_2}_{q})$ is Schur-concave and hence the theorem follows. □

Example 4.2. Set $p = 2, q = 2, \lambda_1 = 1/4, \lambda_2 = 4, \lambda_1^* = 1/2$ and $\lambda_2^* = 2$ in Theorem 4.1. Then we have $\lambda_1^2\lambda_2^2 = (\lambda_1^*)^2(\lambda_2^*)^2$. Fig. 4 plots distribution functions of sample ranges. From Theorem 4.1, it follows that $F_{R(Y)}(t) \ge F_{R(X)}(t)$ for all $t \in \Re_+$, as shown in Fig. 4.

Theorem 4.3. *Let $X_1, X_2, \dots, X_n$ be independent exponential random variables such that X_i has failure rate λ_1 for $i = 1, \dots, p$ and, X_j has failure rate λ_2 for $j = p + 1, \dots, n$, where $p \ge 1$ and $q = n - p \ge 1$. Let $Y_1, Y_2, \dots, Y_n$ be independent exponential random variables such that Y_i has failure rate λ_1^* for $i = 1, \dots, p$ and, Y_j has failure rate λ_2^* for $j = p + 1, \dots, n$. Suppose $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$. If*

$$(\underbrace{\lambda_1, \dots, \lambda_1}_{p}, \underbrace{\lambda_2, \dots, \lambda_2}_{q}) \overset{p}{\succeq} (\underbrace{\lambda_1^*, \dots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \dots, \lambda_2^*}_{q}),$$

then,

$$X_{n:n} - X_{1:n} \ge_{\mathrm{st}} Y_{n:n} - Y_{1:n}.$$

Proof. Assume $(\underbrace{\lambda_1, \dots, \lambda_1}_{p}, \underbrace{\lambda_2, \dots, \lambda_2}_{q}) \overset{p}{\succeq} (\underbrace{\lambda_1^*, \dots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \dots, \lambda_2^*}_{q})$ to hold, we have $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$ and $\lambda_1^p\lambda_2^q \le (\lambda_1^*)^p(\lambda_2^*)^q$. The result holds when $\lambda_1^p\lambda_2^q = (\lambda_1^*)^p(\lambda_2^*)^q$ from Theorem 4.1. In the following, suppose that $\lambda_1^p\lambda_2^q < (\lambda_1^*)^p(\lambda_2^*)^q$. Let $\lambda' = \sqrt[p]{\frac{(\lambda_1^*)^p(\lambda_2^*)^q}{\lambda_2^q}}$. Then, we have $(\lambda')^p\lambda_2^q = (\lambda_1^*)^p(\lambda_2^*)^q$ and $\lambda_1 < \lambda' \le \lambda_2$. Let $V_{n:n} - V_{1:n}$ be the sample range from the independent exponential variables $V_1, V_2, \dots, V_n$ with respective hazard rates

$$(\underbrace{\lambda', \dots, \lambda'}_{p}, \underbrace{\lambda_2, \dots, \lambda_2}_{q}).$$

From Theorem 4.1, it follows that $V_{n:n} - V_{1:n} \ge_{\mathrm{st}} Y_{n:n} - Y_{1:n}$. On the other hand, we have $X_{n:n} - X_{1:n} \ge_{\mathrm{rh}} V_{n:n} - V_{1:n}$ from Theorem 3.4, which implies that $X_{n:n} - X_{1:n} \ge_{\mathrm{st}} V_{n:n} - V_{1:n}$. Hence, the desired result that $X_{n:n} - X_{1:n} \ge_{\mathrm{st}} Y_{n:n} - Y_{1:n}$ follows immediately. □

Example 4.4. Set $p = 3, q = 2, \lambda_1 = 1/4, \lambda_2 = 2, \lambda_1^* = 1/2$ and $\lambda_2^* = 1$ in Theorem 4.3. We then have $(1/4, 1/4, 1/4, 2, 2) \overset{p}{\succeq} (1/2, 1/2, 1/2, 1, 1)$, but $(1/4, 1/4, 1/4, 2, 2) \overset{w}{\not\succeq} (1/2, 1/2, 1/2, 1, 1)$. Fig. 5 gives the plot of distribution functions of two sample ranges. Note that $F_{R(X)}(t)$ is smaller than $F_{R(Y)}(t)$ for all $t \in \Re_+$, which shows the validity of the result in Theorem 4.3. Fig. 6 plots the reversed hazard rate functions $\tilde{r}_{R(X)}(t)$ and $\tilde{r}_{R(Y)}(t)$. It can be seen that the reversed hazard rate functions can not be compared in this case.

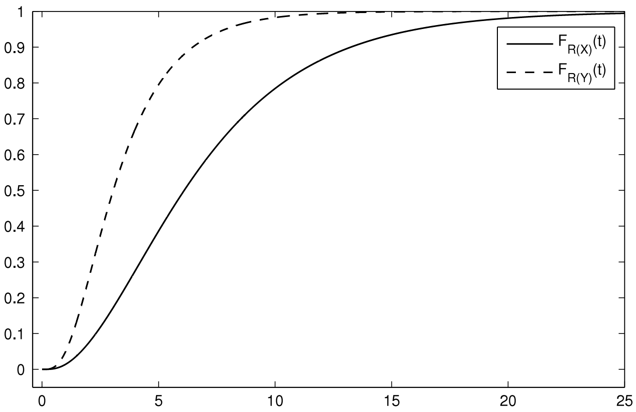

Fig. 5. Plot of $F_{R(X)}(t)$ and $F_{R(Y)}(t)$ when $p = 3, q = 2, \lambda_1 = 1/4, \lambda_2 = 2, \lambda_1^* = 1/2$ and $\lambda_2^* = 1$.

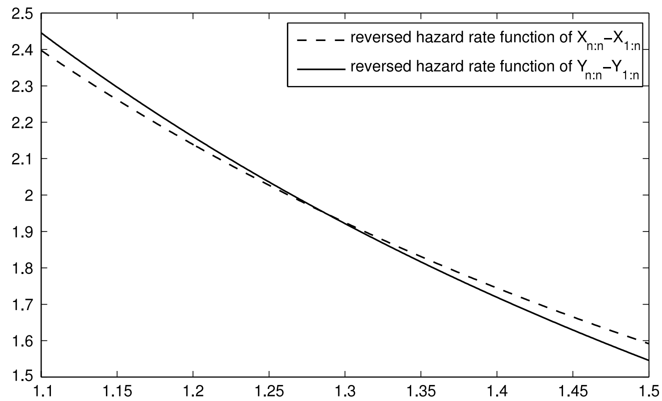

Fig. 6. Plot of $\tilde{r}_{R(X)}(t)$ and $\tilde{r}_{R(Y)}(t)$ when $p = 3, q = 2, \lambda_1 = 1/4, \lambda_2 = 2, \lambda_1^* = 1/2$ and $\lambda_2^* = 1$.

In the following, we present a result for the proportional hazard rates (PHR) model. Independent random variables $X_1, \ldots, X_n$ are said to follow PHR model if, for $i = 1, \ldots, n$, the survival function of X_i can be written as

$$\overline{F}_i(x) = [\overline{F}(x)]^{\lambda_i},$$

where $\overline{F}(x)$ is the survival function of some base random variable X. Let $r(t)$ be the hazard rate function of the baseline distribution F. Then, the survival function of X_i can be written as

$$\overline{F}_i(x) = e^{-\lambda_i R(x)}$$

for $i = 1, \ldots, n$, where $R(x) = \int_0^x r(t)\mathrm{d}t$ is the cumulative hazard rate of X. Many well-known models are special cases of the PHR model such as exponential, Weibull, Pareto, and Lomax et al.

We are now ready to present our result for the PHR model.

Theorem 4.5. *Let $X_1, X_2, \ldots, X_n$ follow a PHR model with survival functions*

$$(\underbrace{[\overline{F}(x)]^{\lambda_1}, \ldots, [\overline{F}(x)]^{\lambda_1}}_{p}, \underbrace{[\overline{F}(x)]^{\lambda_2}, \ldots, [\overline{F}(x)]^{\lambda_2}}_{q}),$$

where $p + q = n$. Let $Y_1, Y_2, \ldots, Y_n$ follow another PHR model with survival functions

$$(\underbrace{[\overline{F}(x)]^{\lambda_1^*}, \ldots, [\overline{F}(x)]^{\lambda_1^*}}_{p}, \underbrace{[\overline{F}(x)]^{\lambda_2^*}, \ldots, [\overline{F}(x)]^{\lambda_2^*}}_{q}).$$

Suppose $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$, we then have

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{m}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longrightarrow X_{n:n} - X_{1:n} \ge_{\mathrm{st}} Y_{n:n} - Y_{1:n}.$$

Proof. In virtue of [6, p. 26], the probability distribution function of $X_{n:n} - X_{1:n}$ is given by, for $t \geq 0$,

$$\begin{aligned} F_{R(X)}(t) &= \sum_{i=1}^{n} \int_0^\infty \lambda_i \overline{F}^{\lambda_i - 1}(u) f(u) \prod_{j=1, j\neq i}^{n} \left[\overline{F}^{\lambda_j}(u) - \overline{F}^{\lambda_j}(u+t)\right] \mathrm{d}u \\ &= p \int_0^\infty \lambda_1 \overline{F}^{\lambda_1 - 1}(u) f(u) \left[\overline{F}^{\lambda_1}(u) - \overline{F}^{\lambda_1}(u+t)\right]^{p-1} \left[\overline{F}^{\lambda_2}(u) - \overline{F}^{\lambda_2}(u+t)\right]^{q} \mathrm{d}u \\ &\quad + q \int_0^\infty \lambda_2 \overline{F}^{\lambda_2 - 1}(u) f(u) \left[\overline{F}^{\lambda_1}(u) - \overline{F}^{\lambda_1}(u+t)\right]^{p} \left[\overline{F}^{\lambda_2}(u) - \overline{F}^{\lambda_2}(u+t)\right]^{q-1} \mathrm{d}u. \end{aligned}$$

Similarly, the distribution function of $Y_{n:n} - Y_{1:n}$ is given by, for $t \geq 0$,

$$\begin{aligned} F_{R(Y)}(t) &= p \int_0^\infty \lambda_1^* \overline{F}^{\lambda_1^* - 1}(u) f(u) \left[\overline{F}^{\lambda_1^*}(u) - \overline{F}^{\lambda_1^*}(u+t)\right]^{p-1} \left[\overline{F}^{\lambda_2^*}(u) - \overline{F}^{\lambda_2^*}(u+t)\right]^{q} \mathrm{d}u \\ &\quad + q \int_0^\infty \lambda_2^* \overline{F}^{\lambda_2^* - 1}(u) f(u) \left[\overline{F}^{\lambda_1^*}(u) - \overline{F}^{\lambda_1^*}(u+t)\right]^{p} \left[\overline{F}^{\lambda_2^*}(u) - \overline{F}^{\lambda_2^*}(u+t)\right]^{q-1} \mathrm{d}u. \end{aligned}$$

Thus, it suffices to show, for $t \geq 0$, $u \geq 0$,

$$\begin{aligned} &\left[\overline{F}^{\lambda_1}(u) - \overline{F}^{\lambda_1}(u+t)\right]^{p-1} \left[\overline{F}^{\lambda_2}(u) - \overline{F}^{\lambda_2}(u+t)\right]^{q-1} \\ &\quad \times \left[p\lambda_1 \overline{F}^{\lambda_1}(u) \left(\overline{F}^{\lambda_2}(u) - \overline{F}^{\lambda_2}(u+t)\right) + q\lambda_2 \overline{F}^{\lambda_2}(u) \left(\overline{F}^{\lambda_1}(u) - \overline{F}^{\lambda_1}(u+t)\right)\right] \\ &\leq \left[\overline{F}^{\lambda_1^*}(u) - \overline{F}^{\lambda_1^*}(u+t)\right]^{p-1} \left[\overline{F}^{\lambda_2^*}(u) - \overline{F}^{\lambda_2^*}(u+t)\right]^{q-1} \\ &\quad \times \left[p\lambda_1^* \overline{F}^{\lambda_1^*}(u) \left(\overline{F}^{\lambda_2^*}(u) - \overline{F}^{\lambda_2^*}(u+t)\right) + q\lambda_2^* \overline{F}^{\lambda_2^*}(u) \left(\overline{F}^{\lambda_1^*}(u) - \overline{F}^{\lambda_1^*}(u+t)\right)\right] \end{aligned}$$

i.e.,

$$\begin{aligned} &\overline{F}^{p\lambda_1 + q\lambda_2}(u) \left[1 - \left(\frac{\overline{F}(u+t)}{\overline{F}(u)}\right)^{\lambda_1}\right]^{p-1} \left[1 - \left(\frac{\overline{F}(u+t)}{\overline{F}(u)}\right)^{\lambda_2}\right]^{q-1} \\ &\quad \times \left\{p\lambda_1 \left[1 - \left(\frac{\overline{F}(u+t)}{\overline{F}(u)}\right)^{\lambda_2}\right] + q\lambda_2 \left[1 - \left(\frac{\overline{F}(u+t)}{\overline{F}(u)}\right)^{\lambda_1}\right]\right\} \\ &\leq \overline{F}^{p\lambda_1^* + q\lambda_2^*}(u) \left[1 - \left(\frac{\overline{F}(u+t)}{\overline{F}(u)}\right)^{\lambda_1^*}\right]^{p-1} \left[1 - \left(\frac{\overline{F}(u+t)}{\overline{F}(u)}\right)^{\lambda_2^*}\right]^{q-1} \\ &\quad \times \left\{p\lambda_1^* \left[1 - \left(\frac{\overline{F}(u+t)}{\overline{F}(u)}\right)^{\lambda_2^*}\right] + q\lambda_2^* \left[1 - \left(\frac{\overline{F}(u+t)}{\overline{F}(u)}\right)^{\lambda_1^*}\right]\right\}. \end{aligned} \tag{5}$$

Denote $\overline{F}_u(t) = \frac{\overline{F}(u+t)}{\overline{F}(u)}$, and it is the survival function of $T_u = T - u \mid T > u$, the residual life of T at time $u \geq 0$. Upon using $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, $p\lambda_1 + q\lambda_2 = p\lambda_1^* + q\lambda_2^*$ and the transform

$$H(t) = -\log \overline{F}_u(t), \quad u > 0,$$

the inequality (5) becomes,

$$\begin{aligned} &[1 - e^{-\lambda_1 H(t)}]^{p-1} [1 - e^{-\lambda_2 H(t)}]^{q-1} \left\{p\lambda_1 [1 - e^{-\lambda_2 H(t)}] + q\lambda_2 [1 - e^{-\lambda_1 H(t)}]\right\} \\ &\leq [1 - e^{-\lambda_1^* H(t)}]^{p-1} [1 - e^{-\lambda_2^* H(t)}]^{q-1} \left\{p\lambda_1^* [1 - e^{-\lambda_2^* H(t)}] + q\lambda_2^* [1 - e^{-\lambda_1^* H(t)}]\right\}. \end{aligned}$$

Using the result of Theorem 3.6 and the fact that the reversed hazard rate implies the usual stochastic order, we can get

$$\begin{aligned} &(1 - e^{-\lambda_1 t})^{p-1} (1 - e^{-\lambda_2 t})^{q-1} \left[p\lambda_1 (1 - e^{-\lambda_2 t}) + q\lambda_2 (1 - e^{-\lambda_1 t})\right] \\ &\leq (1 - e^{-\lambda_1^* t})^{p-1} (1 - e^{-\lambda_2^* t})^{q-1} \left[p\lambda_1^* (1 - e^{-\lambda_2^* t}) + q\lambda_2^* (1 - e^{-\lambda_1^* t})\right]. \end{aligned} \tag{6}$$

Replacing t with $H(t)$ in inequality (6), the desired result follows immediately. □

5. Discussion

Let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier exponential model with parameters

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}),$$

where $p + q = n$, and let $Y_1, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier exponential model with parameters

$$(\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}).$$

In this article, we have established, under the condition $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$, we have

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{w}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longleftrightarrow X_{n:n} - X_{1:n} \ge_{\mathrm{rh}} Y_{n:n} - Y_{1:n} \tag{7}$$

and

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{p}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longrightarrow X_{n:n} - X_{1:n} \ge_{\mathrm{st}} Y_{n:n} - Y_{1:n}. \tag{8}$$

It would be of interest to check whether the results in (7) and (8) can be generalized to the likelihood ratio order and hazard rate order, respectively.

Another natural question is whether, under the condition $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$,

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}) \overset{\mathrm{rm}}{\succeq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda_2^*, \ldots, \lambda_2^*}_{q}) \Longrightarrow X_{n:n} - X_{1:n} \ge_{\mathrm{mrl}} Y_{n:n} - Y_{1:n}$$

also holds. We are currently working on these problems and hope to report these findings in a future paper.

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