

The random matrix regime of Maronna's M-estimator with elliptically distributed samples[☆]

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ABSTRACT

This article demonstrates that the robust scatter matrix estimator $\hat{C}_N \in \mathbb{C}^{N \times N}$ of a multivariate elliptical population $x_1, \dots, x_n \in \mathbb{C}^N$ originally proposed by Maronna in 1976, and defined as the solution (when existent) of an implicit equation, behaves similar to a well-known random matrix model in the limiting regime where the population N and sample n sizes grow at the same speed. We show precisely that $\hat{C}_N \in \mathbb{C}^{N \times N}$ is defined for all n large with probability one and that, under some light hypotheses, $\|\hat{C}_N - \hat{S}_N\| \rightarrow 0$ almost surely in spectral norm, where \hat{S}_N follows a classical random matrix model. As a corollary, the limiting eigenvalue distribution of \hat{C}_N is derived. This analysis finds applications in the fields of statistical inference and signal processing.

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1. Introduction and problem statement

The recent advances in the spectral analysis of large dimensional random matrices, and particularly of matrices of the sample covariance type, have triggered a new wave of interest for (sometimes old) problems in statistical inference and signal processing, which were usually treated under the assumption of a small population size versus a large sample dimension and are now explored assuming similar population and sample sizes. For instance, new source detection schemes have been proposed [9,7,28] based on the works on the extreme eigenvalues of large Wishart matrices [37,17,5,4,6,13]. New subspace methods in large array processing have also been derived [27,22,18] that outperform the original MUSIC algorithm [34] by exploiting statistical inference methods on large random matrices [26,16,39]. Most of these signal processing methods fundamentally rely on the structure of the sample covariance matrix $\frac{1}{n} \sum_{i=1}^n x_i x_i^*$ formed from independent or linearly dependent (say zero mean) population vectors $x_1, \dots, x_n \in \mathbb{C}^N$, the asymptotic spectral properties of which are now well understood [25,35,36,6,17,26]. The field of signal processing however covers a much wider scope than that of sample covariance matrices. Of interest here are the robust scatter matrix estimation techniques (a subclass of M-estimation [40, Chapter 5]) that allow for a better – more robust – empirical approximation of population covariance (or scatter) matrices whenever (i) the probability distribution of the population vectors x_i is heavy-tailed (or is at least far from Gaussian) or (ii) a small proportion of the samples x_i presents an outlier behavior (i.e., follows an unknown distribution, quite different from the distribution of most samples) [20,21,24].

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While classical sample covariance matrices exhibit a rather simple dependence structure (as they are merely the sum of independent or linearly dependent rank-one matrices), robust scatter matrix estimators are usually of a much more complex form which does not allow for standard random matrix analysis. This work specifically considers a widely spread model of robust scatter estimator, proposed in [23], which contains as special cases the maximum-likelihood estimator of the scatter matrix for elliptically distributed population vectors, and which is well-behaved and mostly understood in the classical regime where $n \rightarrow \infty$ while N is fixed. It is in particular shown in [23] that under some conditions the estimator is well-defined as the unique solution of a fixed-point equation and that the robust estimator converges almost surely (a.s.) to a deterministic matrix (which can be the scatter matrix for elliptical distribution of x_i under correct parametrization). In this article, we revisit the study of Maronna's estimator for elliptically distributed samples using a probabilistic approach (as opposed to the statistical approach used classically in robust estimation theory) under the assumption that n and N are both large and of the same order of magnitude. This work follows after Couillet et al. [15] where the simpler case of vector samples x_i with independent entries was explored. The intuition for the proof of the main results follows in particular from the proof of the main theorem in [15].

Studying robust scatter estimators in the large random matrix regime, i.e., as N, n grow large at the same speed, has important consequences in understanding many signal processing algorithms exploiting these estimators [30,31]. It also allows one to derive improved methods for source detection and parameter estimation as in [9,7,27,22,18] for sample covariance matrix-based estimators. Adaptations (and improvements) of these results to robust estimation are currently under investigation.

Before discussing our main results, we first introduce the notations and assumptions taken in this article. We let $x_1, \dots, x_n \in \mathbb{C}^N$ be n random vectors defined by $x_i = \sqrt{\tau_i} A_N y_i$, where $\tau_1, \dots, \tau_n \in \mathbb{R}^+$ and $y_1, \dots, y_n \in \mathbb{C}^{\bar{N}}$ are random and $A_N \in \mathbb{C}^{N \times \bar{N}}$ is deterministic. We denote $c_N \triangleq N/n$ and $\bar{c}_N \triangleq \bar{N}/N$ and shall consider the following growth regime.

Assumption 1. For each N , $c_N < 1$, $\bar{c}_N \geq 1$ and

$$c_- < \liminf_n c_N \leq \limsup_n c_N < c_+$$

where $0 < c_- < c_+ < 1$.

The robust estimator under consideration in this article is Maronna's M-estimator \hat{C}_N defined, when it exists, as a (possibly unique) solution to the equation in $Z \in \mathbb{C}^{N \times N}$

$$Z = \frac{1}{n} \sum_{i=1}^n u \left(\frac{1}{N} x_i^* Z^{-1} x_i \right) x_i x_i^* \quad (1)$$

where u satisfies the following properties:

- (i) $u : [0, \infty) \rightarrow (0, \infty)$ is nonnegative continuous and non-increasing
- (ii) $\phi : x \mapsto xu(x)$ is increasing and bounded with $\lim_{x \rightarrow \infty} \phi(x) \triangleq \phi_\infty > 1$
- (iii) $\phi_\infty < c_+^{-1}$.

Note that (ii) is stronger than Maronna's original assumption [23, Condition (C) p. 53] as ϕ cannot be constant on any open interval. The assumption (iii) is also not classical in robust estimation but obviously compliant with the large n assumption made in classical works (for which $c_+ = 0$). The importance of both assumptions will appear clearly in the proof of the main results.

The statistical hypotheses on x_1, \dots, x_n are detailed below.

Assumption 2. The vectors $x_i = \sqrt{\tau_i} A_N y_i$, $i \in \{1, \dots, n\}$, satisfy the following hypotheses:

- 1. the (random) empirical measure $\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\tau_i}$ satisfies $\int \tau \nu_n(d\tau) \xrightarrow{\text{a.s.}} 1$
- 2. there exist $\varepsilon < 1 - \phi_\infty^{-1} < 1 - c_+$ and $m > 0$ such that, for all large n a.s. $\nu_n([0, m)) < \varepsilon$
- 3. defining $C_N \triangleq A_N A_N^*$, $C_N > 0$ and $\limsup_N \|C_N\| < \infty$
- 4. $y_1, \dots, y_n \in \mathbb{C}^{\bar{N}}$ are independent unitarily invariant complex (or orthogonally invariant real) zero-mean vectors with, for each i , $\|y_i\|^2 = \bar{N}$, and are independent of τ_1, \dots, τ_n .

Item 1 is merely a normalization condition which, along with Item 3, ensures the proper scaling and asymptotic boundedness of the model parameters. Note in particular that Item 1 ensures a.s. tightness of $\{\nu_n\}_{n=1}^\infty$, i.e., for each $\varepsilon > 0$, there exists $M > 0$ such that, with probability one, $\nu_n([M, \infty)) < \varepsilon$ for all n . Item 2 mainly ensures that no heavy mass of τ_i concentrates close to zero; this will ensure the existence of a solution to (1) and avoid technical problems when a solution to (1) exists (and is therefore invertible) but has many eigenvalues close to zero.

Note that Item 4 could be equivalently stated as $y_i = \sqrt{\bar{N}} \frac{\tilde{y}_i}{\|\tilde{y}_i\|}$ with $\tilde{y}_i \in \mathbb{C}^{\bar{N}}$ standard complex Gaussian (or standard real Gaussian). This remark will be used throughout the proofs of the main results which rely in part on random matrix identities for matrices with independent entries.

All these conditions are met in particular if the τ_i are independent and identically distributed (i.i.d.) with common unit mean distribution ν (in which case $\int x \nu_n(dx) \xrightarrow{\text{a.s.}} 1$ by the strong law of large numbers) such that $\nu(\{0\}) = 0$. If in addition

$N = \bar{N}$, then x_1, \dots, x_n are i.i.d. zero-mean complex (or real) elliptically distributed with full rank [29, Theorem 3]. In particular, if $2N\tau_1$ is chi-squared distributed with $2N$ degrees of freedom, x_1 is complex zero mean Gaussian. If $1/\tau_1$ is chi-squared distributed with arbitrary degrees of freedom, x_1 is instead zero mean complex Student distributed (see [29] for further discussions and recent results on elliptical distributions).

For simplicity of exposition, most of the article, and in particular the proofs of the main results, will assume the case of complex x_i ; the results remain however valid in the case of real random variables.

Assumption 3. For each $a > b > 0$, a.s.

$$\limsup_{t \rightarrow \infty} \frac{\limsup_n v_n((t, \infty))}{\phi(at) - \phi(bt)} = 0.$$

Assumption 3 controls the relative speed of the tail of v_n versus the flattening speed of $\phi(x)$ as $x \rightarrow \infty$. Practical examples satisfying **Assumption 3** are:

- There exists $M > 0$ such that, for all n , $\max_{1 \leq i \leq n} \tau_i < M$ a.s. In this case, $v_n((t, \infty)) = 0$ a.s. for $t > M$ while $\phi(at) - \phi(bt) \neq 0$ since ϕ is increasing.
- For $u(t) = (1 + \alpha)/(\alpha + t)$ for some $\alpha > 0$, it is easily seen that it is sufficient that $\limsup_n v_n((t, \infty)) = o(1/t)$ a.s. for **Assumption 3** to hold. In particular, if the τ_i are i.i.d. with distribution ν , $\limsup_n v_n((t, \infty)) = \nu((t, \infty))$ a.s. (for all t continuity points of ν) and, by Markov inequality, it suffices that $\int x^{1+\varepsilon} \nu(dx) < \infty$ for some $\varepsilon > 0$.

The main contribution of this article is twofold: we first present a result on existence and uniqueness of \hat{C}_N as a solution to (1) (**Theorem 1**) and then study the limiting spectral behavior of \hat{C}_N as $N, n \rightarrow \infty$ (**Theorem 2**). With respect to existence and uniqueness, we recall that for $\bar{N} = N$ [23, Theorem 1] ensures the existence and uniqueness of a solution to (1) under the statistical hypothesis that each N -subset of x_1, \dots, x_n spans \mathbb{C}^N and that $\phi_\infty > n/(n - N)$. While the first condition is met with probability one for continuous distributions of x_i , the second condition is restrictive under **Assumption 1** as it imposes $\phi_\infty > 1/(1 - c_-)$ which brings a loss in robustness for c_- close to one.¹ Our first result is a probabilistic alternative to [23, Theorem 1] which states that for all large n , a.s.,² (1) has a unique solution. This result uses the probability conditions on x_1, \dots, x_n and also uses $\phi_\infty < c_+^{-1}$ which, as opposed to [23, Theorem 1], enforces more robust estimators. The uniqueness part of the result also imposes that ϕ be strictly increasing, while Maronna [23, Theorem 1] allows $\phi(x) = \phi_\infty$ for all large x .

As for the large dimensional behavior of \hat{C}_N , in the fixed N large n regime and for i.i.d. τ_i , it is of the form $\hat{C}_N \xrightarrow{\text{a.s.}} V_N$ where V_N is the unique solution to $V_N = E[u(\frac{1}{N} x_1^* V_N^{-1} x_1) x_1 x_1^*]$ [23, Theorem 5]. When the x_i are i.i.d. elliptically distributed and u is such that \hat{C}_N is the maximum-likelihood estimator for C_N , then $V_N = C_N$, leading to a consistent estimator for C_N . In the random matrix regime of interest here, we show that \hat{C}_N does not converge in any classical sense to a deterministic matrix but satisfies $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$ in spectral norm, where \hat{S}_N follows a random matrix model studied in [42, 33, 12]. As such, the spectral behavior of \hat{C}_N is easily analyzed from that of \hat{S}_N for N, n large.

In the next section, we introduce some new notations that simplify the analysis of \hat{C}_N and provide an insight on the derivation of our main result, **Theorem 2**.

Generic notations: We denote $\lambda_1(X) \leq \dots \leq \lambda_N(X)$ the ordered eigenvalues of any Hermitian (or symmetric) matrix X . The superscript $(\cdot)^*$ designates transpose conjugate (if complex) for vectors or matrices. The norm $\|\cdot\|$ is the spectral norm for matrices and the Euclidean norm for vectors. The cardinality of a finite discrete set Ω is denoted by $|\Omega|$. Almost sure convergence is written $\xrightarrow{\text{a.s.}}$. We use the set notation $\mathbb{C}^+ = \{z \in \mathbb{C}, \Im[z] > 0\}$. The Hermitian (or symmetric) matrix order relations are denoted $A \succeq B$ for $A - B$ nonnegative definite and $A \succ B$ for $A - B$ positive definite. The Dirac measure at point $x \in \mathbb{R}$ is denoted by δ_x .

2. Preliminaries

First note from the expression of \hat{C}_N as a (hypothetical) solution to (1) that we can assume $C_N = I_N$ by studying $C_N^{-\frac{1}{2}} \hat{C}_N C_N^{-\frac{1}{2}}$ in place of \hat{C}_N . Therefore, here and in all the major proofs in the article, without generality restriction, we place ourselves under the assumption $C_N = A_N A_N^* = I_N$.

Our objective is to prove that \hat{C}_N is a well behaved solution of (1) (for all large n , a.s.) and to study the spectral properties of \hat{C}_N as N, n grow large. However, the structure of dependence between the rank-one matrices $u(\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i) x_i x_i^*, i = 1, \dots, n$, makes the large dimensional analysis of \hat{C}_N via standard random matrix methods impossible (see e.g., [32, 3, 1]) as these methods fundamentally rely on the independence (or simple dependence) of the structuring rank-one matrices. We propose

¹ As commented in [23], small values of ϕ_∞ induce increased robustness to the expense of accuracy.

² As is common in random matrix theory, the probability space under consideration is that engendered by the growing sequences $\{x_1, \dots, x_n\}_{n=1}^\infty$, with N, n satisfying **Assumption 1**, so that an event E_n holds true “for all large n , a.s.” whenever, with probability one, there exists n_0 for which E_n is true for all $n \geq n_0$, this n_0 possibly depending on the sequence.

here to show that, in the large N, n regime, \hat{C}_N behaves similar to a matrix \hat{S}_N whose structure is more standard and easily analyzed through classical random matrix results. For this we first need to rewrite the fundamental equation (1) in order to exhibit a sufficiently “weak” dependence structure in the expression of \hat{C}_N . This rewriting is performed in Section 2.1. This being done, we then prove that some weakly dependent terms can be well approximated by independent ones in the large N, n regime. Since the final result does not take an insightful form, we provide below in Section 2.2 a hint on how to obtain it intuitively.

2.1. Rewriting (1)

We need to introduce some new notations that will simplify the coming considerations. Write $x_i = \sqrt{\tau_i} A_N y_i \triangleq \sqrt{\tau_i} z_i$ and recall that $C_N = I_N$ for the moment (in particular, $\|z_i\|$ is of order \sqrt{N} for most z_i). If \hat{C}_N is well-defined, we denote $\hat{C}_{(i)} \triangleq \hat{C}_N - \frac{1}{n} u \left(\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i \right) x_i x_i^*$.

Remark that $\hat{C}_{(i)}$ depends on x_i only through the terms $u \left(\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i \right)$, $j \neq i$, in which the term \hat{C}_N is built on x_i . But since x_i is only one among a growing number n of x_j vectors, this dependence structure looks intuitively “weak”. This informal weak dependence between x_i and $\hat{C}_{(i)}$, along with classical random matrix theory considerations, suggests that the quadratic forms $\frac{1}{N} z_i^* \hat{C}_{(i)}^{-1} z_i$, $i = 1, \dots, n$, are all well approximated by $\frac{1}{N} \text{tr} \hat{C}_N^{-1}$ (more precisely, this would roughly be a consequence of Lemmas 5 and 4 in the Appendix if z_i and $\hat{C}_{(i)}$ were truly independent).

With this in mind, let us rewrite \hat{C}_N as a function of $\frac{1}{N} z_i^* \hat{C}_{(i)}^{-1} z_i$ instead of $\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i$, $i = 1, \dots, n$. For this, let $Z \in \mathbb{C}^{N \times n}$ be positive definite such that for each i , $Z_{(i)} \triangleq Z - \frac{1}{n} u \left(\tau_i \frac{1}{N} z_i^* Z^{-1} z_i \right) \tau_i z_i z_i^*$ is positive definite. Using the identity $(A + \tau z z^*)^{-1} z = A^{-1} z / (1 + \tau z^* A^{-1} z)$ for invertible A , vector z , and positive scalar τ , observe that

$$\frac{1}{N} z_i^* Z^{-1} z_i = \frac{\frac{1}{N} z_i^* Z_{(i)}^{-1} z_i}{1 + \tau_i u \left(\tau_i \frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \right) \frac{1}{n} z_i^* Z_{(i)}^{-1} z_i}.$$

Hence,

$$\frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \left(1 - c_N \tau_i u \left(\tau_i \frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \right) \frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \right) = \frac{1}{N} z_i^* Z^{-1} z_i$$

which, by the definition of ϕ , is

$$\frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \left(1 - c_N \phi \left(\tau_i \frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \right) \right) = \frac{1}{N} z_i^* Z^{-1} z_i.$$

Using Assumption 1 and $\phi_\infty < c_+^{-1}$, taking n large enough to have $\phi(x) \leq \phi_\infty < 1/c_N$, this can be rewritten

$$\frac{1}{N} z_i^* Z_{(i)}^{-1} z_i = \frac{\frac{1}{N} z_i^* Z^{-1} z_i}{1 - c_N \phi \left(\tau_i \frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \right)}. \quad (2)$$

Now, since ϕ is increasing, $g : [0, \infty) \rightarrow [0, \infty)$, $x \mapsto x / (1 - c_N \phi(x))$ is increasing, nonnegative, and maps $[0, \infty)$ onto $[0, \infty)$. Thus, g is invertible with inverse denoted g^{-1} . In particular, from (2),

$$\tau_i \frac{1}{N} z_i^* Z^{-1} z_i = g^{-1} \left(\tau_i \frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \right).$$

Call now $v : [0, \infty) \rightarrow [0, \infty)$, $x \mapsto u \circ g^{-1}(x)$. Since g is increasing and nonnegative and u is non-increasing, v is non-increasing and positive. Moreover, $\psi : x \mapsto x v(x)$ satisfies:

$$\psi(x) = x u(g^{-1}(x)) = g(g^{-1}(x)) u(g^{-1}(x)) = \frac{\phi(g^{-1}(x))}{1 - c_N \phi(g^{-1}(x))}$$

which is increasing, nonnegative, and has limit $\psi_\infty^N \triangleq \phi_\infty / (1 - c_N \phi_\infty)$ as $x \rightarrow \infty$. Hence, v and ψ keep the same properties as u and ϕ , respectively.

With these notations, to prove the existence and uniqueness of a solution to (1), it is equivalent to prove that the equation in Z

$$Z = \frac{1}{n} \sum_{i=1}^n \tau_i v \left(\tau_i \frac{1}{N} z_i^* Z_{(i)}^{-1} z_i \right) z_i z_i^*$$

has a unique positive definite solution. But for this, it is sufficient to prove the uniqueness of $d_1, \dots, d_n \geq 0$ satisfying the n equations:

$$d_j = \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v \left(\tau_i d_i \right) z_i z_i^* \right)^{-1} z_j, \quad 1 \leq j \leq n. \quad (3)$$

Indeed, if these d_i are uniquely defined, then so is the matrix

$$\hat{C}_N = \frac{1}{n} \sum_{i=1}^n \tau_i v(\tau_i d_i) z_i z_i^* \quad (4)$$

with $d_i = \frac{1}{N} z_i^* \hat{C}_N^{-1} z_i$, $\hat{C}_{(i)} = \hat{C}_N - \frac{1}{n} u(\frac{1}{N} x_i^* \hat{C}_N^{-1} x_i) x_i x_i^*$ (the existence follows from taking the d_i solution to (3) and write \hat{C}_N as in (4), while uniqueness follows from the fact that (4) cannot be written with a different set of d_i from the uniqueness of the solution to (3)).

This is the approach that is pursued to prove Theorem 1, based on the results from Yates [41]. Eq. (4), which is equivalent to (1) (with \hat{C}_N in place of Z), will be preferably used in the remainder of the article.

2.2. Hint on the main result

Assume here that the d_i above are indeed unique for all large n so that \hat{C}_N is well defined. We provide some intuition on the main result.

From the discussion in Section 2.1, we may expect the terms d_i to be all close to $\frac{1}{N} \text{tr} \hat{C}_N^{-1}$ for N, n large enough. We may also expect $\frac{1}{N} \text{tr} \hat{C}_N^{-1}$ to have a deterministic equivalent γ_N , i.e., there should exist a deterministic sequence $\{\gamma_N\}_{N=1}^\infty$ such that $|\frac{1}{N} \text{tr} \hat{C}_N^{-1} - \gamma_N| \xrightarrow{\text{a.s.}} 0$. Let us say that all this is true. Since $\frac{1}{N} \text{tr} \hat{C}_N^{-1}$ is the Stieltjes transform $\frac{1}{N} \text{tr} (\hat{C}_N - z I_N)^{-1}$ of the empirical spectral distribution of \hat{C}_N at point $z = 0$, and since \hat{C}_N is expected to be close to $\frac{1}{n} \sum_i \tau_i v(\tau_i \gamma_N) z_i z_i^*$ with now $v(\tau_i \gamma_N)$ independent of z_1, \dots, z_n , from classical random matrix works, e.g., [35], we would expect that one such γ_N be given by (recall that $C_N = I_N$)

$$\gamma_N = \left(\frac{1}{n} \sum_{i=1}^n \frac{\tau_i v(\tau_i \gamma_N)}{1 + c_N \tau_i v(\tau_i \gamma_N) \gamma_N} \right)^{-1}$$

if this fixed-point equation makes sense at all. This can be equivalently written as

$$1 = \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{1 + c_N \psi(\tau_i \gamma_N)}. \quad (5)$$

We in fact prove in Section 3 that such a positive γ_N is well defined, unique, and satisfies $\max_{1 \leq i \leq n} |d_i - \gamma_N| \xrightarrow{\text{a.s.}} 0$ (under correct assumptions). Proving this result is the main difficulty of the article.

This convergence, along with (4), will then ensure that for all large n , a.s.

$$\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N = \frac{1}{n} \sum_{i=1}^n v(\tau_i \gamma_N) \tau_i z_i z_i^*$$

with γ_N the unique positive solution to (5). It will then be immediate under Assumption 2-3 to see that the result holds true also for $C_N \neq I_N$.

The major interest of this convergence in spectral norm is that \hat{S}_N is a known and easily manipulable object, as opposed to \hat{C}_N . The result therefore conveys a lot of information about \hat{C}_N among which the fact that its largest and smallest eigenvalues are almost surely bounded and bounded away from zero for all large n (which is not in general the case of $\frac{1}{n} \sum_{i=1}^n x_i x_i^*$ for τ_i with unbounded support).

3. Main results

We now make the statements of Section 2.2 rigorous. The first result ensures the existence and uniqueness of a solution \hat{C}_N to (1) for n large enough.

Theorem 1 (Existence and Uniqueness). Let Assumptions 1 and 2 hold, with $\limsup_N \|C_N\|$ non necessarily bounded. Then, for all large n a.s., (1) has a unique solution \hat{C}_N given by

$$\hat{C}_N = \lim_{\substack{t \rightarrow \infty \\ t \in \mathbb{N}}} Z^{(t)}$$

where $Z^{(0)} > 0$ is arbitrary and, for $t \in \mathbb{N}$,

$$Z^{(t+1)} = \frac{1}{n} \sum_{i=1}^n u \left(\frac{1}{N} x_i^* (Z^{(t)})^{-1} x_i \right) x_i x_i^*.$$

Having defined \hat{C}_N , the main result of the article provides a random matrix equivalent to \hat{C}_N , much easier to study than \hat{C}_N itself.

Theorem 2 (Asymptotic Behavior). Let [Assumptions 1–3](#) hold, and let \hat{C}_N be given by [Theorem 1](#) when uniquely defined as the solution of [\(1\)](#) or chosen arbitrarily if not. Then

$$\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$$

where

$$\hat{S}_N \triangleq \frac{1}{n} \sum_{i=1}^n v(\tau_i \gamma_N) x_i x_i^*$$

and γ_N is the unique positive solution of the equation in γ

$$1 = \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)}$$

with the functions $v : x \mapsto (u \circ g^{-1})(x)$, $\psi : x \mapsto xv(x)$, and $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $x \mapsto x/(1 - c_N \phi(x))$.

Note as an immediate corollary that if $\tau_i = 1$ for each i , one falls essentially back to the simpler setting of Couillet et al. [15]. In this case, the defining equation for γ_N reduces to $\psi(\gamma_N) = (1 - c_N)^{-1}$; since $\psi(x) = \phi(g^{-1}(x))/(1 - c_N \phi(g^{-1}(x)))$, this induces $\phi(g^{-1}(\gamma_N)) = 1$ or equivalently $g^{-1}(\gamma_N) = \phi^{-1}(1)$. From this, we then have $v(\gamma_N) = u(\phi^{-1}(1))$ or more simply $v(\gamma_N) = 1/\phi^{-1}(1)$ and we finally recover the result from Couillet et al. [15, Theorem 1-(II)].

The fact that \hat{C}_N is well approximated by \hat{S}_N , which follows a random matrix model studied extensively in [33,12], has important consequences. From a purely mathematical standpoint, this provides a full characterization of the spectral behavior of \hat{C}_N for large N, n (see in particular [Corollary 1](#)). For application purposes, this first enables the performance analysis in the large N, n horizon of standard signal processing methods already relying on \hat{C}_N (these methods were so far analyzed solely in the fixed N large n regime). A second, more important, consequence for signal processing application is the possibility to fully exploit the structure of \hat{C}_N for large N, n to improve existing robust schemes (see e.g., [14] for an example in array processing). Deriving such improved methods is not the subject of the current article but may be directly accessible from [Theorem 2](#), while performance analysis of these methods may demand supplementary treatment, such as central limit theorems for functionals of \hat{C}_N . Note in passing that \hat{S}_N is not an observable matrix since γ_N and the τ_i 's are not directly readable from the x_i 's so that \hat{S}_N has a purely analytical purpose and cannot be used as a substitute for \hat{C}_N in practice.

Corollary 1 (Spectrum). Let [Assumptions 1–3](#) hold. Then

$$\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i(\hat{C}_N)} - \mu_N \xrightarrow{\text{a.s.}} 0 \quad (6)$$

where the convergence is in the weak probability measure sense, with μ_N a probability measure with continuous density and Stieltjes transform $m_N(z)$ given, for $z \in \mathbb{C}^+$, by

$$m_N(z) = -\frac{1}{z} \frac{1}{N} \sum_{i=1}^N \frac{1}{1 + \tilde{\delta}_N(z) \lambda_i(C_N)}$$

where $\tilde{\delta}_N(z)$ is the unique solution in \mathbb{C}^+ of the equations in $\tilde{\delta}$

$$\begin{aligned} \tilde{\delta} &= -\frac{1}{z} \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{\gamma_N + \psi(\tau_i \gamma_N) \tilde{\delta}} \\ \delta &= -\frac{1}{z} \frac{1}{n} \sum_{i=1}^N \frac{\lambda_i(C_N)}{1 + \lambda_i(C_N) \tilde{\delta}} \end{aligned}$$

and where γ_N is defined in [Theorem 2](#). Besides, the support \mathcal{S}_N of μ_N is uniformly bounded. If $C_N = I_N$, $m_N(z)$ is the unique solution in \mathbb{C}^+ of the equation in m

$$m = \left(-z + \gamma_N^{-1} \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{1 + c \gamma_N^{-1} \psi(\tau_i \gamma_N) m} \right)^{-1}.$$

Also, for each $N_0 \in \mathbb{N}$ and each closed set $\mathcal{A} \subset \mathbb{R}$ with $\mathcal{A} \cap \left(\bigcup_{N \geq N_0} \mathcal{S}_N\right) = \emptyset$,

$$\left| \left\{ \lambda_i(\hat{C}_N) \right\}_{i=1}^N \cap \mathcal{A} \right| \xrightarrow{\text{a.s.}} 0 \quad (7)$$

so that, in particular,

$$\limsup_N \|\hat{C}_N\| < \infty. \quad (8)$$

Proof. Eq. (6) is obtained from the results of Zhang [42] with notations similar to Couillet and Hachem [12]. The characterization of μ_N follows from Couillet and Hachem [12], where more information can be found. The uniform boundedness of the support is a consequence of the boundedness of ψ and γ_N , Lemma 1 in Section 4. Finally, the results (7) and (8) are an application of Paul and Silverstein [33] along with $\limsup_N \|\hat{S}_N\| \leq \nu(0) \limsup_N \|C_N\| \limsup_N \left\| \frac{1}{n} \sum_{i=1}^n x_i x_i^* \right\| < \infty$ by Assumption 2–3 and Bai and Silverstein [2].

A consequence of Theorem 2 and Corollary 1 in the i.i.d. elliptical case is as follows.

Corollary 2 (Elliptical Case). Let Assumptions 1–3 hold and in addition, let τ_i be i.i.d. with law ν and let $c_N \rightarrow c$. Then

$$\left\| \hat{C}_N - \frac{1}{n} \sum_{i=1}^n v(\tau_i \gamma^\infty) x_i x_i^* \right\| \xrightarrow{\text{a.s.}} 0$$

where γ^∞ is the unique positive solution to the equation in γ

$$1 = \int \frac{\psi_c(t\gamma)}{1 + c\psi_c(t\gamma)} \nu(dt)$$

with $\psi_c = \lim_{c_N \rightarrow c} \psi$. Moreover, if $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i(C_N)} \rightarrow \nu^c$ weakly, then

$$\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i(\hat{C}_N)} \xrightarrow{\text{a.s.}} \mu$$

weakly with μ a probability measure with continuous density of bounded support \mathcal{S} , the Stieltjes transform $m(z)$ of which is given for $z \in \mathbb{C}^+$ by

$$m(z) = -\frac{1}{z} \int \frac{1}{1 + \tilde{\delta}(z)t} \nu^c(dt)$$

where $\tilde{\delta}(z)$ is the unique solution in \mathbb{C}^+ of the equations in $\tilde{\delta}$

$$\tilde{\delta} = -\frac{1}{z} \int \frac{\psi_c(t\gamma^\infty)}{\gamma^\infty + \psi_c(t\gamma^\infty)\tilde{\delta}} \nu(dt)$$

$$\delta = -\frac{c}{z} \int \frac{t}{1 + t\tilde{\delta}} \nu^c(dt).$$

Finally, for every closed set $\mathcal{A} \subset \mathbb{R}$ with $\mathcal{A} \cap \mathcal{S} = \emptyset$,

$$\left| \left\{ \lambda_i(\hat{C}_N) \right\}_{i=1}^N \cap \mathcal{A} \right| \xrightarrow{\text{a.s.}} 0.$$

Proof. We use the fact that $\gamma_N \xrightarrow{\text{a.s.}} \gamma^\infty$ (γ_N defined in Theorem 2) which is a consequence of $\psi/(1+c_N\psi)$ being monotonous and γ_N uniformly bounded, Lemma 1. The rest unfolds from classical random matrix techniques.

Figs. 1 and 2 depict the empirical histogram of the eigenvalues of \hat{C}_N and \hat{S}_N , for $N = 500$ and $n = 2500$ with $u(t) = (1+\alpha)/(t+\alpha)$, $\alpha = 0.1$, $C_N = \text{diag}(I_{125}, 3I_{125}, 10I_{250})$, and τ_1, \dots, τ_n i.i.d. with $\Gamma(0.5, 2)$ distribution. In thick line is also depicted the density of μ_N in Corollary 1 which shows an accurate match to the empirical spectrum as predicted by (6). As a comparison, Fig. 3 shows the empirical histogram of the eigenvalues of the sample covariance matrix $\frac{1}{n} \sum_{i=1}^n x_i x_i^*$ under the same parametrization against the deterministic equivalent density for this model in thick line [42]. This graph presents a large eigenvalue spectrum support, seemingly unboundedly growing with N , which is indeed expected according to Couillet and Hachem [12, Proposition 3.4] as τ_1 has unbounded support; this is to be compared against the provably uniformly bounded spectrum of \hat{C}_N (owing again to Couillet and Hachem [12, Proposition 3.4] and the uniform boundedness of $\nu(x)$ and $\|C_N\|$). Also note the gain of separability in the spectrum of \hat{C}_N which exhibits clearly three compact subsets of eigenvalues, reminiscent of the three masses in the eigenvalue distribution of C_N , while $\frac{1}{n} \sum_{i=1}^n x_i x_i^*$ exhibits a single compact set of eigenvalues.

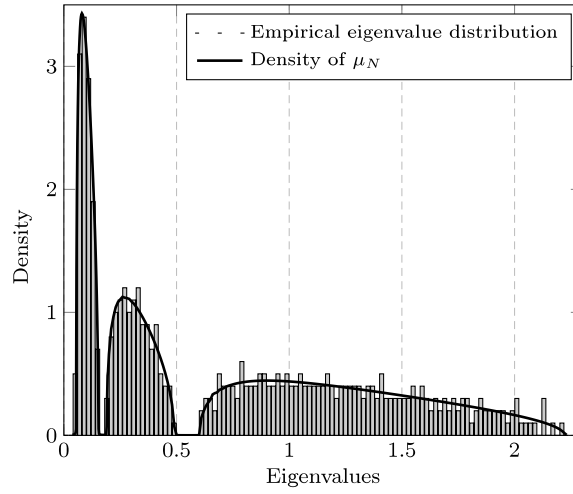


Fig. 1. Histogram of the eigenvalues of \hat{C}_N for $n = 2500$, $N = 500$, $C_N = \text{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(0.5, 2)$ -distribution.

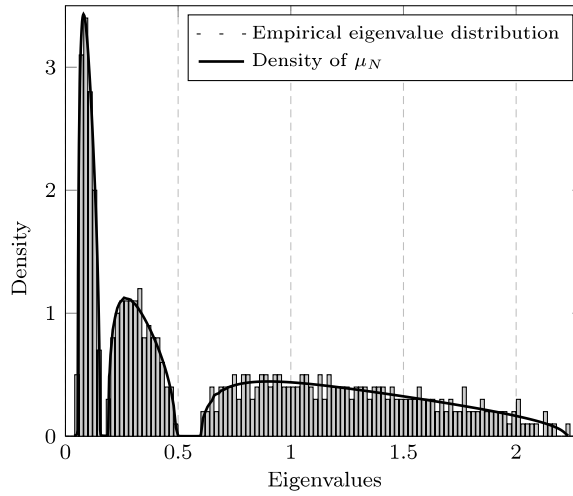


Fig. 2. Histogram of the eigenvalues of \hat{S}_N for $n = 2500$, $N = 500$, $C_N = \text{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(0.5, 2)$ -distribution.

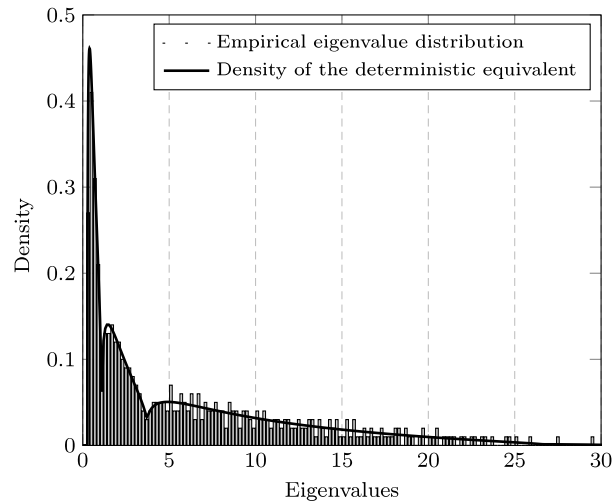


Fig. 3. Histogram of the eigenvalues of $\frac{1}{n} \sum_{i=1}^n x_i x_i^*$ for $n = 2500$, $N = 500$, $C_N = \text{diag}(I_{125}, 3I_{125}, 10I_{250})$, τ_1 with $\Gamma(0.5, 2)$ -distribution.

Both remarks have major consequences from detection and estimation purposes in signal processing applications of robust estimation, where relevant system information is often carried in the largest eigenvalues, ideally found sufficiently far from the “noise” eigenvalues. As such, from a practical standpoint, it is expected that robust estimators would allow for an improved separation between information and noise in impulsive data setting. This behavior is in fact confirmed in the companion article Couillet [10] which extends [Theorem 2](#) to a practical information-plus-impulsive noise array processing setting and shows outstanding performance improvements in detection and parameter estimation. These applied considerations however fall beyond the scope of the present article and shall no longer be discussed here.

In the next section, we present the proofs of [Theorems 1](#) and [2](#).

4. Proof of the main results

For the sake of definition, we take all variables to be complex here although the arguments are also valid for real random variables.

4.1. Proof of [Theorem 1](#)

As mentioned in [Section 2](#), we can assume without generality restriction that $C_N = I_N$. Indeed, if \hat{C}_N is the unique solution to (1) assuming $C_N = I_N$, then, for any other choice of $C_N \succ 0$, $C_N^{\frac{1}{2}} \hat{C}_N C_N^{\frac{1}{2}}$ is the unique solution to the corresponding model in (1). Hence, we only need to prove the result for $C_N = I_N$.

Consider a growing sequence $\{x_1, \dots, x_n\}_{n=1}^{\infty}$ according to [Assumptions 1](#) and [2](#). Since $|\{\tau_i = 0\}| = nv_n(\{0\}) < n(1 - c_+)$ for all large n a.s. ([Assumption 2-2](#)), $n - |\{\tau_i = 0\}| > c_+n > N + 1$ which, along with z_1, \dots, z_n being normalized Gaussian vectors, ensures that $\{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$ spans \mathbb{C}^N for all j for all large n a.s. As long as n is large enough, we can therefore almost surely define $h = (h_1, \dots, h_n)$ with $h_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ given by

$$h_j(q_1, \dots, q_n) = \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v(\tau_i q_i) z_i z_i^* \right)^{-1} z_j.$$

As shown in [Section 2.1](#), in order to show that \hat{C}_N is uniquely defined, it suffices to show that there exists a unique q_1, \dots, q_n such that for each j , $q_j = h_j(q_1, \dots, q_n)$. For this, we show first that h satisfies the following properties with probability one:

- (a) Nonnegativity: For each $q_1, \dots, q_n \geq 0$ and each i , $h_i(q_1, \dots, q_n) > 0$
- (b) Monotonicity: For each $q_1 \geq q'_1, \dots, q_n \geq q'_n$ and each i , $h_i(q_1, \dots, q_n) \geq h_i(q'_1, \dots, q'_n)$
- (c) Scalability: For each $\alpha > 1$ and each i , $\alpha h_i(q_1, \dots, q_n) > h_i(\alpha q_1, \dots, \alpha q_n)$.

Item (a) is obvious since the matrix inverse is well defined for all n large and $z_i \neq 0$ almost surely. Item (b) follows from the fact that, for two Hermitian matrices $A \geq B > 0$, $B^{-1} \geq A^{-1} \succ 0$ [[19](#), Corollary 7.7.4], and from v being non-increasing, entailing h_i to be a non-decreasing function of each q_j . As for Item (c), it follows also from the previous matrix inverse relation and from ψ being increasing, entailing in particular that, for $\alpha > 1$, $\psi(\alpha q_i) > \psi(q_i)$ if $q_i \neq 0$ so that $v(\alpha q_i) > v(q_i)/\alpha$ for $q_i \geq 0$.

According to [Yates \[41, Theorem 2\]](#), h is then a *standard interference function* and, if there exists q_1, \dots, q_n such that for each i , $q_i > h_i(q_1, \dots, q_n)$ (feasibility condition), then there is a unique $\{q_1, \dots, q_n\}$ satisfying $q_i = h_i(q_1, \dots, q_n)$ for each i , which is given by $q_i = \lim_{t \rightarrow \infty} q_i^{(t)}$ with $q_i^{(0)} \geq 0$ arbitrary and, for $t \geq 0$, $q_i^{(t+1)} = h_i(q_1^{(t)}, \dots, q_n^{(t)})$ (which would then conclude the proof). To obtain the feasibility condition, note that the function $q \mapsto \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \psi(\tau_i q) z_i z_i^* \right)^{-1} z_j$

is decreasing and, as $q \rightarrow \infty$, has limit $\frac{1 - c_N \phi_{\infty}}{\phi_{\infty}} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j, \tau_i \neq 0} z_i z_i^* \right)^{-1} z_j$. As $\{\tau_i\}_{i=1}^n$ and $\{z_i\}_{i=1}^n$ are independent and $\limsup_n N/|\{\tau_i \neq 0\}| = \limsup c_N/(1 - v_n(\{0\})) < 1$ a.s. ([Assumptions 2](#) and [1](#)), for all large n a.s., we fall within the hypotheses of [Lemma 6](#) in the [Appendix](#) and we can then write,³

$$\max_{1 \leq j \leq n} \left| (1 - v_n(\{0\})) \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \neq 0} z_i z_i^* \right)^{-1} z_j - 1 \right| \xrightarrow{\text{a.s.}} 0.$$

Assume first that $\tau_j \neq 0$. Then, using the relation

$$\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \neq 0, i \neq j} z_i z_i^* \right)^{-1} z_j = \frac{\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \neq 0} z_i z_i^* \right)^{-1} z_j}{1 - c_N \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \neq 0} z_i z_i^* \right)^{-1} z_j}$$

³ To be more exact, since $|\{\tau_i \neq 0\}|$ is random with probability space \mathcal{T} producing the τ_i 's, [Lemma 6](#) applies only on a subset of probability one of \mathcal{T} . It then suffices to apply Tonelli's theorem [[8](#)] to ensure that [Lemma 6](#) can be extended and still holds with probability one on the product space producing the (τ_i, z_i) .

and the fact that for all large n a.s. $1 - \nu_n(\{0\}) > c_+$, we have

$$\max_{j, \tau_j \neq 0} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \neq 0, i \neq j} z_i z_i^* \right)^{-1} z_j - \frac{1}{1 - \nu_n(\{0\}) - c_N} \right| \xrightarrow{\text{a.s.}} 0.$$

Therefore, using the fact that $\nu_n(\{0\}) < 1 - \phi_\infty^{-1}$ for all n large a.s. (Assumption 2-2), we have that for all j with $\tau_j \neq 0$

$$\frac{1 - c_N \phi_\infty}{\phi_\infty} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \neq 0, i \neq j} z_i z_i^* \right)^{-1} z_j < 1. \quad (9)$$

If instead $\tau_j = 0$, then

$$\max_{j, \tau_j = 0} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \neq 0} z_i z_i^* \right)^{-1} z_j - \frac{1}{1 - \nu_n(\{0\})} \right| \xrightarrow{\text{a.s.}} 0$$

and we find also the inequality (9) for all large n a.s. and for all j with $\tau_j = 0$, using once more $\nu_n(\{0\}) < 1 - \phi_\infty^{-1}$. As such, (9) is valid for all $j \in \{1, \dots, n\}$.

We can then choose n large enough so that (9) holds for all j , after which, taking q sufficiently large,

$$\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \psi(\tau_i q) z_i z_i^* \right)^{-1} z_j < 1$$

which is equivalent to

$$\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} v(\tau_i q) \tau_i z_i z_i^* \right)^{-1} z_j < q$$

for all j , i.e., $h_j(q, \dots, q) < q$. This ensures feasibility for all large n a.s. and concludes the proof.

4.2. Proof of Theorem 2

Similar to the proof of Theorem 1, we can restrict ourselves to the assumption that $C_N = I_N$. The generalization to C_N satisfying Assumption 2-3 will follow straightforwardly. We therefore take $C_N = I_N$ in what follows.

We start the proof by introducing the following fundamental lemmas (note that these lemmas in fact hold true irrespective of $C_N > 0$).

Lemma 1. Let Assumption 1 hold and let $h : [0, \infty) \rightarrow [0, \infty)$ be given by

$$h(\gamma) = \left(\frac{1}{n} \sum_{i=1}^n \frac{\tau_i v(\tau_i \gamma)}{1 + c_N \tau_i v(\tau_i \gamma) \gamma} \right)^{-1} = \begin{cases} \gamma \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)} \right)^{-1}, & \gamma > 0 \\ \frac{1}{v(0)} \left(\frac{1}{n} \sum_{i=1}^n \tau_i \right)^{-1}, & \gamma = 0. \end{cases}$$

Then, for all large n a.s., there exists a unique $\gamma_N > 0$ satisfying $\gamma_N = h(\gamma_N)$, given by

$$\gamma_N = \lim_{t \rightarrow \infty} \gamma_N^{(t)}$$

with $\gamma_N^{(0)} \geq 0$ arbitrary and, for $t \geq 0$, $\gamma_N^{(t+1)} = h(\gamma_N^{(t)})$. Moreover, with probability one,

$$\gamma_- < \liminf_N \gamma_N \leq \limsup_N \gamma_N < \gamma_+$$

for some $\gamma_-, \gamma_+ > 0$ finite.

Proof. As in the proof of Theorem 1, we show that h (scalar-valued this time) is a standard interference function. We show easily positivity, monotonicity and scalability of h . Indeed, for $\gamma \geq 0$, $h(\gamma) > 0$. For $\gamma \geq \gamma' \geq 0$,

$$\frac{h(\gamma) - h(\gamma')}{h(\gamma)h(\gamma')} = \frac{1}{n} \sum_{i=1}^n \frac{\tau_i (v(\tau_i \gamma') - v(\tau_i \gamma)) + (\gamma - \gamma') c_N \tau_i^2 v(\tau_i \gamma) v(\tau_i \gamma')}{(1 + c_N \tau_i v(\tau_i \gamma) \gamma)(1 + c_N \tau_i v(\tau_i \gamma') \gamma')} \geq 0$$

which follows from v being nonnegative decreasing. Finally, for $\alpha > 1$, $\alpha h(0) > h(0)$ and for $\gamma \neq 0$,

$$\begin{aligned} h(\alpha\gamma) &= \alpha\gamma \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \alpha\gamma)}{1 + c_N \psi(\tau_i \alpha\gamma)} \right)^{-1} < \alpha\gamma \left(\frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)} \right)^{-1} \\ &= \alpha h(\gamma) \end{aligned}$$

which follows from $\gamma \mapsto \psi(\tau_i \gamma)(1 + c_N \psi(\tau_i \gamma))^{-1}$ being increasing as long as $\tau_i \neq 0$. It remains to prove the existence of a γ such that $\gamma > h(\gamma)$, inducing by Yates [41, Theorem 2] the uniqueness of the fixed-point γ_N given by $\gamma_N = \lim_{t \rightarrow \infty} \gamma_N^{(t)}$ as stated in the theorem. For this, we use again the fact that $\gamma \mapsto \psi(\tau_i \gamma)(1 + c_N \psi(\tau_i \gamma))^{-1}$ is increasing and that (Assumption 2-2), for all large n a.s.

$$\lim_{\gamma \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)} = \frac{(1 - v_n(\{0\}))\psi_\infty^N}{1 + c_N \psi_\infty^N} = (1 - v_n(\{0\}))\phi_\infty > 1.$$

Therefore, there exists γ_0 (a priori dependent on the set $\{\tau_1, \dots, \tau_n\}$) such that, for all $\gamma > \gamma_0$, $h(\gamma) < \gamma$.

To prove uniform boundedness of γ_N , let $\varepsilon > 0$ and $m > 0$ be such that $(1 - \varepsilon)\phi_\infty > 1$ and $v_n((m, \infty)) > 1 - \varepsilon$ for all n large a.s. (always possible from Assumption 2-2). Then, for all n large a.s.

$$\frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)} > (1 - \varepsilon) \frac{\psi(m\gamma)}{1 + c_N \psi(m\gamma)} \rightarrow (1 - \varepsilon)\phi_\infty > 1$$

as $\gamma \rightarrow \infty$. Similar to γ_0 above, we can therefore choose γ_+ large enough, now independent of n large, such that, a.s. $\gamma \geq \gamma_+ \Rightarrow \gamma > h(\gamma)$, implying $\gamma_N < \gamma_+$ for these n large since $\gamma_N = h(\gamma_N)$. Also, $h(0) > 1/(2v(0))$ for all large n a.s. since $\frac{1}{n} \sum_{i=1}^n \tau_i \xrightarrow{\text{a.s.}} 1$ by Assumption 2. Hence, by the continuous growth of h , we can take $\gamma_- = 1/(2v(0)) > 0$ which is such that $\gamma \leq \gamma_- \Rightarrow h(\gamma) \geq h(0) > \gamma$ for all large n a.s. This implies $\gamma_N > \gamma_-$ for all large n a.s., which concludes the proof.

Remark 1. For further use, note that Lemma 1 can be refined as follows. Let (η, M_η) be couples indexed by η with $0 < \eta < 1$ and $M_\eta > 0$ such that $v_n((M_\eta, \infty)) < \eta$ for all large n a.s. (possible by tightness of v_n). Then, for sufficiently small η , the equation in γ

$$\gamma = \left(\frac{1}{n} \sum_{\tau_i \leq M_\eta} \frac{\tau_i v(\tau_i \gamma)}{1 + c_N \tau_i v(\tau_i \gamma)} \right)^{-1}$$

has a unique solution γ_N^η for all large n a.s. and there exists $\gamma_-, \gamma_+ > 0$ such that, for all $\eta < \eta_0$ small, $\gamma_- < \gamma_N^\eta < \gamma_+$ for all large n a.s.

Proof. The uniqueness is clear as long as $(1 - \eta_0)(1 - \limsup_n v_n(\{0\}))\phi_\infty > 1$ since then, exploiting the fact that $\lim_n \frac{|\{\tau_i \leq M_\eta\}|}{n} > 1 - \eta_0$ a.s.,

$$\lim_{\gamma \rightarrow \infty} \frac{1}{n} \sum_{\tau_i \leq M_\eta} \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)} = \frac{|\{\tau_i \leq M_\eta\}|}{n} (1 - v_n(\{0\}))\phi_\infty > 1$$

for all n large a.s. and the proof follows from the proof of Lemma 1. For uniform boundedness, taking $M_{\eta_0} < M_\eta$ large enough (or equivalently $\eta_0 > \eta$ small enough) such that $\liminf_n \frac{|\{m < \tau_i \leq M_\eta\}|}{n} > \liminf_n \frac{|\{m < \tau_i \leq M_{\eta_0}\}|}{n} > 1 - \varepsilon$ a.s. in the proof of Lemma 1 leads to the same upper bound result for all small $\eta < \eta_0$. As for the lower bound, we still have $h(0) > 1/(2v(0))$ for all large n a.s. independently of η so the result is maintained.

Lemma 2. Let Assumption 1 hold and define γ_N as in Lemma 1. Then, as $n \rightarrow \infty$,

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v(\tau_i \gamma_N) z_i z_i^* \right)^{-1} z_j - \gamma_N \right| \xrightarrow{\text{a.s.}} 0.$$

Proof. We first introduce some notations to simplify readability. First, we will write $z_j = \sqrt{N} A_N \tilde{y}_j / \|\tilde{y}_j\| \triangleq \sqrt{N} \tilde{z}_j / \|\tilde{y}_j\|$ with \tilde{y}_j zero-mean I_N -covariance Gaussian, hence \tilde{z}_j is zero-mean I_N -covariance Gaussian. With this notation, in what follows, we denote $A = \frac{1}{n} \sum_{i=1}^n \tau_i v(\tau_i \gamma_N) z_i z_i^*$, $A_{(j)} = A - \frac{1}{n} \tau_j v(\tau_j \gamma_N) z_j z_j^*$, $\tilde{A} = \frac{1}{n} \sum_{i=1}^n \tau_i v(\tau_i \gamma_N) \tilde{z}_i \tilde{z}_i^*$ and $\tilde{A}_{(j)} = \tilde{A} - \frac{1}{n} \tau_j v(\tau_j \gamma_N) \tilde{z}_j \tilde{z}_j^*$.

We first show that there exists $\eta > 0$ such that, for all large n a.s.

$$\min_{1 \leq j \leq n} \lambda_1(A_{(j)}) > \eta \tag{10}$$

(recall that λ_1 stands for the smallest eigenvalue). For this, take $0 < \varepsilon < 1 - c_+$ and $m > 0$ be such that $\nu_n((m, \infty)) > 1 - \varepsilon$ for all n large a.s. (Assumption 2-2). Using the fact that $xv(x) = \psi(x)$ is non-decreasing and that any subtraction of a nonnegative definite matrix cannot increase the smallest eigenvalue, we have

$$\begin{aligned} \min_{1 \leq j \leq n} \lambda_1(A_{(j)}) &\geq \min_{1 \leq j \leq n} \lambda_1 \left(\frac{1}{n} \sum_{i \neq j, \tau_i \geq m} \frac{\psi(\tau_i \gamma_N)}{\gamma_N} z_i z_i^* \right) \\ &\geq \frac{\psi(m \gamma_N)}{\gamma_N} \min_{1 \leq j \leq n} \lambda_1 \left(\frac{1}{n} \sum_{i \neq j, \tau_i \geq m} z_i z_i^* \right). \end{aligned} \quad (11)$$

Since $\nu_n((m, \infty)) > 1 - \varepsilon$ for all n large a.s.,

$$0 < c_- < \liminf_n \frac{N}{|\{\tau_i \geq m\}|} \leq \limsup_n \frac{N}{|\{\tau_i \geq m\}|} < \frac{c_+}{1 - \varepsilon} < 1.$$

From Lemma 6 in the Appendix (see footnote in the proof of Theorem 1 for details), we can then write

$$\begin{aligned} \min_{1 \leq j \leq n} \lambda_1(A_{(j)}) &\geq \frac{\psi(m \gamma_N)}{\gamma_N} \nu_n((m, \infty)) \min_{1 \leq j \leq n} \lambda_1 \left(\frac{1}{|\{\tau_i \geq m\}|} \sum_{i \neq j, \tau_i \geq m} z_i z_i^* \right) \\ &> \frac{\psi(m \gamma_N)}{\gamma_N} (1 - \varepsilon) \eta' \end{aligned}$$

for some $\eta' > 0$ which, along with the almost sure boundedness of γ_N (Lemma 1) proves (10).

Now that (10) is acquired, let $E_{\tilde{y}_j}$ denote the expectation with respect to \tilde{y}_j (i.e., conditionally on the sigma-field engendered by the \tilde{y}_i , $i \neq j$, and the τ_i) and $\kappa_j \triangleq 1_{\{\lambda_1(A_{(j)}) > \eta\}}$ with η as defined in (10). From Bai and Silverstein [3, Lemma B.26] (which applies here since \tilde{z}_j and $\kappa_j^{1/p} A_{(j)}^{-1}$ are independent), for $p > 2$,

$$E_{\tilde{y}_j} \left[\kappa_j \left| \frac{1}{N} \tilde{z}_j^* A_{(j)}^{-1} \tilde{z}_j - \frac{1}{N} \text{tr} A_{(j)}^{-1} \right|^p \right] \leq \frac{\kappa_j K_p}{N^{\frac{p}{2}}} \left[\left(\frac{\zeta_4}{N} \text{tr} A_{(j)}^{-2} \right)^{\frac{p}{2}} + \frac{\zeta_{2p}}{N^{\frac{p}{2}}} \text{tr} A_{(j)}^{-p} \right]$$

for ζ_ℓ any upper bound on $E[|\tilde{z}_{ij}|^\ell]$ and K_p a constant dependent only on p . From the definition of κ_j , we have $\kappa_j \|A_{(j)}^{-1}\| < \eta^{-1}$, so that, using $\frac{1}{N} \text{tr} B \leq \|B\|$ for nonnegative definite $B \in \mathbb{C}^{N \times N}$,

$$E_{\tilde{y}_j} \left[\kappa_j \left| \frac{1}{N} \tilde{z}_j^* A_{(j)}^{-1} \tilde{z}_j - \frac{1}{N} \text{tr} A_{(j)}^{-1} \right|^p \right] \leq \frac{K_p}{\eta^p N^{\frac{p}{2}}} \left(\zeta_4^{\frac{p}{2}} + \frac{\zeta_{2p}}{N^{\frac{p}{2}-1}} \right).$$

This bound being irrespective of all z_i and τ_i , $i \neq j$, we can take the expectation with respect to all y_i , $i \neq j$, and all τ_i to obtain

$$E \left[\kappa_j \left| \frac{1}{N} \tilde{z}_j^* A_{(j)}^{-1} \tilde{z}_j - \frac{1}{N} \text{tr} A_{(j)}^{-1} \right|^p \right] = \mathcal{O} \left(\frac{1}{N^{\frac{p}{2}}} \right).$$

Taking $p > 4$ and applying the union bound, Markov inequality, and Borel–Cantelli lemma finally shows that

$$\max_{1 \leq j \leq n} \kappa_j \left| \frac{1}{N} \tilde{z}_j^* A_{(j)}^{-1} \tilde{z}_j - \frac{1}{N} \text{tr} A_{(j)}^{-1} \right| \xrightarrow{\text{a.s.}} 0. \quad (12)$$

With the same arguments on κ_j and with the same p as above, now remark that

$$\begin{aligned} E_{\tilde{y}_j} \left[\kappa_j \left| \frac{1}{N} z_j^* A_{(j)}^{-1} z_j - \frac{1}{N} \tilde{z}_j^* A_{(j)}^{-1} \tilde{z}_j \right|^p \right] &= E_{\tilde{y}_j} \left[\kappa_j \left| \frac{1}{N} z_j^* A_{(j)}^{-1} z_j \left(1 - \frac{\|\tilde{y}_j\|^2}{N} \right) \right|^p \right] \\ &\leq \frac{1}{\eta^p} E_{\tilde{y}_j} \left[\left| 1 - \frac{\|\tilde{y}_j\|^2}{N} \right|^p \right] = \mathcal{O} \left(\frac{1}{N^{p/2}} \right) \end{aligned}$$

since $\bar{N} \geq N$, again by Bai and Silverstein [3, Lemma B.26]. Therefore, by the union bound, Markov inequality, and Borel–Cantelli lemma,

$$\max_{1 \leq j \leq n} \kappa_j \left| \frac{1}{N} z_j^* A_{(j)}^{-1} z_j - \frac{1}{N} \tilde{z}_j^* A_{(j)}^{-1} \tilde{z}_j \right| \xrightarrow{\text{a.s.}} 0. \quad (13)$$

Combining (12) and (13) along with the fact that $\min_{1 \leq j \leq n} \kappa_j \xrightarrow{\text{a.s.}} 1$ (from (10)) finally gives

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} z_j^* A_{(j)}^{-1} z_j - \frac{1}{N} \text{tr} A_{(j)}^{-1} \right| \xrightarrow{\text{a.s.}} 0.$$

By (10), $A_{(j)} = (A_{(j)} - \frac{\eta}{2} I_N) + \frac{\eta}{2} I_N$ with $\liminf_n \lambda_1(A_{(j)} - \frac{\eta}{2} I_N) > 0$ a.s., so we are in the conditions of Lemma 4 and we have

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} \text{tr} A_{(j)}^{-1} - \frac{1}{N} \text{tr} A^{-1} \right| \xrightarrow{\text{a.s.}} 0.$$

It remains to find a deterministic equivalent for $\frac{1}{N} \text{tr} A^{-1}$. Similar to above, note first that, for all large n a.s.

$$\left| \frac{1}{N} \text{tr} A^{-1} - \frac{1}{N} \text{tr} \tilde{A}^{-1} \right| \leq \frac{1}{\eta^2} \frac{\psi_\infty}{\gamma_N} \max_{1 \leq j \leq n} \left| \frac{1 - \tilde{N}^{-1} \|\tilde{y}_j\|^2}{\tilde{N}^{-1} \|\tilde{y}_j\|^2} \right| \left\| \frac{1}{n} \sum_{i=1}^n \tilde{z}_i \tilde{z}_i^* \right\|$$

where we used the definition and boundedness of ψ and standard matrix inversion formulas. From Bai and Silverstein [2], the right hand side converges almost surely to zero, so that it is equivalent to consider z_i or \tilde{z}_i . Now, the trace $\frac{1}{N} \text{tr} \tilde{A}^{-1}$ is exactly the Stieltjes transform $\hat{m}_N(z)$ of the matrix \tilde{A} evaluated at point $z = 0$. Since $\lambda_1(\tilde{A}) \geq \lambda_1(\tilde{A}_{(1)}) > \eta$ for all large n a.s. and since $\tau_i v(\tau_i \gamma_N) = \psi(\tau_i \gamma_N) \gamma_N^{-1}$ is uniformly bounded across i and n (from the boundedness of ψ and Lemma 1), from standard random matrix results (e.g., [11]),⁴ we have

$$\hat{m}_N(0) - m_N(0) \xrightarrow{\text{a.s.}} 0$$

where $m_N(0)$ is the unique nonnegative solution to the equation in m (as long as at least one τ_i is non-zero)

$$m = \left(\frac{1}{n} \sum_{i=1}^n \frac{\tau_i v(\tau_i \gamma_N)}{1 + c_N \tau_i v(\tau_i \gamma_N) m} \right)^{-1}.$$

Now, by definition, γ_N coincides with such a solution. By uniqueness of $m_N(0)$, one must then have $m_N(0) = \gamma_N$ so that, gathering all results together,

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} z_j^* A_{(j)}^{-1} z_j - \gamma_N \right| \xrightarrow{\text{a.s.}} 0$$

which completes the proof.

Remark 2. Similar to Remark 1, note that Lemma 2 can be further extended to

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \leq M_\eta, i \neq j} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^* \right)^{-1} z_j - \gamma_N^\eta \right| \xrightarrow{\text{a.s.}} 0$$

for some η small enough, with M_η and γ_N^η defined in Remark 1.

Proof. One shows boundedness of $\lambda_1(\frac{1}{n} \sum_{\tau_i \leq M_\eta, i \neq j} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^*)$ simply by taking η for which $\nu_n((m, M_\eta)) > 1 - \varepsilon$ for all large n a.s. in the proof of Lemma 2. Then it suffices to adapt all derivations by substituting τ_i by zero if $\tau_i > M_\eta$. The result follows straightforwardly.

The two lemmas above are standard random matrix results on x_1, \dots, x_n , independent of the structure of \hat{C}_N . The next lemma introduces a first result on the matrix \hat{C}_N which will be fundamental in what follows. Recall that we denoted $d_i = \frac{1}{N} z_i^* \hat{C}_{(i)}^{-1} z_i$, with $\hat{C}_{(i)} = \hat{C}_N - \frac{1}{n} v(\tau_i d_i) \tau_i z_i z_i^*$.

Lemma 3 (Boundedness of the d_i). *There exist $d_+ > d_- > 0$ such that, for all large n a.s.,*

$$d_- < \liminf_n \min_{1 \leq i \leq n} d_i \leq \limsup_n \max_{1 \leq i \leq n} d_i < d_+.$$

⁴ More precisely, Couillet et al. [11] shows that $\hat{m}_N(z) - m_N(z) \xrightarrow{\text{a.s.}} 0$ for all points z with $\Im[z] > 0$. Using $\lambda_1(\tilde{A}) > \eta$ for all large n a.s., the proof can be generalized to all $z \in \mathbb{C}$ with positive distance to $[\eta, \infty)$ by turning the bounds in $1/|\Im[z]|$ into $1/d(z, [\eta, \infty))$ with d denoting the Hausdorff distance, so for $z = 0$. The existence of $m_N(0)$ is in particular already obtained in the generalization of the existence result of Couillet et al. [11, Appendix A–C], this time for $z = 0$. The proof of uniqueness of $m_N(0)$ can then be checked by standard interference function arguments, where feasibility follows in particular from the right-hand behaving as $c_N m < m$ from Assumption 1.

Proof. Let us denote $d_{\max} = \max_{1 \leq i \leq n} d_i$ and $d_{\min} = \min_{1 \leq i \leq n} d_i$. Take $j \in \{1, \dots, n\}$ arbitrary and, for $0 < \varepsilon < 1 - \phi_\infty^{-1} < 1 - c_+$, take $m > 0$ such that for all large n a.s. $\nu_n([m, \infty)) > 1 - \varepsilon$ (Assumption 2-2). Then, using the fact that v is non-increasing while ψ is non-decreasing,

$$\begin{aligned} \hat{C}_{(j)} &\geq \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} \tau_i v(\tau_i d_i) z_i z_i^* = \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} \frac{\psi(\tau_i d_i)}{d_i} z_i z_i^* \geq \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} \frac{\psi(m d_i)}{d_i} z_i z_i^* \\ &= \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} m v(m d_i) z_i z_i^* \geq m v(m d_{\max}) \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^*. \end{aligned} \quad (14)$$

The right-hand side matrix is invertible for n large since $|\{\tau_i \geq m\}| > nc_+ > N$ for all large n a.s. Therefore, choosing j to be such that $d_{\max} = \frac{1}{N} z_j^* \hat{C}_{(j)}^{-1} z_j$, and using $A \geq B > 0 \Rightarrow B^{-1} \geq A^{-1}$ for Hermitian A, B matrices,

$$d_{\max} \leq \frac{1}{m v(m d_{\max})} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \geq m, i \neq j} z_i z_i^* \right)^{-1} z_j.$$

This implies

$$\psi(m d_{\max}) \leq \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \geq m, i \neq j} z_i z_i^* \right)^{-1} z_j$$

which can be rewritten, from the definition of ψ ,

$$\phi(g^{-1}(m d_{\max})) \leq \frac{\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \geq m, i \neq j} z_i z_i^* \right)^{-1} z_j}{1 + c_N \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\tau_i \geq m, i \neq j} z_i z_i^* \right)^{-1} z_j}. \quad (15)$$

From Lemma 6 in the Appendix and the fact that $\nu_n([m, \infty)) = n^{-1} |\{\tau_i \geq m\}| > 1 - \varepsilon$ for all large n a.s., we then have for all large n a.s.

$$\begin{aligned} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^* \right)^{-1} z_j &= \frac{1}{\nu_n([m, \infty))} \frac{1}{N} z_j^* \left(\frac{1}{|\{\tau_i \geq m\}|} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^* \right)^{-1} z_j \\ &< \frac{1}{1 - \varepsilon} \frac{1}{1 - \frac{c_N}{1 - \varepsilon}} = \frac{1}{1 - c_N - \varepsilon}. \end{aligned}$$

Now, since $t \mapsto t/(1 + c_N t)$ is increasing, for all large n a.s.

$$\frac{\frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^* \right)^{-1} z_j}{1 + c_N \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \geq m}} z_i z_i^* \right)^{-1} z_j} < \frac{1}{1 - c_N - \varepsilon} \frac{1}{1 + c_N \frac{1}{1 - c_N - \varepsilon}} = \frac{1}{1 - \varepsilon}.$$

As $\varepsilon < 1 - \phi_\infty^{-1}$, $(1 - \varepsilon)^{-1} < \phi_\infty$ so that, from the inequality above, we can apply ϕ^{-1} on both sides of (15) to obtain, for all large n a.s.

$$g^{-1}(m d_{\max}) \leq \phi^{-1} \left(\frac{1}{1 - \varepsilon} \right)$$

hence

$$d_{\max} \leq \frac{1}{m} g \left(\phi^{-1} \left(\frac{1}{1 - \varepsilon} \right) \right)$$

from which d_{\max} is uniformly bounded for all large n a.s. by say d_+ .

To proceed to d_{\min} , note similarly that we can write

$$\begin{aligned}\hat{C}_{(j)} &\leq \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} M v(M d_{\min}) z_i z_i^* + \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i v(\tau_i d_i) z_i z_i^* \\ &\leq \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} M v(M d_{\min}) z_i z_i^* + v(0) \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i z_i z_i^*\end{aligned}$$

for any $M > 0$. Selecting j meeting the minimum for d_j , we then have

$$d_{\min} \geq \frac{1}{M v(M d_{\min})} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* + \frac{v(0)}{M v(M d_{\min})} \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i z_i z_i^* \right)^{-1} z_j$$

which, for all large n a.s., satisfies

$$d_{\min} \geq \frac{1}{M v(M d_{\min})} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* + \frac{v(0)}{M v(M d_+)} \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i z_i z_i^* \right)^{-1} z_j$$

or equivalently

$$\psi(M d_{\min}) \geq \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* + \frac{v(0)}{M v(M d_+)} \frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i > M}} \tau_i z_i z_i^* \right)^{-1} z_j.$$

With the same arguments as in the proof of [Lemma 2](#), note that, taking M large enough

$$\lim_n \inf \left\{ \inf_j \lambda_1 \left(\frac{1}{n} \sum_{\substack{i \neq j, \tau_i \leq M}} z_i z_i^* \right) \right\} > 0$$

almost surely (from [Lemma 6](#) and since $\lim_n \inf_n v_n((M, \infty)) \rightarrow 1$ a.s. as $M \rightarrow \infty$). We can then apply [Lemma 5](#) to obtain, along with [Lemma 4](#), Markov inequality, and Borel–Cantelli lemma arguments,

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* + E_M \right)^{-1} z_j - \frac{1}{N} \text{tr} \left(\frac{1}{n} \sum_{\tau_i \leq M} z_i z_i^* + E_M \right)^{-1} \right| \xrightarrow{\text{a.s.}} 0 \quad (16)$$

where we defined $E_M = \frac{v(0)}{M v(M d_+)} \frac{1}{n} \sum_{\tau_i > M} \tau_i z_i z_i^*$. Now, E_M is of maximum rank $|\{\tau_i > M\}|$. Taking M large enough to ensure $v_n((M, \infty)) = |\{\tau_i > M\}|/n < c_- \varepsilon'$ for some $\varepsilon' > 0$ arbitrary, we then have from $|\{\tau_i > M\}|$ applications of [Lemma 4](#)

$$\left| \frac{1}{N} \text{tr} \left(\frac{1}{n} \sum_{\tau_i \leq M} z_i z_i^* + E_M \right)^{-1} - \frac{1}{N} \text{tr} \left(\frac{1}{n} \sum_{\tau_i \leq M} z_i z_i^* \right)^{-1} \right| \leq \varepsilon'.$$

This and (16) give for all large n a.s.

$$\psi(M d_{\min}) \geq \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \neq j \\ \tau_i \leq M}} z_i z_i^* \right)^{-1} z_j + 2\varepsilon'$$

for all large n almost surely. From there, it suffices to proceed similar to the boundedness proof for d_{\max} starting from (15) with inequality signs reverted and accounting for ε' arbitrarily small. This shows finally that d_{\min} is uniformly bounded away from zero and this completes the proof.

Equipped with [Lemmas 1–3](#), we are now in position to develop the core of the proof. For readability, we divide the proof in two parts. In the first part, we will assume that τ_1, \dots, τ_n have a uniformly bounded support. This will greatly simplify the calculus and will allow for a better understanding of the main arguments; in particular, the technical [Assumption 3](#) will be irrelevant in this part. Then in a second part, we relax the boundedness assumption and fully exploit [Assumption 3](#) in a more technical proof.

4.2.1. Bounded τ_i

First assume $\tau_1, \dots, \tau_n \leq M$ a.s. for some $M > 0$. We follow here a similar path as in [15] but slightly more involved. Define

$$e_i \triangleq \frac{v(\tau_i d_i)}{v(\tau_i \gamma_N)} > 0 \quad (17)$$

with γ_N the value given by Lemma 1 and with d_i still defined as $d_i = \frac{1}{N} z_i^* \hat{C}_{(i)}^{-1} z_i$. Up to labeling change, we reorder the e_i 's as $e_1 \leq \dots \leq e_n$. Our goal is to show that $e_1 \xrightarrow{\text{a.s.}} 1$ and $e_n \xrightarrow{\text{a.s.}} 1$ (hence $\max_{1 \leq i \leq n} |e_i - 1| \xrightarrow{\text{a.s.}} 0$), which we will prove by a contradiction argument.

For any $j = 1, \dots, n$, we have

$$\begin{aligned} e_j &= \frac{v\left(\tau_j \frac{1}{N} z_j^* \hat{C}_{(j)}^{-1} z_j\right)}{v(\tau_j \gamma_N)} \\ &= \frac{v\left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v(\tau_i d_i) z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)} \\ &= \frac{v\left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v(\tau_i \gamma_N) e_i z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)} \end{aligned} \quad (18)$$

$$\begin{aligned} &\leq \frac{v\left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v(\tau_i \gamma_N) e_n z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)} \\ &= \frac{v\left(\frac{\tau_j}{e_n} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v(\tau_i \gamma_N) z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)} \end{aligned} \quad (19)$$

where the inequality arises from v being non-increasing and from Horn and Johnson [19, Corollary 7.7.4]. Similarly, for each j ,

$$e_j \geq \frac{v\left(\frac{\tau_j}{e_1} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v(\tau_i \gamma_N) z_i z_i^*\right)^{-1} z_j\right)}{v(\tau_j \gamma_N)}. \quad (20)$$

From Lemma 2, let now $0 < \varepsilon_n < \gamma_N$, $\varepsilon_n \downarrow 0$, be such that, for all large n a.s. and for all $j \leq n$,

$$\gamma_N - \varepsilon_n < \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} \tau_i v(\tau_i \gamma_N) z_i z_i^*\right)^{-1} z_j < \gamma_N + \varepsilon_n.$$

In particular, since v is non-increasing, taking $j = n$ in (19) and applying the left-hand inequality,

$$e_n < \frac{v(e_n^{-1} \tau_n (\gamma_N - \varepsilon_n))}{v(\tau_n \gamma_N)}$$

or equivalently

$$\frac{e_n v(\tau_n \gamma_N)}{v(e_n^{-1} \tau_n (\gamma_N - \varepsilon_n))} < 1. \quad (21)$$

By the definition of ψ , this can be further rewritten

$$(1 - \varepsilon_n \gamma_N^{-1}) \frac{\psi(\tau_n \gamma_N)}{\psi(e_n^{-1} \tau_n \gamma_N (1 - \varepsilon_n \gamma_N^{-1}))} < 1. \quad (22)$$

Assume now that, for some $\ell > 0$, $e_n > 1 + \ell$ infinitely often and let us restrict the sequence e_n to those indexes for which $e_n > 1 + \ell$.

We distinguish two scenarios. First, assume that $\liminf_n \tau_n = 0$. Then, by the definition (17) and since both d_n and γ_N are uniformly bounded (Lemmas 1 and 3), on some subsequence $\{n_j\}$ satisfying $\lim_j \tau_{n_j} = 0$, $e_{n_j} \xrightarrow{\text{a.s.}} 1$, in contradiction with $e_n > 1 + \ell$.

We must then have $\liminf_n \tau_n > \tau_-$ for some $\tau_- > 0$ along with $\tau_n \leq M$ a.s. for some $M > 0$ (bounded τ_i assumption). Then, since γ_N is bounded and bounded away from zero for all large n a.s., so is $\tau_n \gamma_N$. Considering and restricting ourselves to a further subsequence over which $\tau_n \gamma_N \rightarrow x > 0$ and $c_N \rightarrow c$, we then have, with $\psi_c(x) = \lim_{c_N \rightarrow c} \psi(x)$ (recall that ψ depends on c_N through g),

$$\lim_n (1 - \varepsilon_n \gamma_N^{-1}) \frac{\psi(\tau_n \gamma_N)}{\psi(e_n^{-1} \tau_n \gamma_N (1 - \varepsilon_n \gamma_N^{-1}))} \geq \frac{\psi_c(x)}{\psi_c((1 + \ell)^{-1} x)} > 1 \quad (23)$$

which contradicts (22). Gathering the results and reconsidering the initial sequence e_n (i.e., not a subsequence) we then have, for each $\ell > 0$, $e_n \leq 1 + \ell$ for all large n a.s.

Symmetrically, we obtain that, for some $\varepsilon_n \downarrow 0$ and for all large n a.s.

$$\frac{e_1 v(\tau_1 \gamma_N)}{v(e_1^{-1} \tau_1 (\gamma_N + \varepsilon_n))} > 1.$$

From this, we conclude similar to above that, for each $\ell > 0$ small, $e_1 \geq 1 - \ell$, for all large n a.s. so that, finally

$$\max_{1 \leq i \leq n} |e_i - 1| \xrightarrow{\text{a.s.}} 0$$

or, by uniform boundedness of the τ_i and γ_N ,

$$\max_{1 \leq i \leq n} |v(\tau_i d_i) - v(\tau_i \gamma_N)| \xrightarrow{\text{a.s.}} 0.$$

Hence, letting $\ell > 0$ and recalling that $\tau_i v(\tau_i \gamma_N) = \psi(\tau_i \gamma_N) / \gamma_N$, for all large n a.s.

$$(1 - \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{\gamma_N} z_i z_i^* \leq \frac{1}{n} \sum_{i=1}^n v(\tau_i d_i) \tau_i z_i z_i^* \leq (1 + \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{\gamma_N} z_i z_i^*. \quad (24)$$

Therefore, since $\gamma_N > \gamma_-$ and $\|\frac{1}{n} \sum_{i=1}^n z_i z_i^*\| < (1 + \sqrt{c_+})^2$ for all large n a.s. [2],

$$\|\hat{C}_N - \hat{S}_N\| \leq \ell (1 + \sqrt{c_+})^2 \frac{\psi_\infty}{\gamma_-}$$

where $\hat{S}_N = \gamma_N^{-1} \frac{1}{n} \sum_{i=1}^n \psi(\tau_i \gamma_N) z_i z_i^*$. Since ℓ is arbitrary, the difference tends to zero a.s. as $n \rightarrow \infty$, which concludes the proof for $\tau_i < M$ a.s. and for $C_N = I_N$.

If $C_N \neq I_N$ is positive definite, remark simply that neither d_i nor γ_N are affected in their values, so that the effect of C_N first appears in (24) with z_i having $C_N \neq I_N$ as a covariance matrix. But then, in this case, since $\|\frac{1}{n} \sum_{i=1}^n z_i z_i^*\| < (1 + \sqrt{c_+})^2 \limsup_N \|C_N\| < \infty$ (Assumption 2), the last arguments still hold true and the result is also proved for these C_N .

Note the importance of the assumption on ϕ being increasing and not simply non-decreasing (as in [23]) to ensure that (23) is a strict inequality. If this were to be replaced by “ ≥ 1 ”, no contradiction with (22) could be evoked. There does not seem to be any easy way to work this limitation around. Similar reasons explain why Tyler robust estimator discussed in Section 5 cannot be analyzed in the same way as Maronna estimator. All the same, when τ_1, \dots, τ_n have unbounded support with growing n , the left-hand side of (23) may equal one provided $\limsup_n \tau_n = \infty$, which is not excluded. For this reason, a specific treatment is necessary where the set of $\{\tau_i\}_{i=1}^n$ is split into a large bounded set of τ_i and a small set of large τ_i . This is the approach followed in the second part of the proof below.

4.2.2. Unbounded τ_i

We now relax the boundedness assumption on the support of the distribution of τ_1 and use Assumption 3 instead.

Since $\{v_n\}_{n=1}^\infty$ is tight, we can exhibit pairs (η, M_η) with $\eta \downarrow 0$ as $M_\eta \uparrow \infty$ such that, for all large n a.s. $v_n((M_\eta, \infty)) < \eta$. Let us fix such a pair (η, M_η) with η small and restrict ourselves to a subsequence where $v_n((M_\eta, \infty)) < \eta$ for all n . Denote $\mathcal{C}_\eta = \{i, \tau_i \leq M_\eta\}$ with cardinality $|\mathcal{C}_\eta|/n = 1 - v_n((M_\eta, \infty))$.

We follow the same steps as in the previous proof but differentiating between indices in \mathcal{C}_η and indices in \mathcal{C}_η^c . Also we denote

$$e_i^\eta \triangleq \frac{v(\tau_i d_i)}{v(\tau_i \gamma_N^\eta)}$$

where γ_N^η is the unique positive solution to the equation in γ

$$1 = \frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \frac{\psi(\tau_i \gamma)}{1 + c_N \psi(\tau_i \gamma)}.$$

Recall first from [Remark 1](#) that the conclusions of [Lemma 1](#) are still valid and importantly in what follows, that $\gamma_- < \gamma_N^\eta < \gamma_+$ for some $\gamma_-, \gamma_+ > 0$, for all large N irrespective of $\eta < \eta_0$ for some η_0 small. This uniform control of γ_N^η with respect to η plays a key role here. For the moment, we do not make explicit the sufficiently small value of η_0 that is needed in the following; all what will matter is that we can always choose η arbitrarily small from here.

Let $j \in \mathcal{C}_\eta$ and denote ψ_∞ any upper bound on ψ_∞^N for all N . Then, similar to [\(18\)](#), with $e_1^\eta = \min_{i \in \mathcal{C}_\eta} \{e_i^\eta\}$ and $e_n^\eta = \max_{i \in \mathcal{C}_\eta} \{e_i^\eta\}$,

$$\begin{aligned} e_j^\eta &= \frac{v \left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \in \mathcal{C}_\eta \\ i \neq j}} \tau_i v(\tau_i \gamma_N^\eta) e_i^\eta z_i z_i^* + \frac{1}{n} \sum_{i \in \mathcal{C}_\eta^c} \tau_i v(\tau_i d_i) z_i z_i^* \right)^{-1} z_j \right)}{v(\tau_j \gamma_N^\eta)} \\ &\leq \frac{v \left(\tau_j \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \in \mathcal{C}_\eta \\ i \neq j}} \tau_i v(\tau_i \gamma_N^\eta) e_n^\eta z_i z_i^* + \frac{1}{n} \frac{\psi_\infty}{d_-} \sum_{i \in \mathcal{C}_\eta^c} z_i z_i^* \right)^{-1} z_j \right)}{v(\tau_j \gamma_N^\eta)} \\ &= \frac{v \left(\frac{\tau_j}{e_n^\eta} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \in \mathcal{C}_\eta \\ i \neq j}} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^* + \frac{1}{n} \frac{\psi_\infty}{e_n^\eta d_-} \sum_{i \in \mathcal{C}_\eta^c} z_i z_i^* \right)^{-1} z_j \right)}{v(\tau_j \gamma_N^\eta)} \end{aligned}$$

where the first inequality uses $d_i > d_-$ for all large n a.s. ([Lemma 3](#)). Since $e_n^\eta = \frac{v(\tau_n d_n)}{v(\tau_n \gamma_N^\eta)} = \frac{\psi(\tau_n d_n)}{\psi(\tau_n \gamma_N^\eta)} \frac{\gamma_N^\eta}{d_n}$, with the bounds derived previously ([Remark 1](#) and [Lemma 3](#)), e_n^η is almost surely bounded and bounded away from zero for all large n a.s., irrespective of η small enough (if $\liminf_n \tau_n = 0$, the first equality ensures $\liminf_n e_n^\eta > 0$ while if $\limsup_n \tau_n = \infty$, the second equality ensures $\limsup_n e_n^\eta < \infty$). Thus, in particular, $e_n^\eta > e_-$ for some $e_- > 0$ for all large n a.s. From this observation, for all large n a.s.

$$\begin{aligned} e_j^\eta &\leq \frac{v \left(\frac{\tau_j}{e_n^\eta} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \in \mathcal{C}_\eta \\ i \neq j}} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^* + \frac{1}{n} \frac{\psi_\infty}{e_n^\eta d_-} \sum_{i \in \mathcal{C}_\eta^c} z_i z_i^* \right)^{-1} z_j \right)}{v(\tau_j \gamma_N^\eta)} \\ &= \frac{v \left(\frac{\tau_j}{e_n^\eta} \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{\substack{i \in \mathcal{C}_\eta \\ i \neq j}} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^* \right)^{-1} z_j + w_{j,n} \right)}{v(\tau_j \gamma_N^\eta)} \end{aligned} \quad (25)$$

where we defined

$$w_{j,n} \triangleq \frac{1}{N} z_j^* (A_{\eta,(j)} + B_\eta)^{-1} z_j - \frac{1}{N} z_j^* A_{\eta,(j)}^{-1} z_j$$

with

$$A_{\eta,(j)} \triangleq \frac{1}{n} \sum_{\substack{i \in \mathcal{C}_\eta \\ i \neq j}} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^*, \quad B_\eta \triangleq \frac{1}{n} \frac{\psi_\infty}{d_-} \sum_{i \in \mathcal{C}_\eta^c} z_i z_i^*.$$

Note that $A_{\eta,(j)}^{-1}$ is well defined as $A_{\eta,(j)}$ is invertible for all large n a.s. provided η is small enough. Similar to the proof of [Lemma 3](#), note first that, for some $\kappa > 0$ and for all $j \in \mathcal{C}_\eta$, $\lambda_1(A_{\eta,(j)}) > \kappa > 0$ for all large n a.s. Indeed, with the same derivation as [\(14\)](#), for any $m > 0$ satisfying $v_n([m, M_\eta]) > c_+$ for all n a.s. (this may require M_η large enough), $\lambda_1(A_{\eta,(j)}) \geq mv(m\gamma_+) \lambda_1(\frac{1}{n} \sum_{\tau_i \in [m, M_\eta], i \neq j} z_i z_i^*)$ away from zero for all large n a.s., independently of η small enough ([Lemma 6](#)).

Then, since B_η is of maximum rank $|C_\eta^c| = v_n((M_\eta, \infty))$, the successive applications of [Lemmas 5](#) and [4](#) (see the similar steps in the proof of [Lemma 3](#)) lead to

$$\max_{j \in \mathcal{C}_\eta} |w_{nj}| \leq K v_n((M_\eta, \infty)) \quad (26)$$

for some $K > 0$ constant, independent of η .

Now that $w_{j,n}$ is controlled for all $j \in \mathcal{C}_\eta$, we can proceed similar to the proof in the bounded τ_i case. First, for any fixed $\eta > 0$ small enough, [Remark 2](#) ensures that there exists a sequence $\varepsilon_n^\eta \downarrow 0$, such that a.s.

$$\max_{j \in \mathcal{C}_\eta} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\eta, i \neq j} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^* \right)^{-1} z_j - \gamma_N^\eta \right| \leq \varepsilon_n^\eta. \quad (27)$$

Combining [\(25\)](#), [\(26\)](#), and [\(28\)](#), we then have for all large n a.s. and for all $j \in \mathcal{C}_\eta$

$$e_j^\eta \leq \frac{v\left(\frac{\tau_j}{\varepsilon_n^\eta} (\gamma_N^\eta - \varepsilon_n^\eta - K v_n((M_\eta, \infty)))\right)}{v(\tau_j \gamma_N^\eta)} \quad (28)$$

which, for $j = \bar{n}$, is

$$e_{\bar{n}}^\eta \leq \frac{v\left(\frac{\tau_{\bar{n}}}{\varepsilon_n^\eta} (\gamma_N^\eta - \varepsilon_n^\eta - K v_n((M_\eta, \infty)))\right)}{v(\tau_{\bar{n}} \gamma_N^\eta)}.$$

Using the definition of ψ , this reads equivalently

$$\left(1 - \frac{\varepsilon_n^\eta + K v_n((M_\eta, \infty))}{\gamma_N^\eta}\right) \frac{\psi(\tau_{\bar{n}} \gamma_N^\eta)}{\psi\left((e_n^\eta)^{-1} \tau_{\bar{n}} \gamma_N^\eta \left(1 - \frac{\varepsilon_n^\eta + K v_n((M_\eta, \infty))}{\gamma_N^\eta}\right)\right)} < 1$$

which implies, from the growth of ψ ,

$$\left(1 - \frac{\varepsilon_n^\eta + K v_n((M_\eta, \infty))}{\gamma_N^\eta}\right) \frac{\psi(\tau_{\bar{n}} \gamma_N^\eta)}{\psi\left((e_n^\eta)^{-1} \tau_{\bar{n}} \gamma_N^\eta\right)} < 1.$$

Adding $\frac{\varepsilon_n^\eta + K v_n((M_\eta, \infty))}{\gamma_N^\eta} - 1$ on both sides, this further reads

$$\left(1 - \frac{\varepsilon_n^\eta + K v_n((M_\eta, \infty))}{\gamma_N^\eta}\right) \frac{\psi(\tau_{\bar{n}} \gamma_N^\eta) - \psi\left((e_n^\eta)^{-1} \tau_{\bar{n}} \gamma_N^\eta\right)}{\psi\left((e_n^\eta)^{-1} \tau_{\bar{n}} \gamma_N^\eta\right)} < \frac{\varepsilon_n^\eta + K v_n((M_\eta, \infty))}{\gamma_N^\eta}$$

or equivalently, if η is taken small enough (recalling that $\gamma_N^\eta > \gamma_-$ uniformly on η small),

$$\frac{\psi(\tau_{\bar{n}} \gamma_N^\eta) - \psi\left((e_n^\eta)^{-1} \tau_{\bar{n}} \gamma_N^\eta\right)}{\varepsilon_n^\eta + K v_n((M_\eta, \infty))} < \frac{\psi\left((e_n^\eta)^{-1} \tau_{\bar{n}} \gamma_N^\eta\right)}{\gamma_N^\eta \left(1 - \frac{\varepsilon_n^\eta + K v_n((M_\eta, \infty))}{\gamma_N^\eta}\right)} < \frac{2\psi_\infty}{\gamma_-} \quad (29)$$

where the right-most bound holds for all large n a.s. provided η is chosen small enough.

Assume $\limsup_n e_n^\eta > 1 + \ell$ for some $\ell > 0$. Then one must have $\liminf_n \tau_{\bar{n}} > \tau_-$ for [\(28\)](#) to remain valid, with $\tau_- > 0$ independent of η small since $\gamma_- < \gamma_N^\eta < \gamma_+$ for all n large a.s., both bounds being independent of η . Since $\tau_{\bar{n}} \gamma_N^\eta$ belongs to $[\tau_- \gamma_-, M_\eta \gamma_+]$ for all large N a.s., taking the limit of [\(29\)](#) over some converging subsequence over which $\tau_{\bar{n}} \gamma_N^\eta \rightarrow x^\eta \in [\tau_- \gamma_-, M_\eta \gamma_+]$, $c_N \rightarrow c$, and $v_n((M_\eta, \infty))$ converges, ensures that

$$\frac{\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell} x^\eta\right)}{\lim_n v_n((M_\eta, \infty))} \leq K' \quad (30)$$

for $K' > 0$ independent of η , with $\psi_c = \lim_{c_N \rightarrow c} \psi$.

We now operate on η . If $\limsup_{\eta \rightarrow 0} x^\eta < \infty$, the left-hand side in [\(30\)](#) diverges to ∞ as $\eta \rightarrow 0$ so that, starting with an η sufficiently small and taking the limit over n on the subsequence under consideration raises a contradiction. If instead $\limsup_{\eta \rightarrow 0} x^\eta = \infty$, then, since $x^\eta \leq M_\eta \gamma_+$,

$$\frac{\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell} x^\eta\right)}{\lim_n v_n((M_\eta, \infty))} \geq \frac{\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell} x^\eta\right)}{\lim_n v_n\left(\left(\frac{x^\eta}{\gamma_+}, \infty\right)\right)}.$$

Call $y^\eta = g^{-1}(x^\eta)$. Recalling that $\psi_c(t) = \phi(g^{-1}(t))(1 - c\phi(g^{-1}(t)))^{-1}$, we get

$$\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell}x^\eta\right) = \frac{\phi(y^\eta) - \phi\left(g^{-1}\left[\frac{1}{1+\ell}g(y^\eta)\right]\right)}{(1 - c\phi(y^\eta))(1 - c\phi\left(g^{-1}\left[\frac{1}{1+\ell}g(y^\eta)\right]\right))}.$$

Now, letting $\kappa > 0$ small, for all large t , $g(t) < t(1 - c\phi_\infty)^{-1}(1 + \kappa)$ and similarly $g^{-1}(t) < t(1 - c\phi_\infty)(1 + \kappa)$. Hence, letting κ small enough, for all large y^η , we have, say,

$$\phi\left(g^{-1}\left[\frac{1}{1+\ell}g(y^\eta)\right]\right) < \phi\left(\frac{1}{1+\frac{1}{2}\ell}y^\eta\right).$$

Moreover, using $0 < 1 - c\phi(t) < 1$, we have $(1 - c\phi(t))^{-1} > 1$. Using these results now gives, for all large y^η ,

$$\begin{aligned} \frac{\psi_c(x^\eta) - \psi_c\left(\frac{1}{1+\ell}x^\eta\right)}{\lim_n v_n\left(\left(\frac{x^\eta}{\gamma_+}, \infty\right)\right)} &> \frac{\phi(y^\eta) - \phi\left(\frac{1}{1+\frac{1}{2}\ell}y^\eta\right)}{\lim_n v_n\left(\left(\frac{y^\eta}{\gamma_+(1-c\phi(y^\eta))}, \infty\right)\right)} \\ &> \frac{\phi(y^\eta) - \phi\left(\frac{1}{1+\frac{1}{2}\ell}y^\eta\right)}{\lim_n v_n\left(\left(\frac{y^\eta}{\gamma_+}, \infty\right)\right)}. \end{aligned}$$

Since $y^\eta \rightarrow \infty$ as $x^\eta \rightarrow \infty$, from [Assumption 3](#), the right-hand side must go to ∞ as $x^\eta \rightarrow \infty$, or equivalently as $\eta \rightarrow 0$. Therefore, taking η sufficiently small from the beginning and then bringing n large on the subsequence under study leads to a contradiction. Consequently, we must have $\limsup_n e_n^\eta \leq 1 + \ell$ a.s. A similar reasoning shows that $\liminf_n e_1^\eta \geq 1 - \ell$ a.s., for any given $\ell > 0$. We conclude that

$$\max_{j \in \mathcal{C}_\eta} |e_j^\eta - 1| \xrightarrow{\text{a.s.}} 0.$$

We now have to deal with e_j^η for $j \in \mathcal{C}_\eta^c$. For such a j ,

$$d_j = \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \tau_i v(\tau_i \gamma_N^\eta) e_i^\eta z_i z_i^* + \frac{1}{n} \sum_{i \in \mathcal{C}_\eta^c, i \neq j} \frac{\psi(\tau_i d_i)}{d_i} z_i z_i^* \right)^{-1} z_j.$$

But then, from the same reasoning as with the $w_{j,n}$ above (using in particular the uniform boundedness of d_i) and from $\max_{i \in \mathcal{C}_\eta} |e_i^\eta - 1| \xrightarrow{\text{a.s.}} 0$, we have

$$\max_{j \in \mathcal{C}_\eta^c} \left| d_j - \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \tau_i v(\tau_i \gamma_N^\eta) z_i z_i^* \right)^{-1} z_j \right| < K v_n((M_\eta, \infty)) < K \eta$$

for some $K > 0$ independent of η , which further implies from [Remark 2](#) that for all large n a.s. and for all $j \in \mathcal{C}_\eta^c$,

$$\gamma_N^\eta - K \eta \leq d_j \leq \gamma_N^\eta + K \eta.$$

Using the definition $e_j^\eta = \frac{\psi(\tau_j d_j)}{\psi(\tau_j \gamma_N^\eta)} \frac{\gamma_N^\eta}{d_j}$, the uniform bounds on γ_N^η , and the continuous growth of ψ shows finally that, a.s.

$$\limsup_n \max_{j \in \mathcal{C}_\eta^c} \{|e_j^\eta - 1|\} \leq \eta'$$

for some $\eta' > 0$ with $\eta' \rightarrow 0$ as $\eta \rightarrow 0$.

Gathering the results for $j \in \mathcal{C}_\eta$ and $j \in \mathcal{C}_\eta^c$, we therefore conclude that, for each $\ell > 0$, there exists $\eta > 0$ small enough such that a.s.

$$1 - \ell < \liminf_n \min_{1 \leq i \leq n} e_i^\eta \leq \limsup_n \max_{1 \leq i \leq n} e_i^\eta < 1 + \ell.$$

For such η small, we then have, by definition of e_i^η and from $\tau_i v(\tau_i \gamma_N^\eta) = \psi(\tau_i \gamma_N^\eta)/\gamma_N^\eta$,

$$(1 - \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N^\eta)}{\gamma_N^\eta} z_i z_i^* \leq \frac{1}{n} \sum_{i=1}^n v(\tau_i d_i) \tau_i z_i z_i^* \leq (1 + \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N^\eta)}{\gamma_N^\eta} z_i z_i^*. \quad (31)$$

It now remains to show that, for each $\varepsilon > 0$, there exists $\eta > 0$ for which $|\gamma_N^\eta - \gamma_N| < \varepsilon$ for all n large a.s. For this, observe that, by definition of γ_N and γ_N^η ,

$$1 = \frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \frac{\psi(\tau_i \gamma_N^\eta)}{1 + c_N \psi(\tau_i \gamma_N^\eta)} = \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{1 + c_N \psi(\tau_i \gamma_N)}$$

so that, since $\psi/(1 + c_N \psi)$ is increasing, we obtain $\gamma_N \leq \gamma_N^\eta$ and

$$\frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \frac{\psi(\tau_i \gamma_N)}{1 + c_N \psi(\tau_i \gamma_N)} = \frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \frac{\psi(\tau_i \gamma_N^\eta) - \psi(\tau_i \gamma_N)}{(1 + c_N \psi(\tau_i \gamma_N))(1 + c_N \psi(\tau_i \gamma_N^\eta))} \geq 0.$$

Take an interval $[m, M]$, $M < M_\eta$ (chosen once for all, independently of M_η large), with $v_n([m, M]) > \kappa > 0$ for all large n a.s. (possible from [Assumption 2-2](#)). Then we can further write

$$\begin{aligned} \frac{1}{n} \sum_{i \in \mathcal{C}_\eta} \frac{\psi(\tau_i \gamma_N)}{1 + c_N \psi(\tau_i \gamma_N)} &\geq \frac{1}{(1 + c_+ \psi_\infty)^2} \frac{1}{n} \sum_{\tau_i \in [m, M]} (\psi(\tau_i \gamma_N^\eta) - \psi(\tau_i \gamma_N)) \\ &\geq \frac{\kappa}{2(1 + c_+ \psi_\infty)^2} \min_{x \in [m, M]} (\psi(x \gamma_N^\eta) - \psi(x \gamma_N)) \end{aligned}$$

with the second inequality valid for all large n a.s. Now, for sufficiently small η , the left-hand side can be made arbitrarily small. Since γ_N and γ_N^η are uniformly bounded and bounded away from zero (irrespective of η small), if $|\gamma_N^\eta - \gamma_N|$ were uniformly away from zero for all η small, so would be the right-hand side, which is in contradiction with our previous statement. Therefore, for each $\varepsilon > 0$, one can choose η so that $|\gamma_N - \gamma_N^\eta| < \varepsilon$ for all n large a.s.

Now, by uniform continuity of ψ on bounded intervals along with the fact that $\psi(x) \uparrow \psi_\infty$, from [\(31\)](#), taking η small enough, for all large n a.s.

$$(1 - \ell)^2 \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{\gamma_N} z_i z_i^* \leq \hat{C}_N \leq (1 + \ell)^2 \frac{1}{n} \sum_{i=1}^n \frac{\psi(\tau_i \gamma_N)}{\gamma_N} z_i z_i^* \quad (32)$$

which therefore implies, with the same arguments as in the case τ_i bounded, that $\|\hat{C}_N - \hat{S}_N\| \xrightarrow{\text{a.s.}} 0$, when $C_N = I_N$. The arguments of the case τ_i bounded still hold for $C_N \neq I_N$ satisfying [Assumption 2-3](#). This completes the proof.

5. Conclusion

This article introduces a large dimensional analysis for robust estimators of scatter matrices of the Maronna-type from elliptically distributed samples. We specifically showed that, under mild assumptions, the Maronna estimator behaves similar to a classical sample covariance matrix model as both the population and sample sizes grow large. This study opens new roads in the analysis of signal processing methods based on robust scatter matrix estimation. In a similar manner as in [\[23, Theorem 6\]](#), it is believed that second order statistics for well behaved functionals of \hat{C}_N can be further analyzed, which would provide more information on the asymptotic fluctuations of $\hat{C}_N - \hat{S}_N$. The mathematical treatment developed in the proofs of our present results however shows some strong limitations for hypothetical extensions to other robust scatter matrix estimates. In particular, the important Tyler robust estimator [\[38,30\]](#), given by the unique solution (up to a scale factor) to [\(1\)](#) for $u(x) = 1/x$, cannot be analyzed from the present method which relies essentially on $\phi(x) = xu(x)$ being increasing. Although extensive simulations suggest that similar conclusions hold for Tyler estimator, there is to this day no approach to tackle this problem.

Appendix. Some lemmas

Lemma 4 (Rank-One Perturbation (Silverstein and Bai [\[35, Lemma 2.6\]](#))). Let $v \in \mathbb{C}^N$, $A, B \in \mathbb{C}^{N \times N}$ nonnegative definite, and $x > 0$. Then

$$\left| \text{tr} B (A + v v^* + x I_N)^{-1} - \text{tr} B (A + x I_N)^{-1} \right| \leq x^{-1} \|B\|.$$

Lemma 5 (Trace Lemma (Bai and Silverstein [\[3, Lemma B.26\]](#))). Let $A \in \mathbb{C}^{N \times N}$ be non-random and $y = [y_1, \dots, y_N]^T \in \mathbb{C}^N$ be a vector of independent entries with $E[y_i] = 0$, $E[|y_i|^2] = 1$, and $E[|y_i|^\ell] \leq \zeta_\ell$ for all $\ell \leq 2p$, with $p \geq 2$. Then,

$$E[|y^* A y - \text{tr} A|^p] \leq C_p \left((\zeta_4 \text{tr} A A^*)^{\frac{p}{2}} + \zeta_{2p} \text{tr} (A A^*)^{\frac{p}{2}} \right)$$

for C_p a constant depending on p only.

Lemma 6. Let $z_1, \dots, z_n \in \mathbb{C}^N$ be independent unitarily invariant vectors with $\|z_i\|^2 = N$. Then, if $0 < \liminf_n N/n \leq \limsup_n N/n < 1$,

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i=1}^n z_i z_i^* \right)^{-1} z_j - 1 \right| \xrightarrow{\text{a.s.}} 0$$

or equivalently

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} z_j^* \left(\frac{1}{n} \sum_{i \neq j} z_i z_i^* \right)^{-1} z_j - \frac{1}{1 - \frac{N}{n}} \right| \xrightarrow{\text{a.s.}} 0.$$

Moreover, there exists $\varepsilon > 0$ such that, for all large n a.s.

$$\lambda_1 \left(\frac{1}{n} \sum_{i=1}^n z_i z_i^* \right) \geq \min_{1 \leq j \leq n} \left\{ \lambda_1 \left(\frac{1}{n} \sum_{\substack{1 \leq i \leq n \\ i \neq j}} z_i z_i^* \right) \right\} > \varepsilon.$$

Proof. For readability, we denote $F = \frac{1}{n} \sum_{i=1}^n z_i z_i^*$, $F_{(j)} = F - \frac{1}{n} z_j z_j^*$, $\tilde{F} = \frac{1}{n} \sum_{i=1}^n \tilde{z}_i \tilde{z}_i^*$, and $\tilde{F}_{(j)} = \tilde{F} - \frac{1}{n} \tilde{z}_j \tilde{z}_j^*$, where we recall the relation $z_i = \sqrt{\tilde{N}} \tilde{z}_i / \|\tilde{y}_i\|$ for \tilde{z}_i zero mean I_N -covariance Gaussian and \tilde{y}_i zero mean $I_{\tilde{N}}$ -covariance Gaussian (non-independent). From the proof of Couillet et al. [15, Lemma 2], we have:

$$\max_{1 \leq j \leq n} \left| \frac{1}{N} \tilde{z}_j^* \tilde{F}^{-1} \tilde{z}_j - 1 \right| \xrightarrow{\text{a.s.}} 0$$

and, there exists $\varepsilon > 0$ such that, for all large n a.s.

$$\lambda_1(\tilde{F}) \geq \min_{1 \leq j \leq n} \lambda_1(\tilde{F}_{(j)}) > \varepsilon. \quad (\text{A.1})$$

Now,

$$\min_{1 \leq j \leq n} \lambda_1(F_{(j)}) \geq \frac{\min_{1 \leq j \leq n} \lambda_1(\tilde{F}_{(j)})}{\max_{1 \leq j \leq n} \tilde{N}^{-1} \|\tilde{y}_j\|^2}.$$

Since $\max_{1 \leq j \leq n} N^{-1} \|\tilde{z}_j\|^2 \xrightarrow{\text{a.s.}} 1$ a.s. from standard probability results, we have that for all large n a.s.

$$\lambda_1(F) \geq \min_{1 \leq j \leq n} \lambda_1(F_{(j)}) > \varepsilon/2$$

which already gives the second part of the lemma. Using only the outer inequality of (A.1), we now have, for all large n a.s.

$$\begin{aligned} \max_{1 \leq j \leq n} \left| \frac{1}{N} \tilde{z}_j^* F^{-1} \tilde{z}_j - \frac{1}{N} \tilde{z}_j^* \tilde{F}^{-1} \tilde{z}_j \right| &= \max_{1 \leq j \leq n} \left| \frac{1}{N} \tilde{z}_j^* F^{-1} (\tilde{F} - F) \tilde{F}^{-1} \tilde{z}_j \right| \\ &\leq \max_{1 \leq j \leq n} \left\{ \frac{1}{n} \sum_{k=1}^n \left| 1 - \frac{\tilde{N}}{\|\tilde{y}_k\|^2} \right| \left| \frac{1}{N} \tilde{z}_j^* \tilde{F}^{-1} \tilde{z}_k \right|^2 \right\} \\ &\leq \max_{1 \leq k \leq n} \left| 1 - \frac{\tilde{N}}{\|\tilde{y}_k\|^2} \right| \frac{1}{N} \left(\max_{1 \leq k \leq n} \|\tilde{z}_k\| \right)^2 \frac{4}{\varepsilon^2} \\ &\xrightarrow{\text{a.s.}} 0. \end{aligned}$$

Finally, for all large n a.s.

$$\begin{aligned} \max_{1 \leq j \leq n} \left| \frac{1}{N} \tilde{z}_j^* F^{-1} \tilde{z}_j - \frac{1}{N} \tilde{z}_j^* F^{-1} z_j \right| &= \max_{1 \leq j \leq n} \left\{ \left| \frac{1}{N} \tilde{z}_j^* F^{-1} \tilde{z}_j \right| \left| 1 - \frac{\tilde{N}}{\|\tilde{y}_j\|^2} \right| \right\} \\ &\leq \frac{2}{\varepsilon} \max_{1 \leq k \leq n} \left| 1 - \frac{\tilde{N}}{\|\tilde{y}_k\|^2} \right| \max_{1 \leq j \leq n} \frac{1}{N} \|\tilde{z}_j\|^2 \\ &\xrightarrow{\text{a.s.}} 0. \end{aligned}$$

The proof is concluded by putting these results together.

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