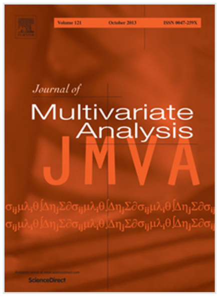

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Characterizations of the class of bivariate Gompertz distributions

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Abstract

The main goal of this article is to characterize the class of bivariate Gompertz distributions recently derived by Marshall & Olkin (2015) through functional equations. As a by-product, new properties of these distributions are obtained and discussed.

Keywords: Bivariate Gompertz distribution, characterization, copula, functional equation, hazard gradient vector, reliability, Sibuya's dependence function.
AMS 2010 subject classification: 39A60, 62E10, 62H10.

1. Introduction and preliminaries

This paper is motivated by a recent article of Marshall & Olkin [12] in which the authors derived bivariate Gompertz distributions based on a bivariate version of a functional equation due to Kaminsky [5]. Specifically, let X be a non-negative continuous random variable whose survival function S_X satisfies, for all $x \geq 0$ and $t \geq 0$, the aging equation

$$\frac{S_X(x+t)}{S_X(t)} = \{S_X(x)\}^{\phi(t)} \tag{1}$$

for some function $\phi : [0, \infty) \to [0, \infty)$. Kaminsky [5] shows that X must then have a Gompertz distribution defined, for all $x \geq 0$, by

$$S_X(x) = \exp\{-c(e^{ax} - 1)\} \tag{2}$$

with parameters $a, c > 0$; furthermore, ϕ is then given, for all $t \geq 0$, by $\phi(t) = e^{at}$. When $\phi(t) = 1$ for all $t \geq 0$, the functional equation (1) reduces to the well-known lack of memory property that characterizes the exponential distribution.

If parameters a and c in (2) are replaced by $-a$ and $-c$, respectively, the resulting expression is

$$S_X(x) = \exp\{c(e^{-ax} - 1)\}, \tag{3}$$

termed "negative Gompertz distribution" by Marshall & Olkin [10]. In this case, $S_X(x) \to e^{-c} \neq 0$ as $x \to \infty$, i.e., $S_X(x)$ in (3) is not a proper survival function.

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A slightly different characterization of the Gompertz distribution is given by Marshall & Olkin [10]. The difference comes from the fact that Kaminsky [5] defines the survival function of a random variable X by $\Pr(X \geq x)$ while Marshall & Olkin [10] adopt the usual one, i.e., $S_X(x) = \Pr(X > x)$. The following result takes this minor adjustment into account.

Theorem 1. *Suppose that X is an absolutely continuous non-negative random variable such that $S_X(x) > 0$ for all $x \geq 0$. Then S_X satisfies Equation* (1) *for some $a > 0$ and some function ϕ that does not depend on x if and only if either*

(i) $S_X(x) = e^{-ax}$ *and* $\phi(t) = 1$*, or*

(ii) S_X *is a Gompertz distribution* (2) *and* $\phi(t) = e^{at}$*, or*

(iii) S_X *is given by Equation* (3) *and* $\phi(t) = e^{-at}$.

The exponential and Gompertz survival functions are thus solutions of Kaminsky's functional equation (1). The negative Gompertz law (3) is also a solution, but the corresponding survival function is not proper.

The proof of Theorem 1 (ii) is given in Marshall & Olkin [10]. However, it is not clear (at least to the present author) how these authors arrived at Equation (18) from the first relation on p. 373. Luckily, a brilliant demonstration of the same statement can be found in Marshall & Olkin [11]; see their Proposition 6.

Marshall & Olkin [12] investigated two bivariate versions of Kaminsky's functional equation (1) obtaining the corresponding bivariate Gompertz distributions. Below, we revisit the analysis of the stronger version they consider, which pertains to the functional equation

$$\frac{S(x+z, y+t)}{S(z,t)} = \{S(x,y)\}^{\phi(z,t)}, \tag{4}$$

where $S(x,y) = \Pr(X > x, Y > y)$ is the joint survival function of non-negative continuous random variables X and Y and $\phi : [0,\infty)^2 \to [0,\infty)$ is some function of z and t only. Under the assumption that S has Gompertz marginals given by (2), Marshall & Olkin [12] show in their Proposition 3.1 that the solution of the functional equation (4) is the bivariate Gompertz distribution given, for all $x, y \geq 0$, by

$$S(x,y) = \exp\{-c(e^{ax+by} - 1)\} \tag{5}$$

with parameters $a, b > 0$ and $c \geq 1$. Note in passing that there exist many other versions of the bivariate distribution with Gompertz marginals in the literature; see, e.g., Mardia [8] and Marshall & Olkin [12].

The purpose of this note is to characterize the class (5) of bivariate Gompertz distributions by relaxing the requirement that the margins should be Gompertz. Section 2 also provides a second characterization of (5), but based on a specific product representing the sum of components of the underlying hazard gradient vector. We conclude with a discussion presenting other properties of bivariate Gompertz laws (5). In particular, it happens that (5) is generated by the Gumbel–Barnett copula.

2. Characterizations of bivariate Gompertz law (5)

This section reports two characterizations of the bivariate Gompertz distribution (5) based on a two-dimensional version of Kaminsky's functional equation (4). A bivariate version of Theorem 1 is given first. A conditional specification of the bivariate Gompertz law will result as a consequence. The second characterization of (5) is related to the underlying hazard vector.

Let X be a non-negative absolutely continuous random variable with density f_X and, for all $x \geq 0$, let $R_X(x) = -\ln\{S_X(x)\}$ be the corresponding hazard function (cumulative hazard rate). Denote by $r_X(x) = f_X(x)/S_X(x)$ the force of mortality (failure, hazard rate) function which uniquely determines the distribution. The hazard rate r_X can be equivalently defined by

$$r_X(x) = -\frac{d}{dx}\ln\{S_X(x)\} = \frac{d}{dx}R_X(x),$$

where $R_X(x) = \int_0^x r_X(u)du$.

It is assumed hereafter that Y is also a non-negative absolutely continuous random variable. A function $S : [0,\infty)^2 \to [0,1]$ is a proper joint survival function defined by $S(x,y) = \Pr(X > x, Y > y)$ if and only if

$$\lim_{x,y\to 0} S(x,y) = 1, \quad \lim_{x,y\to\infty} S(x,y) = 0, \tag{6}$$

and

$$\frac{\partial^2}{\partial x \partial y} S(x,y) \geq 0$$

if $S(x,y)$ is differentiable in a neighborhood of (x,y). Let $R(x,y) = -\ln\{S(x,y)\}$ be a bivariate hazard function. The above inequality can be rewritten as

$$\frac{\partial^2}{\partial x \partial y} S(x,y) = S(x,y)\{r_1(x,y) r_2(x,y) - r_{12}(x,y)\} \geq 0, \tag{7}$$

where

$$r_1(x,y) = \frac{\partial}{\partial x} R(x,y), \quad r_2(x,y) = \frac{\partial}{\partial y} R(x,y), \quad \text{and} \quad r_{12}(x,y) = \frac{\partial^2}{\partial x \partial y} R(x,y).$$

2.1. First characterization

The first characterization calls on the notion of dependence function, which is different from the concept of copula. Let R_X and R_Y be the marginal hazard functions (cumulative failure rates) of X and Y respectively, i.e., for all $x, y \geq 0$, $S_X(x) = \exp\{-R_X(x)\}$ and $S_Y(y) = \exp\{-R_Y(y)\}$. Then one can write

$$S(x,y) = \exp\{-R_X(x) - R_Y(y) + D(x,y)\},$$

where $D(x,y)$ is the so-called dependence function of the random vector (X,Y); see Sibuya [15]. Equivalently, $D(x,y)$ can be defined, for all $x, y \geq 0$, by

$$D(x,y) = \ln\left\{\frac{S(x,y)}{S_X(x) S_Y(y)}\right\} \tag{8}$$

and may serve as a (local) measure of dependence between random variables X and Y. When X and Y are non-negative random variables, $D(x,y)$ satisfies the boundary conditions $D(0,y) = D(x,0) = 0$.

Marshall & Olkin [12] noted that the solutions of the functional equation (4) are not guaranteed to be proper survival functions, so conditions (6) and (7) must be verified. For example, the bivariate analogue of (3) given by

$$S(x,y) = \exp\{c(e^{-ax-by} - 1)\} \tag{9}$$

is a solution of (4), but it is not a proper survival function; see Remark 1(d).

The following characterizing statement is an exact bivariate analogue of Theorem 1.

Theorem 2. *The joint survival function S satisfies* (4) *for some function $\phi(z,t) : [0,\infty)^2 \to [0,\infty)$ that does not depend on $x \geq 0$ and $y \geq 0$ if and only if either*

(i) *$S(x,y) = e^{-ax-by}$ and $\phi(z,t) = 1$ for $z, t \geq 0$ and some $a, b > 0$, or*

(ii) *$S(x,y)$ is given by* (5) *and $\phi(z,t) = e^{az+bt}$ for $z, t \geq 0$ and some $a, b > 0, c \geq 1$, or*

(iii) *$S(x,y)$ is given by* (9) *and $\phi(z,t) = e^{-az-bt}$ for $z, t \geq 0$ and some $a, b, c > 0$.*

Observe that Sibuya's dependence function $D(x,y)$ specified by (8) exhibits interesting connections with important cases of dependencies as follows. One has $D(x,y) \equiv 0$ if and only if X and Y are independent, corresponding to item (i). Also, $D(x,y) \leq 0$ if and only if X and Y are negative quadrant dependent (NQD), i.e., $S(x,y) \leq S_X(x) S_Y(y)$ for all $x, y \geq 0$, which is the case in item (ii). Similarly, $D(x,y) \geq 0$ if and only if X and Y are positive quadrant dependent (PQD), linked to item (iii) of Theorem 2.

Proof. Assume first that the functional equation (4) is satisfied. Substitute $x = t = 0$ in (4) to get

$$\frac{S(z,y)}{S(z,0)} = \{S(0,y)\}^{\phi(z,0)}, \quad \text{i.e.,} \quad \frac{S(z,y)}{S_X(z)S_Y(y)} = \{S_Y(y)\}^{\phi(z,0)-1}$$

for all $z, y \geq 0$. Taking logarithms on both sides of the last identity and using (8), one finds that, for all $z, y \geq 0$,

$$D(z,y) = \{\phi(z,0) - 1\} \ln\{S_Y(y)\}.$$

Similarly, setting $y = z = 0$ in (4) allows one to deduce that, for all $x, t \geq 0$,

$$D(x,t) = \{\phi(0,t) - 1\} \ln\{S_X(x)\}.$$

Given that these two forms of Sibuya's dependence function must be equal, one has, for all $x, y \geq 0$,

$$D(x,y) = \{\phi(x,0) - 1\} \ln\{S_Y(y)\} = \{\phi(0,y) - 1\} \ln\{S_X(x)\}. \tag{10}$$

We will consider three basic cases, corresponding to items (i), (ii) and (iii) of Theorem 2.

Case 1. Assume that X and Y are independent, i.e., $D(x,y) = 0$ for all $x, y \geq 0$. It follows from (10) that, for all $x, y \geq 0$,

$$\phi(x,0) - 1 = \phi(0,y) - 1 = 0$$

and hence $\phi(x,y) \equiv 1$ in (4). Therefore, one has, for all $x, y, z, t \geq 0$,

$$\frac{S(x+z, y+t)}{S(z,t)} = S(x,y).$$

This functional equation was studied by Marshall & Olkin [9] and the corresponding solution is given by $S(x,y) = e^{-ax-by}$ for all $x, y \geq 0$ and some $a, b > 0$.

Case 2. Let X and Y be NQD, by which we mean $D(x,y) < 0$ for all $x, y \geq 0$. Given that $\ln\{S_X(x)\} \leq 0$ and $\ln\{S_Y(y)\} \leq 0$ always hold, it follows from (10) that $\phi(x,0) - 1 \geq 0$ and $\phi(0,y) - 1 \geq 0$ for all $x, y \geq 0$. In addition,

$$\ln\{S_X(x)\} = -c\{\phi(x,0) - 1\} \quad \text{and} \quad \ln\{S_Y(y)\} = -c\{\phi(0,y) - 1\}$$

should be satisfied for some constant $c > 0$. This means that, for some $c > 0$,

$$S_X(x) = \exp[-c\{\phi(x,0) - 1\}] \quad \text{and} \quad S_Y(y) = \exp[-c\{\phi(0,y) - 1\}].$$

Substituting these expressions into the right-hand side of $S(x,y) = S_X(x)\{S_Y(y)\}^{\phi(x,0)}$, one deduces that

$$S(x,y) = \exp[-c\{\phi(x,0)\phi(0,y) - 1\}]. \tag{11}$$

Now set $z = 0$ in (4) to obtain $S(x, y+t)/S_Y(t) = \{S(x,y)\}^{\phi(0,t)}$ and apply (11) to conclude that, for all $y, t \geq 0$,

$$\phi(0, y+t) = \phi(0,y)\phi(0,t).$$

The latter identity is a Cauchy functional equation with solution $\phi(0,y) = e^{by}$ for some $b > 0$; see, e.g., Proposition A.2 in Marshall & Olkin [10], p. 702. The trivial solution $\phi(0,y) = 0$ for all $y \geq 0$ can be rejected because $\phi(0,y) \geq 1$ always holds. Therefore, $S_Y(y) = \exp[-c\{e^{by} - 1\}]$ for some $b, c > 0$.

By analogy, putting $t = 0$ in (4) one gets $\phi(x,0) = e^{ax}$ and hence $S_X(x) = \exp[-c\{e^{ax} - 1\}]$ for some $a, c > 0$. Thus, substituting $\phi(x,0) = e^{ax}$ and $\phi(0,y) = e^{by}$ in (11) we get to the bivariate Gompertz distribution (5). Applying condition (7) implies that $c \geq 1$. Clearly, from (5) one can obtain Gompertz marginal distributions $S_X(x) = \exp[-c\{e^{ax} - 1\}]$ and $S_Y(y) = \exp[-c\{e^{by} - 1\}]$, but with parameter $c \geq 1$.

Case 3. Assume that X and Y are PDQ, i.e., $D(x,y) > 0$. Because $\ln\{S_X(x)\} \leq 0$ and $\ln\{S_Y(y)\} \leq 0$, it follows from (10) that $\phi(x,0) - 1 \leq 0$ and $\phi(0,y) - 1 \leq 0$ for all $x, y \geq 0$. Therefore,

$$\ln\{S_X(x)\} = c\{\phi(x,0) - 1\} \quad \text{and} \quad \ln\{S_Y(y)\} = c\{\phi(0,y) - 1\}$$

should be fulfilled for some constant $c > 0$. Similar steps as in Case 2 lead to the conclusion that $S(x, y)$ is given by (9). Hence, if (4) is satisfied, then statements (i), (ii) and (iii) are true.

To complete the proof, it remains to check that if (i), (ii) or (iii) are fulfilled, then (4) is valid as well. This is easily done. □

Remark 1. One can extract the following consequences from the proof.

(a) A joint distribution with independent marginals satisfies (4) if and only if the marginals are exponentially distributed; see item (i).

(b) The bivariate Gompertz distribution (5) is NQD (see Case 2), but not PQD as stated by Marshall & Olkin [12] in their Proposition 3.2.

(c) Univariate Gompertz distributions (2) with a parameter $c \in (0, 1)$ cannot serve as marginals of the bivariate Gompertz distribution (5).

(d) The joint survival function $S(x, y)$ given by (9) is not proper, given that it does not satisfy conditions (6). In fact, $\lim_{x,y\to\infty} S(x, y) = e^{-c} \neq 0$ for $c > 0$.

Observe that the function $\phi(z, t)$ in (4) cannot be arbitrary, but should be separable, i.e., represented as a product of two continuous functions of z and t only. Setting $x = 0$ in (4), one gets

$$\frac{S(z, y + t)}{S(z, t)} = \{S(0, y)\}^{\phi(z,t)} = \{S_Y(y)\}^{\phi(z,t)}.$$

Using (11), $S_Y(y) = \exp[-c\{\phi(0, y) - 1\}]$ and $\phi(0, y + t) = \phi(0, y)\phi(0, t)$ in the latter relation imply that, for all $z, t \geq 0$,

$$\phi(z, t) = \phi(z, 0)\phi(0, t).$$

Let $\alpha(z) = \phi(z, 0)$ and $\beta(t) = \phi(0, t)$. So, hereafter we will consider the function $\phi(z, t)$ in (4) as a product

$$\phi(z, t) = \alpha(z)\beta(t), \quad \text{with} \quad \alpha(0) = \beta(0) = 1.$$

Remark 2. Theorem 2 suggests that the bivariate Gompertz distribution (5) can be derived via conditional specifications. More precisely, substituting either $x = t = 0$ or $y = z = 0$ in (4), one gets

$$\Pr(Y > y \mid X > x) = \{\Pr(Y > y)\}^{\alpha(x)} \quad \text{and} \quad \Pr(X > x \mid Y > y) = \{\Pr(X > x)\}^{\beta(y)},$$

respectively. These relations can be written as

$$\Pr(Y > y \mid X > x) = \exp\{-R_Y(y)\alpha(x)\} \tag{12}$$

and

$$\Pr(X > x \mid Y > y) = \exp\{-R_X(x)\beta(y)\}, \tag{13}$$

where R_X and R_Y are the cumulative hazard functions of X and Y, respectively.

The model specified by (12) and (13) is a particular case of a bivariate model introduced by Navarro & Sarabia [13], who defined proportional hazard rate conditional distributions. Applying Theorem 3.1 from Navarro & Sarabia [13], one can conclude that the most general bivariate survival function satisfying (12) and (13) is given, for all $x, y \geq 0$ and some $c \geq 1$, by

$$S(x, y) = \exp\{-R_X(x) - R_Y(y) - c^{-1}R_X(x)R_Y(y)\}.$$

Therefore, the bivariate Gompertz distribution (5) can be generated under additional assumption that the marginal distributions are Gompertz distributed. Thus, we proved the following.

Corollary 1. *If* (12) – (13) *are satisfied and if S has Gompertz marginals* (2) *with $c \geq 1$, then S is given by* (5).

In fact, the latter statement is equivalent to Proposition 3.1 of Marshall & Olkin [12].

2.2. A characterization via hazard gradient vector

The hazard gradient vector $(r_1(x,y), r_2(x,y))$ uniquely determines the joint distribution of (X,Y) via

$$S(x,y) = \exp\left\{-\int_0^x r_1(t,0)dt - \int_0^y r_2(x,t)dt\right\}; \tag{14}$$

see, e.g., Johnson & Kotz [4]. Pinto & Kolev [14] characterized a class of distributions having a Sibuya-type bivariate lack of memory property through the relation given, for all $x, y \geq 0$, by

$$r_1(x,y) + r_2(x,y) = A_1(x) + A_2(y)$$

for some continuous non-decreasing functions A_1, A_2. In Theorem 3, the bivariate Gompertz distribution (5) will be characterized by a specific product form representation of the sum $r_1(x,y) + r_2(x,y)$.

Theorem 3. *The proper joint survival function S satisfies, for all $x, y, z, t \geq 0$, the functional equation*

$$\frac{S(x+z, y+t)}{S(z,t)} = \{S(x,y)\}^{\alpha(z)\beta(t)} \tag{15}$$

for some differentiable functions $\alpha, \beta : [0,\infty) \to [1,\infty)$ if and only if, for some $E > 0$,

$$r_1(x,y) + r_2(x,y) = EA_1(x)A_2(y), \tag{16}$$

where the functions A_1 and A_2 necessarily take the form $A_1(x) = \alpha(x) = e^{ax}$ and $A_2(y) = \beta(y) = e^{by}$ for some $a, b > 0$.

Proof. Let the functional equation (15) be satisfied for some differentiable functions $\alpha, \beta : [0,\infty) \to [1,\infty)$. Relation (10) can be rewritten, for all $x, y \geq 0$, as

$$\{\alpha(x) - 1\}\{-\ln S_Y(y)\} = \{\beta(y) - 1\}\{-\ln S_X(x)\}. \tag{17}$$

Differentiating (17) with respect to y gives

$$\{\alpha(x) - 1\}r_Y(y) = \beta'(y)\{-\ln S_X(x)\},$$

where r_Y is the hazard rate of Y. From the proof of Theorem 2, it is known hat $S(x,y)$ is a proper survival function if (X,Y) is NQD. Substitute $-\ln S_X(x) = c\{\alpha(x) - 1\}$ with $c > 0$ and $\alpha(x) \geq 1$ in the last expression to obtain

$$r_Y(y) = c\beta'(y).$$

Putting $x = t = 0$ in (15), taking logarithms and differentiating with respect to y gives $r_2(x,y) = \alpha(x)r_Y(y)$ and replacing $r_Y(y) = c\beta'(y)$ in the last equation, one gets

$$r_2(x,y) = c\alpha(x)\beta'(y).$$

Similarly, $r_1(x,y) = c\alpha'(x)\beta(y)$, and therefore

$$r_1(x,y) + r_2(x,y) = c\{\alpha'(x)\beta(y) + \alpha(x)\beta'(y)\}.$$

Hence, in order for (16) to be valid, the equation

$$EA_1(x)A_2(y) = c\{\alpha'(x)\beta(y) + \alpha(x)\beta'(y)\}$$

should be satisfied for all functions and constants involved. This is possible only if

$$\alpha'(x)\beta(y) = aA_1(x)A_2(y) \quad \text{and} \quad \alpha(x)\beta'(y) = bA_1(x)A_2(y)$$

for some $a, b > 0$. The solution of this system of differential equations is

$$\alpha(x) = A_1(x) = e^{ax} \quad \text{and} \quad \beta(y) = A_2(y) = e^{by}.$$

Finally, the constant E is determined by $E = c(a + b)$.

Inversely, assume that (16) is fulfilled with $A_1(x) = \alpha(x) = e^{ax}$, $A_2(y) = \beta(y) = e^{by}$ and $E = c(a + b)$ for some $a, b, c > 0$. Retracing the steps backwards, one deduces that

$$r_1(x, y) = ca \exp\{ax + by\} \quad \text{and} \quad r_2(x, y) = cb \exp(ax + by).$$

In view of (14) and the above expressions, one can reconstruct the joint survival function S, viz.

$$S(x, y) = \exp\left\{-\int_0^x cae^{at}dt - \int_0^y cbe^{ax+bt}dt\right\},$$

which gives the bivariate Gompertz distribution (5) with $c \geq 1$. However, the latter is characterized by the functional equation (15), according to Theorem 2 (ii) and the proof is complete. □

Remark 3.

(a) If X and Y are independent and satisfy (15), it follows from Theorem 2 (i) that $S(x, y) = \exp(-ax-by)$. In this case $r_1(x, y) + r_2(x, y) = a + b$, i.e., (16) is fulfilled with $A_1(x) = A_2(y) = c = 1$ for all $x, y \geq 0$. Conversely, if the sum $r_1(x, y) + r_2(x, y)$ is a constant for all $x, y \geq 0$, then X and Y are necessarily independent and exponentially distributed; see Kulkarni [6].

(b) If S is given by (9), then relation (16) is fulfilled with $A_1(x) = e^{-ax}$, $A_2(x) = e^{-bx}$ and $E = -c(a + b)$ for some $a, b, c > 0$, but the corresponding survival function is improper as noted in Remark 1(d).

The components of the hazard gradient vector r_1 and r_2 are failure rates of the conditional distributions $\Pr(X > x \mid Y > y)$ and $\Pr(Y > y \mid X > x)$, respectively; see, e.g., Johnson and Kotz [4]. As a consequence of Corollary 1, the expressions $r_1(x, y) = c\alpha'(x)\beta(y)$ and $r_2(x, y) = c\alpha(x)\beta'(y)$ obtained in the proof of Theorem 3 will lead to (5) if the marginals are Gompertz.

3. Discussion

Along with few other distributions, the Gompertz distribution (2) satisfies the so-called law of uniform seniority; see, e.g., Greville [3]. For independent and identically distributed random variables X and Y, this law states that, for all $x, y, z \geq 0$,

$$\Pr(X > x + z, Y > y + z \mid X > x, Y > y) = \Pr(X > w + z \mid X > w), \tag{18}$$

where w does not depend upon z.

If X and Y are Gompertz random variables given by (2), w is a solution of the equation $e^{aw} = e^{ax} + e^{ay}$. Thus, two Gompertz variates of different ages (x and y) may be replaced by a single Gompertz variate of age $w = w(x, y)$ in the above conditional probabilities. Equation (18) can be rewritten, for all $x, y, z \geq 0$, as

$$\frac{S_X(x + z)}{S_X(x)} \frac{S_Y(y + z)}{S_Y(y)} = \frac{S_X\{w(x, y) + z\}}{S_X\{w(x, y)\}}.$$

A simple reliability interpretation in terms of series systems based on the latter relation is given by Marshall & Olkin [10]; see their Proposition A.1 on p. 370 as well.

By relaxing the independence assumption between the variables in (18), one is led to the following bivariate version of the law of uniform seniority, valid for all $x, y, z, t \geq 0$,

$$\frac{S(x + z, y + t)}{S(x, y)} = \frac{S\{z + w(x, y), t + w(x, y)\}}{S\{w(x, y), w(x, y)\}} \tag{19}$$

for some function $w : [0, \infty)^2 \to [0, \infty)$ of x and y only. Alternative extensions are also possible.

Relation (19) suggests the following reliability interpretation and related question. Suppose that two items survive to respective ages x and y. Under what condition does the series system composed by (dependent) aged components have the same life distribution of some age $w = w(x, y)$, independent of z and t?

Let us further assume that S is defined by the bivariate Gompertz distribution (5). After some algebra, one can conclude that the answer to the above question, and the solution of functional equation (19), is given by

$$w(x, y) = \frac{a}{a+b}\, x + \frac{b}{a+b}\, y.$$

Observe that this expression is not a characterizing property of the bivariate Gompertz law. However, relation (19) may serve as a starting point for new reliability and actuarial applications of bivariate Gompertz laws and for introducing a weaker version of the bivariate, setting the clock back to zero property (satisfied by (5) and few other bivariate distributions); see Kulkarni & Patkure [7].

From Sklar's representation theorem (see, e.g., [1]), one can see that the unique copula associated with the bivariate Gompertz distribution (5) is the Gumbel–Barnett copula, defined for all $u, v \in (0, 1)$, by

$$C(u, v) = uv \exp\left\{-\frac{\ln(u)\ln(v)}{c}\right\}.$$

Of course, infinitely many other types of bivariate Gompertz distributions could be generated through copulas; see, e.g., [2] for an overview of copula models. For example, if the marginals are Gompertz distributed, the Gumbel–Hougaard copula with parameter $c \in (0, 1]$, defined for all $u, v \in (0, 1)$, by

$$C(u, v) = \exp\{-(|\ln u|^{1/c} + |\ln v|^{1/c})^c\}$$

will generate the bivariate Gompertz distribution considered by [8].

The characterizations presented herein are based on the strong version of bivariate Kaminsky's functional equation (4). As a consequence, it was established that the function $\phi(z, t)$ in (4) cannot be arbitrary, but should be presented as a product $\alpha(z)\beta(t)$ of two differentiable functions of z and t only.

Finally, it would be valuable to investigate further the weak version of bivariate Kaminsky's functional equation specified, for all $x, y, t \geq 0$, by

$$\frac{S(x+t, y+t)}{S(t, t)} = \{S(x, y)\}^{\psi(t)},$$

where $\psi : [0, \infty) \to [0, \infty)$ is some function of t only. The detailed analysis presented by Marshall & Olkin [12] demonstrates that corresponding solutions offer more interesting possibilities for applications.

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