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Weighted Approximation of the Renewal Spacing Processes

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In this paper, we provide a weighted approximation for the renewal spacing empirical and quantile processes. Some linear bounds for the empirical distribution and quantile functions are also given. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let ω, ω_1, $\omega_2, \ldots,$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.) with mean $0 < E\omega < \infty$ and common distribution function $F(x) := P(\omega/E\omega \leqslant x)$. We define the partial sums

$$S_i := \omega_1 + \omega_2 + \cdots + \omega_i, \qquad i = 1, 2, \ldots, \tag{1.1}$$

and the renewal spacings

$$D_{i,n} := \omega_i/S_n, \qquad 1 \leqslant i \leqslant n. \tag{1.2}$$

It is well known that if F is exponential, then the vector $\{S_i/S_n: 1 \leqslant i \leqslant n-1\}$ has the same distribution as the order statistics of a uniform sample of size $n-1$, and therefore, $\{D_{i,n}: 1 \leqslant i \leqslant n-1\}$ has the same distribution as the uniform spacings (see, e.g., [17]).

Now, define the quantile function

$$Q(s) := \inf\{x: F(x) \geqslant s\}$$

and the empirical distribution function of the normalized renewal spacings by

$$\mathbb{D}_n(s) := n^{-1} \sum_{i=1}^{n} \mathbb{I}\{nD_{i,n} \leqslant Q(s)\}, \tag{1.3}$$

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and the corresponding quantile process

$$\mathbb{D}_n^{\leftarrow}(t) := \inf\{s : \mathbb{D}_n(s) \geqslant t\}.$$

Motivated by the fact that many test statistics based on renewal spacings may be expressed as a functional of $\mathbb{D}_n$ or $\mathbb{D}_n^{\leftarrow}$ when F is exponential, the limiting properties of the processes

$$a_n(s) := n^{1/2}(\mathbb{D}_n(s) - s)$$

and

$$b_n(s) := n^{1/2}(\mathbb{D}_n^{\leftarrow}(s) - s)$$

have been subject of investigations over the years by many authors. If F is the exponential or some gamma distribution, these include Shorack [21], Rao and Sethuraman [19], Durbin [11], Beirlant [2], Einmahl and Van Zuijlen [13], Aly, Beirlant, and Horváth [1], Beirlant and Horváth [3], and Csörgő and Horváth (1986). Beirlant *et al.* [4] obtained a weighted approximation for b_n (see their Lemma 3.8) in the same spirit of M. Csörgő *et al.* (Cs–Cs–H–M) [6] or for a more general result by Einmahl and Mason [12]. However, the case of a general F is of interest since it enables us to get the limiting distribution of many statistics for testing whether a renewal process is a Poisson process or not under the null hypothesis, but also under a fixed alternative. Pyke [18] gives a weak convergence result for a_n and b_n.

The processes a_n and b_n are also related to the empirical process with estimated parameter. Since (1.1) is cale invariant, if F belongs to a parametric family with a constant coefficient of variation $E\omega/(\operatorname{Var}\omega)^{1/2}$, $\mathbb{D}_n$ may be viewed as the empirical d.f., where the mean or the standard deviation is estimated. For example, this situation occurs if F belongs to a family concentrated on $(0, \infty)$ and indexed by a scale parameter, like

$$\{F(x) = 1 - (k/x)^a, k > 0, a > 0, x \geqslant k\}$$

or if F is a Gamma distribution indexed by a scale parameter. Another example is given by

$$\{F(x) = \mathcal{N}(\theta, \theta^2) : \theta > 0\}.$$

More general empirical processes with parameter estimated have been studied by Durbin [11], M. Csörgő and Révèsz [9], and M. Csörgő [5].

The Cs–Cs–H–M [6] weighted approximation has been a very useful tool to obtain the asymptotic distribution of a large variety of statistics based on the usual empirical distribution and quantile functions of a sample. See, for instance, the volume edited by Hahn, Mason, and

Weiner [14]. The aim of this paper is to obtain such approximations for the processes a_n and b_n and to show, as S. Csörgő and Mason [8] did for the booststrap, that a "metatheorem" holds (see also [7, Section 17] for an implicit but less powerful metatheorem) in the sense that: all results proved using the weighted approximation and some linear bounds for the uniform empirical distribution or quantile processes hold for the renewal spacings (up to a change of the approximating process).

2. Weighted Approximation and Linear Bounds for the Renewal Spacing Process

We require the following assumptions:

A1. $0 < \operatorname{Var} \omega < \infty$;

A2. The distribution function F of ω is twice differentiable, except maybe at a finite number of points;

A3. $\|x^2 f'(x)\| < \infty$, where $\|\cdot\|$ denotes the sup-norm and $f := F'$ is the density function of F;

A4. $\|xf(x)\, F(x)^{-1/2} (1 - F(x))^{-1/2}\| < \infty$.

In fact, A3 is generally satisfied under A1 if the density function f is bounded and ultimately concave or convex.

The limiting behavior of the processes a_n and b_n are different according to the expectation of ω is zero or not. If $E\omega \neq 0$, then there is no loss of generality in assuming

A5. $E\omega = 1$.

Finally, with B denoting a Brownian bridge, we define the Gaussian process

$$\Gamma_B(s) := B(s) - Q(s)\, f(Q(s)) \int_0^1 B(t)\, dQ(t).$$

If g is a function from $(0, 1)$ into $\mathbb{R}$, we define

$$\|g\|_n := \sup_{1/n \leq s \leq (n-1)/n} |g(s)|.$$

Our first theorem gives the weighted approximation of the process a_n:

Theorem 2.1. *Assume* A1–A5. *If for some* $\nu \in [0, 1/4)$ *we have*

$$\int [F(x)(1 - F(x))]^{(1/2) - \nu}\, dx < \infty, \tag{2.1}$$

then one can construct a sequence of Brownian bridges $(B_n)_{n\geqslant 1}$ *such that*

$$n^{\nu}\left\|\frac{a_n(s)-\Gamma_{B_n}(s)}{(s(1-s))^{(1/2)-\nu}}\right\|_n = O_p(1). \tag{2.2}$$

To obtain an analogous result for b_n, we shall establish a Bahadur–Kiefer type representation. For this, recall that a function g is regularly varying at $a\pm$ (with index $\rho\in\mathbb{R}$) iff $\lim_{s\to 0} g(a\pm\lambda s)/g(a\pm s)=\lambda^{\rho}$ for any $\lambda>0$. We need the following assumptions:

B1. $\|F(x)^{1/2}(1-F(x))^{1/2}\, xf'(x)/f(x)\|<\infty$;

B2. $\|xf(x)\|<\infty$;

B3. $Q(u)f'(Q(u))/f(Q(u))$ and its derivative are regularly varying at $0+$ and $1-$.

If $\max_{1\leqslant i\leqslant n}\omega_i$ and $\min_{1\leqslant i\leqslant n}\omega_i$ normalized by a sequence of affine transformations admit a nondegenerate limiting distribution function and f' is ultimately concave or convex, then (see [20, pp. 85–93]) B1 and B3 hold.

THEOREM 2.2. *Assume* A1–A5 *and* B1–B3. *If for some* $\nu\in[0,1/4)$ *the condition* (2.1) *holds and*

$$\|(\log|\log(F(1-F))|)^{1/2}\,(F(1-F))^{2\nu-1/2}\,(1+(f'/f))\|<\infty, \tag{2.3}$$

then one can construct a sequence $(\omega_i)_{i\geqslant 1}$ *of the desired type on a probability space such that*

$$n^{\nu+(1/2)}\left\|\frac{a_n(s)+b_n(s)}{(s(1-s))^{(1/2)-\nu}}\right\|_n = O_p(1). \tag{2.4}$$

Combining Theorems 2.1 and 2.2, we obtain the approximation for the quantile process:

THEOREM 2.3. *Under the assumptions of Theorem* 2.2 *and on the probability space of Theorem* 2.2,

$$n^{\nu}\left\|\frac{b_n(s)+\Gamma_{B_n}(s)}{(s(1-s))^{(1/2)-\nu}}\right\|_n = O_p(1). \tag{2.5}$$

Remark 2.1. The weighted approximation of Cs–Cs–H–M [6] and Mason and Van Zwet [16] suggest that (2.2) and (2.5) hold also for $1/4<\nu<1/2$ but for different Brownian motions. However, our proof (see Lemma 3.1) allows only $0\leqslant\nu<1/4$.

Our next goal is to provide linear bounds for $\mathbb{D}_n$ and $\mathbb{D}_n^{\leftarrow}$. For this, we denote

$$D_{(1),n} \leqslant \cdots \leqslant D_{(n),n}$$

the order statistics of the renewal spacings $D_{1,n}, ..., D_{n,n}$.

THEOREM 2.4. (i) *Under the assumptions of Theorem* 2.1,

$$\sup_{0<s\leqslant 1} \frac{\mathbb{D}_n(s)}{s} + \sup_{0\leqslant s<1} \frac{1-\mathbb{D}_n(s)}{1-s} = O_p(1),$$

$$\sup_{F(nD_{(1),n})\leqslant s\leqslant 1} \frac{s}{\mathbb{D}_n(s)} + \sup_{0\leqslant s<F(nD_{(n),n})} \frac{1-s}{1-\mathbb{D}_n(s)} = O_p(1).$$

(ii) *Under the assumptions of Theorem* 2.3, *for any* $\lambda>0$, *we have*

$$\sup_{\lambda/n\leqslant s\leqslant 1} \frac{\mathbb{D}_n^{\leftarrow}(s)}{s} + \sup_{0\leqslant s\leqslant 1-(\lambda/n)} \frac{1+\mathbb{D}_n^{\leftarrow}(s)}{1-s} = O_p(1)$$

$$\sup_{0<s<1} \frac{s}{\mathbb{D}_n(s)} + \sup_{0<s<1} \frac{1-s}{1-\mathbb{D}_n^{\leftarrow}(s)} = O_p(1).$$

3. PROOFS

We first prove a lemma which will be used later.

LEMMA 3.1. *Under the assumptions of Theorem* 2.1, *we have*

$$n^{\nu} \left\| \frac{B_n(F((S_n/n)\,Q(s))) - B_n(s)}{(s(1-s))^{(1/2)-\nu}} \right\|_n = O_p(1), \tag{3.1}$$

Proof of Lemma 3.1. Our proof follows exactly Mason [15] and can also be derived equivalently following M. Csörgő and Horváth [8″]: Let $\varepsilon>0$ and denote $\Delta_n := (S_n-n)/n$. Since $n^{1/2}\Delta_n$ is asymptotically $\mathcal{N}(0,\sigma^2)$, we have

$$P\left(n^{\nu} \left\| \frac{B_n(F((S_n/n)\,Q(s))) - B_n(s)}{s(1-s))^{(1/2)-\nu}} \right\|_n > u\right)$$
$$\leqslant P\left(n^{\nu} \left\| \frac{B_n(F((S_n/n)\,Q(s))) - B_n(s)}{(s(1-s))^{(1/2)-\nu}} \right\|_n > u;\ \Delta_n \leqslant \lambda/n^{1/2}\right) + \varepsilon \tag{3.2}$$

provided λ is large enough. From Mason [15] we have

$$P\left(n^{\nu} \max_{1 \leqslant i \leqslant n} \sup_{|s-(i/n)| \leqslant 1/(2n)} \frac{|B_n(s)-B_n(i/n)|}{(s(1-s))^{(1/2)-\nu}} \geqslant u\right) \leqslant \varepsilon$$

for u sufficiently large. Using the same technique, we obtain

$$P\left(\sup_{|s-(i/n)| \leqslant 1/(2n)} n^{1/2} \frac{|B_n(F((S_n/n)\,Q(s)))-B(i/n)|}{i^{(1/2)-\nu}} \geqslant u;\, \Delta_n \leqslant \lambda/n^{1/2}\right)$$
$$\leqslant P\left(\sup_{|t-(i/n)| \leqslant \mu_{i,n}} |B((i/n)+t)-B(i/n)| \geqslant u\mu_{i,n}^{1/2} \frac{i^{(1/2)-\nu}}{n^{1/2}} \mu_{i,n}^{-1/2}\right), \tag{3.3}$$

where

$$\mu_{i,n} := \sup_{\substack{|s-(i/n)| \leqslant 1/(2n) \\ |t| \leqslant \lambda}} |F((1+n^{-1/2}t)\,Q(s))-(i/n)|$$
$$\geqslant \sup_{|s-(i/n)| \leqslant 1/(2n)} |F((S_n/n)\,Q(s))-(i/n)| \quad \text{if } |\Delta_n| \leqslant \lambda/n^{1/2}.$$

As in Mason [15], we bound (3.3) by

$$Au^{-1}i^{\nu-(1/2)}n^{1/2}\mu_{i,n}^{1/2} \exp\left(-\frac{u^2}{8}\frac{i^{1-(2\nu)}}{n\mu_{i,n}}\right). \tag{3.4}$$

To bound $\mu_{i,n}$, expand $F((1+tn^{-1/2})\,Q(s))$ to obtain

$$\begin{aligned} &F((1+tn^{-1/2})\,Q(s))-(i/n) \\ &\quad = s-(i/n)+tn^{-1/2}Q(s)\,f(Q(s)) \\ &\qquad + \frac{t^2}{2n} Q(s)^2 f'(Q((1+\theta(n,s,t)\,n^{-1/2})\,Q(s))) \end{aligned}$$

for some $\theta(n, s, t) \in [0, 1]$.

Since $|s-(i/n)| \leqslant 1/(2n)$ and $|t| \leqslant \lambda$, we obtain from A3 and A4 that for some constants c_1 and c_2 (independant of i and n) we have

$$\begin{aligned} \mu_{i,n} &\leqslant n^{-1}c_1 + n^{-1/2}\lambda \sup_{|s-(i/n)| \leqslant 1/(2n)} |Q(s)\,f(Q(s))| \\ &\leqslant n^{-1}c_1 + \lambda n^{-1}i^{1/2} \leqslant c_2 i^{1/2}/n. \end{aligned}$$

Hence, (3.4) is upper bounded by

$$Au^{-1}i^{\nu-(1/4)} \exp(-u^2 i^{(1/2)-2\nu}/8),$$

which is the term of a convergent series for any fixed $u > 0$. Therefore (3.2) may be taken arbitrarily small, provided u is sufficiently large and ε is small enough, and this proves the lemma. ∎

Proof of Theorem 2.1. We can assume that $\omega_i = Q(U_i)$ where $U_1, U_2, \ldots$ is a sequence of uniform r.v. over (0, 1). Let us define the uniform empirical distribution function

$$\mathbb{F}_n(x) := n^{-1} \sum_{i=1}^{n} \mathbb{I}\{U_i \leqslant x\}. \tag{3.5}$$

Using (1.2) and (1.3), we can write $\mathbb{D}_n$ as

$$\mathbb{D}_n(s) = \mathbb{F}_n\left(F\left(\frac{S_n}{n} Q(s)\right)\right). \tag{3.6}$$

We will first approximate $F((S_n/n)\, Q(s))$ and then use a weighted approximation for $\mathbb{F}_n$. Let $\Delta_n := (S_n - n)/n$. Then

$$F\left(\frac{S_n}{n} Q(s)\right) = s + \Delta_n Q(s)\, f(Q(s)) + (\Delta_n^2/2)\, Q(s)^2 f'[(1 + \theta_n(s)\, \Delta_n)\, Q(s)], \tag{3.7}$$

where $\theta_n(s) \in [0, 1]$. Observe the following facts:

(a) $\lim_{n \to \infty} \sup_{0 \leqslant s \leqslant 1} |\theta_n(s)\, \Delta_n| = 0$ a.s. under A1,

(b) $\|Q^2 f'\| = \|x^2 f'(x)\| < \infty$ under A3,

(c) $\left\| \dfrac{Q(s)\, f(Q(s))}{s^{1/2}(1-s)^{1/2}} \right\| = \left\| \dfrac{x f(x)}{F(x)^{1/2}\,(1 - F(x))^{1/2}} \right\| < \infty$ under A4,

and by the central limit theorem

(d) $n^{\nu} \left\| \dfrac{\Delta_n}{(s(1-s))^{(1/2)-\nu}} \right\|_n = O_p(1)$.

These facts with (3.3) imply that

$$\left(F\left(\frac{S_n}{n} Q(s)\right)\left[1 - F\left(\frac{S_n}{n} Q(s)\right)\right]\right)\Big/ s(1-s) \tag{3.8}$$

is bounded away from 0 and ∞ in probability, for s on $[1/n, 1-(1/n)]$ and, moreover

$$n^{\nu + (1/2)} \left\| \frac{F((S_n/n)\, Q(s)) - (s + \Delta_n Q(s)\, f(Q(s)))}{(s(1-s))^{(1/2)-\nu}} \right\|_n = O_p(1). \tag{3.9}$$

We may assume that the random variable $U_1, U_2, \ldots$, are defined on the probability space where the weighted approximations by Cs–Cs–H–M [6] or Mason and Van Zwet [16] hold. Therefore, there exists a sequence of Brownian bridges $(B_n)_{n \geqslant 1}$, such that for any $0 \leqslant \nu < 1/4$

$$n^{\nu} \left\| \frac{n^{1/2}(\mathbb{F}_n(x) - x) - B_n(x)}{(x(1-x))^{(1/2)-\nu}} \right\|_n = O_p(1). \tag{3.10}$$

Combining (3.6)–(3.10), we get

$$n^{\nu} \left\| \frac{a_n(s) - n^{1/2} \Delta_n Q(s) f(Q(s)) - B_n(F(S_n/n)\, Q(s)))}{(s(1-s))^{(1/2)-\nu}} \right\|_n = O_p(1). \tag{3.11}$$

From (3.11) and (3.1) we deduce

$$n^{\nu} \left\| \frac{a_n(s) - n^{1/2} \Delta_n Q(s) f(Q(s)) - B_n(s)}{(s(1-s))^{(1/2)-\nu}} \right\|_n = O_p(1). \tag{3.12}$$

To end the proof, note that

$$n^{1/2} \Delta_n = n^{1/2} \int_0^1 Q(t)\, d(\mathbb{F}_n(t) - t),$$

which after integration by parts becomes

$$n^{1/2} \Delta_n = -\int_0^1 n^{1/2}(\mathbb{F}_n(t) - t)\, dQ(t).$$

This, in conjunction with (3.10) gives

$$n^{1/2} \Delta_n = -\int_0^1 (B_n(t) + O_p(n^{-\nu})(t(1-t))^{(1/2)-\nu})\, dQ(t), \tag{3.13}$$

where the $O_p(\cdot)$ is uniform in t. Assumption (2.1) ensures that

$$\int_0^1 (t(1-t))^{(1/2)-\nu}\, dQ(t) = \int [F(x)(1-F(x))]^{(1/2)-\nu}\, dx < \infty. \tag{3.14}$$

Assumption A4 implies that

$$\left\| \frac{Q(s) f(Q(s))}{(s(1-s))^{(1/2)-\nu}} \right\|_n \leqslant \left\| \frac{x f(x)}{(F(x)(1-F(x)))^{1/2}} \right\| < \infty. \tag{3.15}$$

Equations (3.13)–(3.15) imply

$$n^{\nu}\left\| Q(s)\, f(Q(s))\, \frac{n^{1/2}\Delta_n + \int_0^1 B_n(t)\, dQ(t)}{(s(1-s))^{(1/2)-\nu}} \right\|_n = O_p(1), \tag{3.16}$$

which with (3.12) gives (2.2). ∎

Proof of Theorem 2.2. We first define the inverse

$$\mathbb{G}_n(y) := \inf\{x\colon \mathbb{F}_n(x) \geqslant y\}$$

where $\mathbb{F}_n$ is defined in (3.1). Then, an easy calculation shows that

$$\mathbb{D}_n^{\leftarrow}(s) = F\left(\frac{n}{S_n}\, Q(\mathbb{G}_n(s))\right) = F((1-\Delta_n n/S_n)\, Q(\mathbb{G}_n(s))). \tag{3.17}$$

Therefore,

$$\begin{aligned} \mathbb{D}_n^{\leftarrow}(s) &= \mathbb{G}_n(s) - \Delta_n \frac{n}{S_n}\, Q(\mathbb{G}_n(s))\, f(Q(\mathbb{G}_n(s))) \\ &\quad + (\Delta_n^2/2)(n/S_n)^2\, Q(\mathbb{G}_n(s))^2 f'\left(\left(1-\theta_n(s)\,\Delta_n \frac{n}{S_n}\right) Q(s)\right) \end{aligned} \tag{3.18}$$

with $0 \leqslant \theta_n(s) \leqslant 1$. Since we assume $E\omega = 1$, there exists a random n_0 a.s. finite such that for all $n \geqslant n_0$,

$$1 - \theta_n(s)\,\Delta_n \frac{n}{S_n} \in [1/2, 3/2]. \tag{3.19}$$

Moreover, the central limit theorem yields

$$n\Delta_n^2 = O_p(1) \tag{3.20}$$

and A3 implies

$$\|Q(u)^2 f'(\lambda Q(u))\| = \lambda^{-1} \|x^2 f'(x)\| < \infty. \tag{3.21}$$

From (3.18)–(3.19) we deduce

$$n^{\nu}\left\| \frac{n^{1/2}\mathbb{D}_n^{\leftarrow}(s) - \mathbb{G}_n(s) + \Delta_n(n/S_n)\, Q(\mathbb{G}_n(s))\, f(Q(\mathbb{G}_n(s)))}{(s(1-s))^{(1/2)-\nu}} \right\|_n = O_p(1). \tag{3.22}$$

We recall here Corollary 2.3 in Cs–Cs–H–M [6], which asserts

$$n^{\nu+(1/2)}\left\| \frac{\mathbb{F}_n(s) + \mathbb{G}_n(s) - 2s}{(s(1-s))^{(1/2)-\nu}} \right\|_n = O_p(1). \tag{3.23}$$

Combining (3.22) and (3.23) gives

$$n^{\nu+(1/2)}\left\|\frac{\mathbb{D}_n^{\leftarrow}(s)+\mathbb{F}_n(s)+\Delta_n(n/S_n)\,Q(\mathbb{G}_n(s))\,f(Q(\mathbb{G}_n(s)))}{(s(1-s))^{(1/2)-\nu}}\right\|_n$$
$$=O_p(1). \tag{3.24}$$

Now, we use (3.10), (3.12), and (3.24) to obtain

$$n^{\nu+(1/2)}\left\|\frac{\mathbb{D}_n^{\leftarrow}(s)+\mathbb{D}_n(s)-2s-\Delta_n Q(s)\,f\circ Q(s)+\Delta_n(n/S_n)\,Q(\mathbb{G}_n(s))\,f(Q(\mathbb{G}_n(s)))}{(s(1-s))^{(1/2)-\nu}}\right\|_n$$
$$=O_p(1). \tag{3.25}$$

It remains for us to study

$$\begin{aligned}(n/S_n)\,&Q(\mathbb{G}_n(s))\,f(Q(\mathbb{G}_n(s)))-Q(s)\,f(Q(s))\\ &=-\Delta_n(n/S_n)\,Q(\mathbb{G}_n(s))\,f(Q(\mathbb{G}_n(s)))\\ &\quad+(\mathbb{G}_n(s)-s)(Qf(Q))'\,(s+\theta(n,s)(\mathbb{G}_n(s)-s))\end{aligned}$$

for some $0\leqslant\theta(n,s)\leqslant 1$. Observe that

$$n^{\nu}\Delta_n\left\|\frac{Q(\mathbb{G}_n(s))\,f(Q(\mathbb{G}_n(s)))}{(s(1-s))^{(1/2)-s}}\right\|_n=O_p(1),$$

since $\|Q(s)\,f(Q(s))\|=\|xf(x)\|<\infty$ by B2 and $n^{1/2}\Delta_n=O_p(1)$ by A1 and the central limit theorem. Moreover, using (3.10), B3, and the linear bounds for $\mathbb{G}_n$ given in Wellner [22], we have

$$\begin{aligned}n^{\nu}&\left\|\frac{(\mathbb{G}_n(s)-s)(Qf(Q))'(s+\theta(n,s)(\mathbb{G}_n(s)-s))}{(s(1-s))^{(1/2)-\nu}}\right\|_n\\
&\leqslant n^{\nu-(1/2)}\left\|\frac{n^{1/2}(\mathbb{G}_n(s)-s)-B_n(s)}{(s(1-s))^{(1/2)-\nu}}\,(Qf(Q))'\,(s+\theta(n,s)(\mathbb{G}(s)-s))\right\|_n\\
&\quad+n^{\nu-(1/2)}\left\|\frac{B_n(s)}{(s(1-s))^{(1/2)-\nu}}\,((Qf(Q))'\,(s+\theta(n,s)(\mathbb{G}(s)-s))\right\|_n \qquad (3.26)\\
&\leqslant n^{-1/2}O_p(1)\,\|(Qf(Q))'\,(s)\|_n\\
&\quad+n^{\nu-(1/2)}\left\|\frac{B_n(s)}{(s(1-s))^{(1/2)-\nu}}\,(Qf(Q))'\,(s)\right\|_n O_p(1). \qquad (3.27)\end{aligned}$$

Since by Darling and Erdős [10],

$$\left\| \frac{B_n(s)}{(s(1-s))^{1/2}} \right\|_n = O_p(\log \log n)^{1/2}, \tag{3.28}$$

both (3.27) and (3.28) are $O_p(1)$ provided

$$n^{-1/2} \|(Qf(Q))'(s)\|_n = O(1) \tag{3.29}$$

and

$$(\log \log n)^{1/2} n^{\nu-(1/2)} \|(s(1-s))^{\nu}(Qf(Q))'(s)\|_n = O(1). \tag{3.30}$$

Assumptions B1 and (2.3) imply that (3.29) and (3.30) hold. Hence (3.26) is $O_p(1)$ and with (3.25), this gives (2.4). ∎

Proof of Theorem 2.3. Equations (2.2) and (2.4) give (2.5). ∎

Proof of Theorem 2.5. (i) Using (3.6), we have

$$\mathbb{D}_n(s) = \frac{\mathbb{F}_n(F((S_n/n)\,Q(s))}{F((S_n/n)\,Q(s))}\, F\left(\frac{S_n}{n} Q(s)\right).$$

Now, Wellner [22] provides the needed linear bounds on $\mathbb{F}_n$, and (3.4) gives the conclusion.

(ii) Using (3.18) and A3 we have

$$\left\| \mathbb{D}_n^{\leftarrow}(s) - \mathbb{G}_n(s) + \Delta_n \frac{n}{S_n} Qf(Q(\mathbb{G}_n(s))) \right\|_n = O_p(n^{-1}).$$

Next, the linear bounds on $\mathbb{G}_n$ given in Wellner [22], combined with the fact that $Qf(Q)$ is regularly varying at $0+$ and $1-$ (under B3), leads easily to linear bounds for $Qf(Q(\mathbb{G}_n(s)))$, and the results follow. ∎

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