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Unitary Actions of Levy Flows of Diffeomorphisms

DAVID APPLEBAUM

The Nottingham Trent University,
Nottingham, NG1 4BU England

A stochastic integral representation is obtained for unitary operators induced by a class of flows of diffeomorphisms of a smooth manifold which are driven by stochastic processes with stationary and independent increments. © 1994 Academic Press, Inc.

1. INTRODUCTION

In this paper we consider a class of stochastic flows which are obtained as the solution of stochastic integral equations driven by a multi-dimensional Lévy process. Although our class is somewhat more restrictive than that considered by Fujiwara and Kunita [7, 8], we feel that it is worthy of separate consideration as

(i) it is large enough to include flows driven by a finite number of independent Brownian motions or Poisson processes;

(ii) its structure is closely related to that of multiplicative processes taking values in a Lie group as described by Holevo in [9] (see also [6]).

This class has been studied from a probabilistic point of view by Kunita and the author in [2]. Here our motivation is more operator theoretic and falls within the programme (see [3–5, 19]) of developing a symbiosis between stochastic differential geometry and quantum probability (see also below).

Our main result is to obtain a stochastic integral form for the family of unitary operators implementing the flow on a certain Hilbert space which is specified in Section 4, thus generalising results obtained in [3] (see also [19]). Further motivation for this analysis is as follows.

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(a) Reformulation of a problem in a Hilbert space context can often lead to fruitful gains. Unitary representations of deterministic flows were first discussed by Koopman in [13]. Their application to ergodic theory is described in [17].

(b) In [4], the author initated the study of quantum stochastic flows on manifolds. These are defined by algebraic stochastic differential equations driven by the Fock space processes of creation, conservation, and annihilation where the coefficients are complex vector fields on the manifold. The building blocks for constructing these flows were unitary operator valued stochastic differential equations of the type discussed below; however, in [4] we were only able to consider the Brownian case. The results of this paper will then allow us to construct a far larger class of such quantum flows (see [5] for some results in this direction).

Notation. If M is a manifold then $C_0(M)$ is the space of smooth functions on M which vanish at ∞ and $C_K^\infty(M)$ is the space of smooth functions with compact support. Càdlàg means right continuous with left limits. Einstein summation convention is used where appropriate.

2. Geometric Background

Let M be a smooth, orientable real manifold of dimension $d < \infty$ equipped with a volume form μ_v and let μ be the corresponding Borel measure on M. We denote by $\xi = (\xi_t, t \in \mathbb{R})$ the flow of diffeomorphisms of M determined infinitesimally by the complete, smooth vector field Y, so that for each $t \in \mathbb{R}$, $\xi_t = \exp(tY)$ where exp is the exponential map from $\mathbb{R}$ into $\mathrm{Diff}(M)$. We denote by $j = (j_t, t \in \mathbb{R})$ the induced flow of automorphisms of $C^\infty(M)$ given by

$$j_t(f) = f \circ \xi_t \tag{2.1}$$

for each $f \in C^\infty(M)$, $t \in \mathbb{R}$ and note that each j_t leaves $C_K^\infty(M)$ invariant. Let $\mathfrak{H}_0$ denote the intrinsic Hilbert space of M [1], i.e., $\mathfrak{H}_0$ is the space of half-densities $f\mu_v^{1/2}$ where $f \in L^2(M, \mu)$ with inner product given by

$$\langle f\mu_v^{1/2}, g\mu_v^{1/2} \rangle = \int \bar{f} g \, d\mu, \tag{2.2}$$

where $f, g \in L^2(M, \mu)$. We use P to denote the canonical unitary isomorphism between $L^2(M, \mu)$ and $\mathfrak{H}_0$ given by

$$P(f) = f\mu_v^{1/2} \tag{2.3}$$

We denote by $\mathfrak{D}_0$ the dense subspace of $\mathfrak{H}_0$ comprising smooth half-densities of the form $f\mu_v^{1/2}$ where $f \in C_K^\infty(M)$.

Let $V = (V(t), t \in \mathbb{R})$ denote the strongly continuous one-parameter group of unitary operators in $\mathfrak{H}_0$ given by continuous extension of the prescription

$$V(t) f\mu_v^{1/2} = f(\xi_t)(\xi_t^*(\mu_v))^{1/2} \tag{2.4}$$

for $t \in \mathbb{R}$, $f \in C_K^\infty(M)$, where ξ_t^* denotes the pullback of ξ_t to the bundle of d-forms. To see that each $V(t)$ is unitary, we write $W(t) = P^{-1}V(t)P$ and observe that

$$W(t)f = f(\xi_t)\sqrt{\frac{d\mu(\xi_t)}{d\mu}} \tag{2.5}$$

for each $f \in C_K^\infty(M)$.

We denote by $-iT_Y$ the infinitesimal generator of V. $\mathfrak{D}_0$ is an invariant core for the skew-adjoint operator T_Y on which it acts as $Y + \frac{1}{2}\operatorname{div}(Y)$, where $\operatorname{div}(Y)$ is the divergence of Y with respect to μ_v.

For the remainder of this paper $Y_1, ..., Y_n$ denotes n complete, smooth vector fields on M and $\mathfrak{L}$ denotes the Lie algebra which they generate. We make the assumption that $\dim(\mathfrak{L}) < \infty$ in which case it is shown in [15] that every member of $\mathfrak{L}$ is itself complete. For $x = (x^1, ..., x^n) \in \mathbb{R}^n$ we let $\xi(x)$ denote the diffeomorphism $\exp(x^j Y_j)$ and $V(x)$ denote the corresponding unitary operator $\exp(x^j T_{Y_j})$, where this latter exponential is understood in the sense of functional calculus.

3. Levy Flows

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space. For fixed $s \in \mathbb{R}^+$, let $\Phi = \{\Phi_{s,t};\ t \geqslant s\}$ be a family of measurable maps from $M \times \Omega$ into M and define, for each $\omega \in \Omega$, $\Phi_{s,t}^\omega: M \to M$ by

$$\Phi_{s,t}^\omega(x) = \Phi_{s,t}(x, \omega)$$

for each $x \in M$. We say that Φ is a *Lévy flow* if the following conditions are satisfied:

(i) $\Phi_{s,t}^\omega$ is a diffeomorphism of M for all $t \geqslant s$ and almost all $\omega \in \Omega$.

(ii) We have

$$\Phi_{s,t}^\omega \circ \Phi_{r,s}^\omega = \Phi_{r,t}^\omega \qquad \text{for all} \quad r \leqslant s \leqslant t$$

and

$$\Phi_{s,s}^\omega(x) = x \qquad \text{for all} \quad s \in \mathbb{R}^+,\ x \in M$$

for almost all $\omega \in \Omega$.

(iii) For each $n \in \mathbb{N}$, $x_1, \ldots, x_n \in M$, $s \leqslant t_1 < t_2 < \cdots < t_{n+1}$, the random variables $\Phi_{t_i, t_{i+1}}(x_i, \cdot)$, for $1 \leqslant i \leqslant n$ are independent.

(iv) For every $x \in M$ and $\omega \in \Omega$ and for all $t \geqslant s$, the map $t \to \Phi_{s,t}(x, \omega)$ is càdlàg.

If "càdlàg" is replaced by "continuous" in (iv) we obtain the notion of (forward) Brownian flow as described in [14]. All Brownian flows can be obtained by solving stochastic differential equations (sde's) driven by a possibly infinite number of Brownian motions. To discuss a similar approach to Lévy flows we first recall the notion of n-dimensional Lévy process. This is a process $X = (X(t),\ r \in \mathbb{R}^+)$ taking values in $\mathbb{R}^n$ with stationary and independent increments which is continuous in probability and for which $X(0) = 0$. It is shown in [16] that such a process has a unique modification which is càdlàg and is also a Lévy process. We work throughout with this version.

The canonical construction of X is as follows. We take Ω to be the space of all càdlàg functions ω from $\mathbb{R}^+$ to $\mathbb{R}^n$ with $\omega(0) = 0$ and $\mathfrak{F}$ to be the smallest σ-algebra for which all the mappings $\omega \to \omega(t)$ are measurable for all $t \in \mathbb{R}^+$. Now define

$$X(t)\omega = \omega(t); \tag{3.1}$$

then there exists a probability measure P on $(\Omega, \mathfrak{F})$ such that X is a Lévy process on $(\Omega, \mathfrak{F}, P)$ adapted to the filtration $\mathfrak{F}_t = \sigma\{X(s);\ 0 \leqslant s \leqslant t\}$.

Each $X(t)$ can be realised as a self-adjoint multiplication operator on $L^2(\Omega, \mathfrak{F}, P)$ with dense domain

$$\mathfrak{D}(X(t)) = \left\{\Psi \in L^2(\Omega, \mathfrak{F}, P);\ \int_\Omega \omega(t)^2\, |\Psi(\omega)|^2\, dP(\omega) < \infty\right\}$$

for each $t \in \mathbb{R}^+$. For each $t \in \mathbb{R}^+$, let $\Delta X(t) = X(t) - X(t-)$, then since $P(\Delta X(t) = 0) = 1$, we see that for every $\Psi \in \mathfrak{D}(X(t))$ we have $\mathbb{E}(|\Delta X(t)\Psi|^2) = 0$. It follows that the map $t \to X(t)\Psi$ is continuous from $\mathbb{R}^+$ into $L^2(\Omega, \mathfrak{F}, P)$.

The Lévy–Itô decomposition of X (see, e.g., [12]) is as follows. Writing $X = (X^1, \ldots, X^n)$, for $1 \leqslant i \leqslant n$ and for each $t \in \mathbb{R}^n$ we have

$$\begin{aligned} X^i(t) = b^i t + \sigma^i_j B^j(t) + \int_0^{t+} \int_{|x| \geqslant 1} x^i N(dt, dx) \\ + \int_0^{t+} \int_{|x| < 1} x^i \tilde{N}(dt, dx), \end{aligned} \tag{3.2}$$

where $b = (b^1, \ldots, b^n) \in \mathbb{R}^n$, $\sigma = (\sigma^i_j)$ is a real $m \times n$ matrix where $m \leqslant n$, $B = (B^1, \ldots, B^m)$ is an m-dimensional standard Brownian motion, N is a

Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^n - \{0\})$ independent of B with associated Lévy measure ν on $\mathbb{R}^n - \{0\}$ so that

$$\nu(\mathfrak{U}) = \mathbb{E}(N(1, \mathfrak{U})) \tag{3.3}$$

where $\mathfrak{U}$ is a Borel set in $\mathbb{R}^n - \{0\}$ with $0 \notin \bar{\mathfrak{U}}$ and finally $\tilde{N}$ is the compensator defined by

$$\tilde{N}(t, \mathfrak{U}) = N(t, \mathfrak{U}) - t\nu(\mathfrak{U}) \tag{3.4}$$

for each $t \in \mathbb{R}^+$ where $\mathfrak{U}$ is as above. $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$.

Using the notation of Section 2, we introduce the closeable linear operator $\mathscr{G}$ on $C_0(M)$ with domain $C_K^2(M)$ given by

$$\begin{aligned}\mathscr{G}(f)(p) &= m^i Y_i(f)(p) + \frac{1}{2} a^{ij} Y_i Y_j(f)(p) \\ &\quad + \int_{\mathbb{R}^n - \{0\}} \left(f(\xi(x)(p)) - f(p) - \frac{x^j}{1 + |x|^2} Y_j(f)(p) \right) \nu(dx) \end{aligned} \tag{3.5}$$

for $p \in M$, $f \in C_K^2(M)$ where $m = (m^1, \quad, m^n) \in \mathbb{R}^n$ and a is the non-negative definite $n \times n$ matrix $\sigma^{\mathrm{T}}\sigma$. The closure of $\mathscr{G}$ generates a Markov semigroup $T = (T_t, t \in \mathbb{R}^+)$ on $C_0(M)$. We note that when $\mathfrak{L}$ is abelian, then $\xi : x \to \xi(x)$ is a homomorphism from $\mathbb{R}^n$ into Diff(M). In this case we see by (3.5) and [11] that T induces a weakly continuous convolution semigroup of probability measures on Im(ξ).

Now for each $s \in \mathbb{R}^+$, $t \geqslant s$ consider the stochastic integral equation given by

$$\begin{aligned} f(\Phi_{s,t}(p)) &= f(p) + \int_s^t \sigma_j^i (Y_i f)(\Phi_{s,u-}(p))\, dB^j(u) \\ &\quad + \int_s^{t+} \int_{|x| \geqslant 1} [f(\xi(x) \circ \Phi_{s,u-}(p)) - f(\Phi_{s,u-}(p)]\, N(du, dx) \\ &\quad + \int_s^{t+} \int_{|x| < 1} [f(\xi(x) \circ \Phi_{s,u-}(p)) - f(\Phi_{s,u-}(p))]\, \tilde{N}(du, dx) \\ &\quad + \int_s^t \bigg[b^i (Y_i f)(\Phi_{s,u-}(p)) + \frac{1}{2} a^{ij} (Y_i Y_j f)(\Phi_{s,u-}(p)) \\ &\quad + \int_{|x|<1} [(f(\xi(x) \circ \Phi_{s,u-}(p)) - f(\Phi_{s,u-}(p)) \\ &\quad - x^j (Y_j f)(\Phi_{s,u-}(p)))\, \nu(dx)] \bigg] du \end{aligned} \tag{3.6}$$

for each $f \in C^\infty(M)$, $p \in M$.

In [2] it is shown that a unique solution exists to (3.6) giving rise to a Lévy flow of diffeomorphisms as required.

Conditions for the existence and uniqueness of solutions to a more general class of equations than (3.6) have been discussed in [7] for the case of $M = \mathbb{R}^d$ and [8] for the case of M compact. We remark that if we take for $1 \leqslant i \leqslant n$

$$m^i = b^i - \int_{|x| \geqslant 1} \frac{x^i}{1+|x|^2}\, \nu(dx) + \int_{|x|<1} \frac{x^i\,|x|^2}{1+|x|^2}\, \nu(dx),$$

then for each $t \in \mathbb{R}^+$, $f \in C_0(M)$, $p \in M$, we have

$$T_t(f)(p) = \mathbb{E}(f(\Phi_{0,t}(p))). \tag{3.7}$$

For a simple example take $n = 1$ and X to be the canonical one-dimensional Lévy process given by (3.1) then it is not difficult to verify using Itô's formula (see [12]) that Φ is a Lévy flow which satisfies an equation of the type (3.6) where

$$\Phi_{s,t}(p, \omega) = \xi_{\omega(t) - \omega(s)}(p). \tag{3.8}$$

For this case we see by (3.7) and (3.8) that we may write T as

$$T_t(f)(p) = \int_{\mathbb{R}} f(\xi_x(p))\, p_t(dx), \tag{3.9}$$

where each $p_t = P \circ X(t)^{-1}$.

In the case where $\mathfrak{L}$ is abelian, solutions to (3.5) which are again Lévy flows may be obtained by taking products of flows of the form (3.8) wherein each term samples from mutually independent one-dimensional Lévy processes so that, in particular, $m = n$ and σ is diagonal in (3.2).

We now return to the general case. Let $\Lambda^p(M)$ denote the space of smooth p-forms on M where $0 \leqslant p \leqslant d$ and for each $1 \leqslant i \leqslant n$, let L_i denote the Lie derivative with respect to the vector field Y_i acting on $\Lambda^p(M)$. For a given Lévy flow Φ we define its stochastic pullback Φ^* to $\Lambda^p(M)$ in the usual way by

$$\Phi^*_{s,t}(\alpha)(p, \omega) = ((\Phi^{\omega}_{s,t})^*\,(\alpha))(p) \tag{3.10}$$

for each $t \geqslant s$, $\alpha \in \Lambda^p(M)$, $p \in M$, $\omega \in \Omega$.

For notational convenience, in the sequel we write each such $\Phi^*_{s,t}(\alpha)$ as $\alpha_{s,t}$ and for each $x \in \mathbb{R}^n$, $\xi(x)^*\,(\alpha)_{s,t}$ means $(\Phi^*_{s,t} \circ \xi(x)^*)\alpha$. We now extend (3.6) to $\Lambda^p(M)$ as follows.

LEMMA 1. *For all* $t \geqslant s$, $\alpha \in \Lambda^p(M)$, $p \in M$ *we have*

$$\alpha_{s,t}(p) = \alpha(p) + \int_s^t \sigma_j^i L_i(\alpha)_{s,u-}(p)\, dB^j(u)$$

$$+ \int_s^{t+} \int_{|x| \geqslant 1} [\xi(x)^* (\alpha)_{s,u-}(p)) - \alpha_{s,u-}(p)]\, N(du, dx)$$

$$+ \int_s^{t+} \int_{|x| < 1} [\xi(x)^* (\alpha)_{s,u-}(p)) - \alpha_{s,u-}(p)]\, \tilde{N}(du, dx)$$

$$+ \int_s^t [b^i L_i(\alpha)_{s,u-}(p) + \frac{1}{2} a^{ij}(L_i L_j(\alpha))_{s,u-}(p)$$

$$+ \int_{|x|<1} [\xi(x)^* (\alpha)_{s,u-}(p))$$

$$- \alpha_{s,u-}(p) - x^i (L_i(\alpha))_{s,u-}(p)]\, \nu(dx)]\, du.$$

Proof. This is a straightforward extension of the methods used in pp. 205–207 of [14]. ∎

We now take α in (3.11) to be our volume form μ_v and note that for $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} L_i(\mu_v) &= \mathrm{div}(Y_i)\, \mu_v \\ L_i L_j(\mu_v) &= [(\mathrm{div}(Y_i)\, \mathrm{div}(Y_j) + Y_i(\mathrm{div}(Y_j))]\, \mu_v. \end{aligned} \tag{3.12}$$

To keep the notation concise we write $\beta_{s,t} = (\Phi_{s,t}^*(\mu_v))^{1/2}$, $\xi(x)^* (\beta)_{s,t} = (\Phi_{s,t}^* \circ \xi(x)^* (\mu_v))^{1/2}$, $c_j(p)_{s,t} = (\mathrm{div}(Y_j))(\Phi_{s,t}(p))$ and $d_{i,j}(p)_{s,t} = (Y_i(\mathrm{div}(Y_j)))(\Phi_{s,t}(p))$ for each $t \geqslant s$, $1 \leqslant j \leqslant n$ and $p \in M$.

We now extend 3.6 to the half-density μ_v.

COROLLARY 2. *For all* $t \geqslant s$, $p \in M$, *we have*

$$\beta_{s,t}(p) = \mu_v^{1/2}(p) + \frac{1}{2} \int_s^t \sigma_j^i c_i(p)_{s,u-}\, \beta_{s,u-}(p)\, dB^j(u)$$

$$+ \int_s^{t+} \int_{|x| \geqslant 1} [\xi(x)^* (\beta)_{s,u-}(p)) - \beta_{s,u-}(p)]\, N(du, dx)$$

$$+ \int_s^{t+} \int_{|x| < 1} [\xi(x)^* (\beta)_{s,u-}(p)) - \beta_{s,u-}(p)]\, \tilde{N}(du, dx)$$

$$+ \int_s^t \left[\frac{1}{2} b^i c_i(p)_{s,u-}\, \beta_{s,u-}(p)\right.$$

$$+a^{ij}\left\{\frac{1}{8}\,c_i(p)_{s,u-}\,c_j(p)_{s,u-}+\frac{1}{4}\,d_{i,j}(p)_{s,u-}\right\}$$
$$+\int_{|x|<1}\Big[\xi(x)^*\,(\beta)_{s,u-}(p))-\beta_{s,u-}(p)$$
$$\left.-\frac{1}{2}\,x^ic_i(p)_{s,u-}\,\beta_{s,u-}(p)\Big]\,\nu(dx)\right]du. \tag{3.13}$$

Proof. Follows by Itô's formula from (3.11) using (3.12). ∎

4. The Flow of Unitaries

Let $\mathfrak{h}$ denote the complex separable Hilbert space $L^2(\Omega, \mathfrak{F}, P; \mathfrak{h}_0)$ and let $\mathfrak{D}$ be the dense subspace of $\mathfrak{h}$ comprising those $\Psi \in \mathfrak{h}$ for which $\psi(\omega)=f(\omega)\,\mu_v^{1/2}\in\mathfrak{D}_0$ for all $\omega\in\Omega$ where f is a square integrable map from Ω into $C_K^\infty(M)$. In the sequel we often extend linear operators on $\mathfrak{h}_0$ to the whole of $\mathfrak{h}$ in the obvious manner. We do not feel it necessary to identify such extensions by additional notation. Now define a family of unitary operators $U=(U(s,t),\ t\geqslant s)$ on $\mathfrak{h}$ by continuous extension of the prescription

$$(U(s,t)\,\Psi)(\omega)=(f(\omega)\circ\Phi_{s,t}^\omega)((\Phi_{s,t}^\omega)^*\,(\mu_v))^{1/2} \tag{4.1}$$

for each $\omega\in\Omega$, where $\Psi\in\mathfrak{D}$ is as above. We note that the map $t\to U(s,t)$ is strongly càdlàg.

Define, for each $t\geqslant s$, a family of homomorphisms from $C_K^\infty(M)\to L^\infty(M\times\Omega,\mu\times P)$ by

$$J_{s,t}(f)=f\circ\Phi_{s,t} \tag{4.2}$$

for each $f\in C_K^\infty(M)$ (cf. (2.1)) then it is easy to see that each

$$J_{s,t}(f)=U(s,t)\,fU(s,t)^{-1}, \tag{4.3}$$

where $J_{s,t}(f)$ in (4.3) is to be understood as a multiplication operator.

It is easily verified that each $U(s,t)^{-1}$ has a similar action to (4.1) wherein each $\Phi_{s,t}^\omega$ is replaced by $(\Phi_{s,t}^\omega)^{-1}$ so that these operators are associated to a backwards Lévy flow.

A simple example of (4.1) where U has an explicit form is obtained by taking Φ to be the flow given by (3.8) where we find that for each $\Psi\in\mathscr{D}$, $\omega\in\Omega$,

$$(U(s,t)\,\Psi)(\omega)=\exp((\omega(t)-\omega(s))\,T_Y)\,\Psi(\omega). \tag{4.4}$$

We need a specific class of operator valued stochastic integrals which we now describe. Let $C = (C(t),\ t \in \mathbb{R}^+)$ be a family of densely defined linear operators on $\mathfrak{h}$ for which $\mathfrak{D} \subseteq \mathrm{Dom}(C(t))$ for each $t \in \mathbb{R}^+$ and such that for each $\Psi \in \mathfrak{D}$, the vector valued process $C(t)\Psi$ is adapted to $\mathfrak{F}_t$. We denote by $\mathcal{M}$ any semimartingale which is adapted to $\mathfrak{F}_t$. Now define the operator valued stochastic integral $I_c(t) = \int_0^t C(u)\, d\mathcal{M}(u)$, for each $t \in \mathbb{R}^+$ by

$$\left(\int_0^t C(u)\, d\mathcal{M}(u)\right)\psi = \int_0^t (C(u)\psi)\, d\mathcal{M}(u) \tag{4.5}$$

for each $\psi \in \mathfrak{D}$. Clearly $\psi \in \mathrm{Dom}(I_c(t))$ whenever the right hand side of (4.5) exists. We consider two examples. First, if M is a standard Brownian motion we see that for each $t \in \mathbb{R}^+$, $\psi \in \mathrm{Dom}(I_c(t))$ if $\int_0^t \mathbb{E}\, \|C(u)\psi\|^2\, du < \infty$ where $\|\cdot\|$ here denotes the norm in $\mathfrak{h}_0$. Second, let $D_p = \{s \in \mathbb{R}^+,\ \Delta X(s) \neq 0\}$ and let $p = (p(t),\ t \in D_p)$ be the Poisson point process defined by $p(t) = \Delta X(t)$. If $C_x = (C(t, x),\ t \in \mathbb{R}^+)$ is a family of adapted operator valued processes of the type described above indexed by $x \in \mathbb{R}^n - \{0\}$, we define for $t \in \mathbb{R}^+$, $I_c(t) = \int_0^t \int_{|x| > 1} C(t, x)\, dN(t, x)$ then $\Psi \in \mathrm{Dom}(I_c(t))$ if $\mathbb{E} \sum_{s \leqslant t,\, s \in D_p} \|C(s, p(s))\Psi\|^2 < \infty$. By similar arguments to those used in Section 3, we see that in this last example the map $t \to I_c(t)\Psi$ is continuous from $\mathbb{R}^+$ to $\mathfrak{h}$.

Our main result is the following unitary flow valued version of (3.6) where all stochastic integrals are understood in the sense of (4.5) on the domain $\mathfrak{D}$. Here I denotes the identity operator in $\mathfrak{h}$.

THEOREM 3. *For all $t \geqslant s$, the map $t \to U(s, t)$ is strongly continuous. Moreover, it admits the following stochastic differential*

$$\begin{aligned} dU(s, t) = U(s, t) \Bigg[\!\!\Bigg[& \sigma^{ij} T_{Y_i}\, dB_j(t) + \int_{|x| \geqslant 1} (V(x) - I)\, N(dt, dx) \\ & + \int_{|x| < 1} (V(x) - I)\, \tilde{N}(dt, x) \\ & + \left[b^i T_{Y_i} + \frac{1}{2}\, a^{ij} T_{Y_i} T_{Y_j} + \int_{|x| < 1} (V(x) - I - x^i T_{Y_i})\, \nu(dx) \right] dt \Bigg]\!\!\Bigg]. \end{aligned} \tag{4.6}$$

Proof. For each $f \in C_K^\infty(M)$ we write $g_{s,t} = f \circ \Phi_{s,t} - f$ and $\gamma_{s,t} = \beta_{s,t} - \mu_v^{1/2}$. We now rewrite (4.1) as

$$U(s, t)\Psi = f\beta_{s,t} + (f \circ \Phi_{s,t})\, \mu_v^{1/2} + g_{s,t}\gamma_{s,t} - \Psi.$$

We apply Itô's product formula in integral form to $g_{s,t}\gamma_{s,t}$ to obtain

$$g_{s,t}\gamma_{s,t} = \int_s^{t+} g_{s,u-}\, d\gamma_{s,u-} + \int_s^{t+} dg_{s,u-}\, \gamma_{s,u-} + \int_s^{t+} dg_{s,u}\, d\gamma_{s,u-}.$$

Now substitute in this expression from (3.6) and (3.13) and observe that the Itô correction term is given by

$$\begin{aligned}\int_s^{t+} dg_{s,u-}\, d\gamma_{s,u-} &= \int_s^{t+}\int_{|x|\geq 1} [f(\xi(x)\circ\Phi_{s,u-})-f(\Phi_{s,u-})] \\ &\quad\times[\xi(x)^*(\beta)_{s,u-}-\beta_{s,u-}]\, N(du,dx) \\ &\quad+\int_s^{t+}\int_{|x|<1} [f(\xi(x)\circ\Phi_{s,u-})-f(\Phi_{s,u-})] \\ &\quad\times[\xi(x)^*(\beta)_{s,u-}-\beta_{s,u-}]\, \tilde{N}(du,dx) \\ &\quad+\int_s^{t}\left[\frac{1}{2}a^{ij}(Y_i(f))(\Phi_{s,u-})\, c_i(\cdot)_{s,u-}\,\beta_{s,u-}\right. \\ &\quad+\int_{|x|<1} [f(\xi(x)\circ\Phi_{s,u-})-f(\Phi_{s,u-})] \\ &\quad\left.\times[\xi(x)^*(\beta)_{s,u-}-\beta_{s,u-}]\, \nu(dx)\right] dt.\end{aligned}$$

Now collect together the terms, observing that for $1\leqslant i,\ j\leqslant n$, we have

$$T_{Y_i}T_{Y_j}=Y_iY_j+\tfrac{1}{2}Y_i(\operatorname{div}(Y_j))+\operatorname{div}(Y_j)\,Y_i+\tfrac{1}{4}\operatorname{div}(Y_i)\operatorname{div}(Y_j),$$

then we see that each $U(s,t)-I$ is a sum of operator valued stochastic integrals of the form $I_c(t)$ discussed above. Since each of these is continuous from $\mathbb{R}^+$ to $\mathfrak{h}$ it follows that $t\to U(s,t)\Psi$ is continuous for $\Psi\in\mathfrak{D}$. Since each $U(s,t)$ is bounded we obtain our required strong continuity by a standard $\varepsilon/3$ argument (see, e.g., [17, p. 27]). The precise form (4.6) now follows from the differential form of the expression obtained above. ∎

From a quantum probabilistic point of view, as described in the introduction, it is natural to consider (4.6) as a stochastic differential equation with initial condition $U_{s,s}=I$ from which, if it has a unique unitary solution, the flow can be reconstructed as multiplication operators using (4.2).

We consider two examples of our equation.

(i) *Brownian Flows*

Here we taken $m=n$, $\nu=0$, and write (for convenience) $X^0=b^iY_i$ and $X^j=\sigma_i^jY^i$ then in this case $t\to\Phi_{s,t}$ is continuous and (3.6) may be written in differential form, using Stratonovitch notation, as

$$d\Phi_{s,t}=X^j(\Phi_{s,t})\circ dB_j(t)+X^0(\Phi_{s,t})\,dt, \tag{4.7}$$

and the corresponding version of (4.6) is

$$dU(s, t) = U(s, t)\left(T_{x_j}\, dB^j(t) + \left(T_{x_0} + \tfrac{1}{2} \sum_{j=1}^{n} (T_{x_j})^2\right) dt\right). \tag{4.8}$$

Under our assumptions on the Y_j's, it is shown in [14, pp. 194–195] that (4.7) has a unique solution which is a Brownian flow of diffeomorphisms thus we may deduce by (4.1) the unitarity of solution to (4.8) (with initial value as above). We note that in this case the equation (4.8) has been independently derived from (4.7) in [19] using a different method.

(ii) *Poisson Flows*

Let $N = (N_1, ..., N_n)$ be a family of independent Poisson processes with N_j having intensity λ_j. In (3.6) we take $\sigma = 0$ and $b_j = \lambda_j$ for $1 \leqslant j \leqslant n$. To describe the measure ν we write $\mathbb{R}^n = \bigcup_{j=1}^n \hat{\mathbb{R}}_j$ where each $\hat{\mathbb{R}}_j = \{(0, ..., 0, x^{(j)}, 0, ...,), x \in \mathbb{R}\}$ and let ν be the decomposable measure [18, p. 281] which is given on each $\hat{\mathbb{R}}_j$ by $\nu((a, b) - \{0\}) = 0$ if $-\infty < a, b \leqslant 1$ and $\nu((a, b)) = \lambda_j$ otherwise. Equation (3.6) then becomes

$$f(\Phi_{s,t}) = \sum_{j=1}^{n} \int_s^t (f \circ \xi_j \circ \Phi_{s,u-} - f \circ \Phi_{s,u-})\, dN_j(u) \tag{4.9}$$

and (4.6) takes the form

$$dU(s, t) = \sum_{j=1}^{n} U(s, t)(V_j - I)\, dN_j(u), \tag{4.10}$$

where $\xi_j = \exp(Y_j)$ and $V_j = \exp(T_{Y_j})$.

We now use the canonical isomorphism between $\mathfrak{h}$ and $\mathfrak{h}_0 \otimes L^2(\Omega, \mathfrak{F}, P)$ to identify these spaces and invoke the theory of [10] to identify $L^2(\Omega, \mathfrak{F}, P)$ with symmetric Fock space over $L^2(\mathbb{R}^+)$, then it follows immediately from the results of [10] that (4.10) has a unique unitary solution. We note that this particular result remains valid even if we drop our earlier condition that $\dim(\mathfrak{L}) < \infty$.

Many of the results of this paper can be generalised to the case where the Poisson random measure N is defined on $G - \{e\}$, where G is a connected finite-dimensional Lie subgroup of $\mathrm{Diff}(M)$ with identity element e. For further examples, see [2].

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