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Central Limit Theorem, Weak Law of Large Numbers for Martingales in Banach Spaces, and Weak Invariance Principle—A Quantitative Study

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This article deals with quantitative results by involving the standard modulus of continuity in Banach spaces. These concern convergence in distribution for Banach space-valued martingale difference sequences and weak convergence of the distribution of random polygonal lines to the Wiener-measure on $C([0, 1])$. A general theorem is given with applications to the central limit theorem and weak law of large numbers for Banach space-valued martingales. Another general theorem is presented on the weak invariance principle with an application to a central limit theorem for real-valued martingales. The exposed results generalize earlier related results of Butzer, Hahn, Kirschfink, and Roeckerath.

INTRODUCTION

Here we present generalizations of some results from Butzer *et al.* (1983) (Part A and Butzer and Kirschfink (1986) (Part B). The improvement in this work is that our inequalities involve the standard modulus of continuity of certain Fréchet derivative of the acting function in the associated weak convergences. This is achieved through the implementation of some general results from Anastassiou [2] (1986) and Lemmas 1 and 2. The corresponding theorems in the above-mentioned references can involve only functions whose certain Fréchet derivative belongs to a Lipschitz class of functions.

However, the author considers the work of the above-mentioned researchers to be pioneering in this direction of research and he feels that

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his improvement was made possible only because of their pre-existing important work.

About Part A. Let $(X_i, \mathscr{F}_i)_{i \in \mathbb{Z}_+}$ be a martingale difference sequence of Banach space B-valued random variables, defined on the probability space $(\underline{\Omega}, \mathscr{A}, P)$. Denote $S_n := \sum_{i=1}^n X_i$. Let Z be a B-valued random variable which is φ-decomposable and $f: B \to \mathbb{R}$ be a $(r-1)$-times uniformly continuously Fréchet differentiable and bounded function, $r \in \mathbb{N} - \{1\}$. Here E stands for the expectation. The random variables X_i, $i \in \mathbb{Z}_+$ do not have to be independent.

In Theorem 1, under the moment condition (21), we find an estimate for $|E(f(\varphi(n) \cdot S_n)) - E(f(Z))|$, see ineqality (22), which involves $\omega_1(f^{(r-1)}, h)$ —the modulus of continuity, h, depends on X_i, Z, and the normalizing function φ. Applications of this theorem are Theorem 2, the weak law of large numbers, and Theorem 3, the central limit theorem for martingale difference sequences on Banach spaces.

About Part B. Here we study the weak convergence of the distribution of the random polygonal lines $S_n(\omega, t)$, see (46), to the Wiener-measure W_R on $C = C([0, 1])$. In this work again, the random variables X_i determining S_n do not have to be independent. This is possible due to a moment condition, that is, relation (65) of our main theorem, Theorem 4. There we estimate $|E(f(S_n(t))) - E(f(W(t)))|$, see inequality (67), where $f \in C_C^{r-1}$ and W is the Wiener process. Inequality (67) involves $\omega_1(f^{(r-1)}, h)$, $r \in \mathbb{N} - \{1\}$, where h depends on X_i, W, and some parameters.

When we consider a more concrete dependency structure among the random variables X_i, see Definition 3, we are able to simplify the moment condition (65) and prove Theorem 5, a central limit theorem for possibly dependent random variables. A direct application of this theorem is Theorem 6, which is the central limit theorem for a martingale difference sequence of real valued random variables.

PART A

Preliminaries

We need the following auxiliary results:

LEMMA 1. *Let $(V_1, \|\cdot\|_1)$, $(V_2, \|\cdot\|_2)$ be real normed vector spaces and Q be a subset of V_1 which is star-shaped relative to the fixed point $x_0 \in Q$. Consider $f: Q \to V_2$ with the properties*

$$f(x_0) = 0 \qquad \textit{and} \qquad \|s - t\|_1 \leqslant h \Rightarrow \|f(s) - f(t)\|_2 \leqslant w;\ w, h > 0. \quad (1)$$

Then

$$\|f(t)\|_2 \leqslant w \cdot \left\lceil \frac{\|t-x_0\|_1}{h} \right\rceil, \qquad \forall t \in Q. \tag{2}$$

(Here $\lceil \cdot \rceil$ is the *ceiling* function, i.e., for $x \geqslant 0$, $\lceil x \rceil$ is defined to be the least integer greater than or equal to x.)

Proof. When $\|t-x_0\|_1 \leqslant h$ then $\|f(t)\|_2 \leqslant w$. For any other $t \in Q$ there is $n \in \mathbb{N}$: $(n-1)\,h < \|t-x_0\|_1 \leqslant nh$. Observe that

$$t - x_0 = \sum_{k=0}^{n-1} \Delta_k,$$

where

$$\Delta_k := \frac{(n-k)\,t + kx_0}{n} - \frac{(n-(k+1))\,t + (k+1)\,x_0}{n} = \frac{t-x_0}{n}.$$

Now that $\|\Delta_k\|_1 \leqslant h$, we get

$$\|f(t)\|_2 \leqslant \sum_{k=0}^{n-1} \left\| f\left(\frac{(n-k)\,t+kx_0}{n}\right) - f\left(\frac{(n-(k+1))\,t+(k+1)\,x_0}{n}\right)\right\|_2 \leqslant n \cdot w. \qquad \text{Q.E.D.}$$

LEMMA 2. *Let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous convex function such that $\phi(0)=0$. Then*

$$\phi(x) + \phi(y) \leqslant \phi(x+y), \qquad \forall x, y \in \mathbb{R}_+. \tag{3}$$

That is, ϕ is superadditive.

Proof. Easy. Q.E.D.

For $t \in \mathbb{R}$ we define

$$\phi_0(t) := \left\lceil \frac{|t|}{h} \right\rceil, \qquad h > 0,$$

and for $r \geqslant 2$ integer we define

$$\phi_{r-1}(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \cdot \frac{(|t|-s)^{r-2}}{(r-2)!} \cdot ds. \tag{4}$$

ϕ_{r-1} is continuous and convex on $\mathbb{R}$ and strictly increasing on $\mathbb{R}^+$. Furthermore

$$\phi_{r-1}(t) \leqslant \left(\frac{|t|^r}{r!\,h} + \frac{|t|^{r-1}}{2(r-1)!} + \frac{h\,|t|^{r-2}}{8(r-2)!}\right), \qquad \forall t \in \mathbb{R}. \tag{5}$$

For more details on ϕ_{r-1} see Anastassiou [2] (1986).

DEFINITION 1. Let $(V_1, \|\cdot\|_1)$, $(V_2, \|\cdot\|_2)$ be real normed vector spaces and Q be a subset of V_1. For a continuous bounded function $f: Q \to V_2$ we define its (first) modulus of continuity

$$\omega_1(f, h) := \sup\{\|f(x) - f(y)\|_2 \colon \text{all } x, y \in Q, \|x-y\|_1 \leqslant h, h>0\}. \tag{6}$$

From now on in Part A, for the convenience of the reader to make comparisons, we follow exactly the same notations as in Butzer *et al.* (1983).

Let B be a real Banach space with a normalized basis $(e_k)_{k\in\mathbb{N}}$, and norm $\|\cdot\|_B$, $(X_i)_{i\in\mathbb{N}}$ be a sequence of B-valued integrable random variables (r.v.s) defined on a common probability space $(\Omega, \mathscr{A}, P)$, and let $(\mathscr{F}_i)_{i\in\mathbb{Z}_+}$ be an increasing sequence of sub-σ-algebras of $\mathscr{A}$ so that X_i is $\mathscr{F}_i$-measurable $\forall i \in \mathbb{N}$. Then $(X_i, \mathscr{F}_i)_{i\in\mathbb{Z}_+}$, $X_0 = 0$ is called a *martingale difference sequence* (MDS) if

$$E(X_i \mid \mathscr{F}_{i-1}) = 0 \qquad \text{a.s.} \quad (i \in \mathbb{N}). \tag{7}$$

This implies $EX_i = 0$. Let $S_n := \sum_{i=1}^n X_i$; the above property is equivalent to $(S_n, \mathscr{F}_n)_{n\in\mathbb{Z}_+}$ being a martingale, that is

$$E(S_n \mid \mathscr{F}_{n-1}) = S_{n-1} \qquad \text{a.s.} \quad (n \in \mathbb{N}). \tag{8}$$

Since B is a real Banach space with countable basis $(e_k)_{k\in\mathbb{N}}$, for each $x \in B$ there exists a unique sequence of real numbers $(d_k)_{k\in\mathbb{N}}$ so that

$$x = \sum_{k=1}^{\infty} d_k e_k. \tag{9}$$

This defines the sequence of coefficient functionals $(e_k^*)_{k\in\mathbb{N}}$ associated with the basis $(e_k)_{k\in\mathbb{N}}$, defined by $e_k^*(x) := d_k$, $k \in \mathbb{N}$, so that $e_k^* \in B^*$. These spaces B are separable, e.g., $L_p[0,1]$, l^p, $1 \leqslant p < \infty$, $C[0,1]$, c_0.

Let $B^j := B \times \cdots \times B$ be the j-fold product endowed with the max-norm $\|c\|_{B^j} := \max_{1\leqslant k\leqslant j} \|x_k\|_B$, where $c := (x_1, ..., x_j) \in B^j$. Then the space $\mathscr{L}_j := \mathscr{L}_j(B^j, \mathbb{R})$ of all real-valued multilinear continuous functions $g: B^j \to \mathbb{R}$ is a Banach space with norm

$$\|g\|_{\mathscr{L}_j} := \sup_{\|c\|_{B^j}=1} |g(c)| = \sup_{\substack{c\in B^j\\ x_k\neq 0}} \frac{|g(x)|}{\|x_1\|_B \cdots \|x_j\|_B}.$$

Let $f\colon B \to \mathbb{R}$ be a function with $\|f\|_\infty := \sup_{x\in B} |f(x)|$, whose Fréchet derivatives $f^{(j)}\colon B \to \mathscr{L}_j$ exist and are continuous for $1 \leqslant j \leqslant r$, $r \in \mathbb{N}$.

Then we have Taylor's formula

$$f(x+y) = f(y) + \sum_{j=1}^{r} \frac{f^{(j)}(y)[x]^j}{j!} + \frac{1}{(r-1)!}$$
$$\cdot \int_0^1 (1-t)^{r-1} \cdot \{f^{(r)}(y+tx)[x]^r - f^{(r)}(y)[x]^r\} \cdot dt, \quad (10)$$

where $x, y \in B$ and $[x]^j := (x, ..., x) \in B^j$. Furthermore, by multilinearity and continuity of $f^{(j)}$ and (9), one has for a j-times continuously differentiable function f that

$$f^{(j)}(y)(x, ..., x) = \sum_{v_1 = 1, ..., v_j = 1}^{\infty} e^*_{v_1}(x) \cdots e^*_{v_j}(x) \cdot f^{(j)}(y)(e_{v_1}, ..., e_{v_j}), \quad (11)$$

where $v_k \in \mathbb{N}$, $1 \leqslant k \leqslant j$, $y \in B$.

To shorten (11), we introduce the following notations for $v = (v_1, ..., v_j) \in \mathbb{N}^j$:

$$|v| := j, \qquad x^v := \prod_{k=1}^{j} e^*_{v_k}(x), \quad f^{[v]}(\cdot) := f^{(j)}(\cdot)(e_{v_1}, ..., e_{v_j})\colon B \to \mathbb{R}. \quad (12)$$

Thus (11) is rewritten as

$$f^{(j)}(y)([x]^j) = \sum_{|v|=j} x^v \cdot f^{[v]}(y), \qquad (x, y \in B). \quad (13)$$

We need the following families of functions ($r \in \mathbb{N}$): $C^0_B = C_B := \{f\colon B \to \mathbb{R};\ f$ uniformly continuous and bounded on $\mathbb{R}\}$, $C_B(\mathscr{L}_r) := \{g\colon B \to \mathscr{L}_r;\ g$ uniformly continuous and bounded on $\mathscr{L}_r\}$,

$$C^r_B := \{f \in C_B;\ f^{(j)} \in C_B(\mathscr{L}_j),\ 1 \leqslant j \leqslant r\}.$$

Smoothness of $f\colon B \to \mathbb{R}$ is estimated through ($r \in \mathbb{N}$)

$$\omega_1(f^{(r-1)}, h) := \sup_{\substack{x, y \in B \\ \|x-y\|_B \leqslant h}} \|f^{(r-1)}(x) - f^{(r-1)}(y)\|_{\mathscr{L}_{r-1}}, \quad h > 0. \quad (14)$$

We assume that

$$\omega_1(f^{(r-1)}, h) \leqslant w, \qquad w > 0. \quad (15)$$

Let $x_0 \in B$ be such that $f^{(r-1)}(x_0) = 0$. Then from Lemma 1 we have

$$\|f^{(r-1)}(x)\|_{\mathscr{L}_{r-1}} \leqslant \omega_1(f^{(r-1)}, h) \cdot \left\lceil \frac{\|x - x_0\|_B}{h} \right\rceil, \qquad \forall x \in B. \tag{16}$$

For an arbitrary probability space $(\underline{\Omega}, \mathscr{A}, P)$, let us consider a B-valued r.v. $Z: \underline{\Omega} \to B$; B is endowed with the Borel σ-algebra $\mathscr{B}_B$. Z has distribution P_Z on $\mathscr{B}_B$ defined by $P_Z(B) := P(\{\omega \in \underline{\Omega} : Z(\omega) \in B\})$ for any $B \in \mathscr{B}_B$, and expectation $E(Z) := \int_{\underline{\Omega}} Z(\omega)\, P(d\omega)$, which is a Bochner integral.

For our main result in Part A we need the following lemma due to Butzer *et al.* (1983).

LEMMA 3. *Let X, Y be two B-valued r.v.s with $E(\|X\|_B^j) < \infty$ for some $j \in \mathbb{N}$, and $X^v = \prod_{k=1}^{j} e_{v_k}^*(X)$. Then $f \in C_B^j$ implies that*

$$E(f^{(j)}(Y)[X]^j) = \sum_{|v| = j} E(X^v \cdot f^{[v]}(Y)). \tag{17}$$

DEFINITION 2. Let $\varphi: \mathbb{N} \to \mathbb{R}^+$ such that $\varphi(n) \to 0$ as $n \to \infty$. The B-valued r.v. Z is said to be φ-decomposable, if for each $n \in \mathbb{N}$ there exist n independent r.v. Z_i, $1 \leqslant i \leqslant n$, not depending on n, such that

$$P_Z = P_{\varphi(n) \cdot \sum_{i=1}^{n} Z_i}. \tag{18}$$

For a comparison of the concepts of φ-decomposability and infinite divisibility, see the related discussion on p. 292 of Butzer *et al.* (1983). One can easily see that

$$E(X_i \mid \mathscr{F}_{i-1}) = \sum_{k=1}^{\infty} E(e_k^*(X_i) \mid \mathscr{F}_{i-1}) \cdot e_k \qquad \text{a.s.}$$

Also $(X_i, \mathscr{F}_i)_{i \in \mathbb{Z}_+}$ is an MDS, iff

$$E(e_k^*(X_i) \mid \mathscr{F}_{i-1}) = 0 \qquad (k, i \in \mathbb{N}) \qquad \text{a.s.} \tag{19}$$

iff (we write $(X_i)^v \equiv X_i^v$)

$$E(X_i^v \mid \mathscr{F}_{i-1}) = 0 \qquad (|v| = 1, i \in \mathbb{N}) \qquad \text{a.s.} \tag{20}$$

Main Results

Here we present the main theorem of Part A.

THEOREM 1. *Let $(X_i, \mathscr{F}_i)_{i \in \mathbb{Z}_+}$ be an MDS, Z be a φ-decomposable r.v. with $E(Z) = 0$, and $E(\|Z\|_B^r) < \infty$, $r \in \mathbb{N} - \{1\}$. Assume that $E(\|X_i\|_B^r) < \infty$ $\forall i \in \mathbb{N}$; also assume that*

$$E(X_i^v \mid \mathscr{F}_{i-1}) = E(Z_i^v) \qquad a.s. \quad (1 \leqslant |v| \leqslant r-1, i \in \mathbb{N}). \tag{21}$$

Let $f \in C_B^{r-1}$. *Then*

$$|E(f(\varphi(n)\cdot S_n)) - E(f(Z))| \leqslant \omega_1(f^{(r-1)}, h)\cdot h^{r-1}\cdot\left\{\frac{\varphi^{r-1}(n)}{r!} + \frac{(\varphi(n))^{(r-1)^2/r}}{2\cdot(r-1)!} + \frac{(\varphi(n))^{((r^2-3r+2)/r)}}{8\cdot(r-2)!}\right\}, \tag{22}$$

where

$$h := \left(\varphi(n)\cdot E\left(\left(\sum_{i=1}^{n} (\|X_i\|_B + \|Z_i\|_B)\right)^r\right)\right)^{1/r}. \tag{23}$$

We proceed as in the proof of Theorem 1 in Butzer *et al.* (1983).

Proof. Note that $f(\varphi(n)\cdot S_n)$ and $f(Z)$ are real integrable r.v.s for any $f \in C_B$. Let Z_i be independent r.v.s chosen independently of $\mathscr{F}_i$ such that (18) is fulfilled. Putting

$$R_{n,i} := \sum_{k=1}^{i-1} X_k + \sum_{k=i+1}^{n} Z_k, \qquad 1 \leqslant i \leqslant n, \quad n \in \mathbb{N},$$

for $f \in C_B^{r-1}$ a double application of Taylor's formula (10) produces

$$f(\varphi(n)\cdot S_n) - f\left(\varphi(n)\cdot\sum_{i=1}^{n} Z_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{r-1} [f^{(j)}(\varphi(n)\cdot R_{n,i})[\varphi(n)\cdot X_i]^j - f^{(j)}(\varphi(n)\cdot R_{n,i})[\varphi(n)\cdot Z_i]^j] + \mathbb{R}, \tag{24}$$

where

$$\mathbb{R} := I_1 - I_2, \tag{25}$$

with

$$I_1 := \sum_{i=1}^{n} I_{1i} \qquad I_2 := \sum_{i=1}^{n} I_{2i}. \tag{26}$$

Here

$$I_{1i} := \frac{1}{(r-2)!}\cdot\int_0^1 (1-t)^{r-2}\cdot[f^{(r-1)}(\varphi(n)\cdot R_{n,i} + t\cdot\varphi(n)\cdot X_i)[\varphi(n)\cdot X_i]^{r-1} - f^{(r-1)}(\varphi(n)\cdot R_{n,i})[\varphi(n)\cdot X_i]^{r-1}]\cdot dt, \tag{27}$$

and

$$I_{2i} := \frac{1}{(r-2)!}\cdot\int_0^1 (1-t)^{r-2}\cdot[f^{(r-1)}(\varphi(n)\cdot R_{n,i} + t\cdot\varphi(n)\cdot Z_i)[\varphi(n)\cdot Z_i]^{r-1} - f^{(r-1)}(\varphi(n)\cdot R_{n,i})[\varphi(n)\cdot Z_i]^{r-1}]\cdot dt.$$

We observe the following

$$\begin{aligned}
&|[f^{(r-1)}(\varphi(n)\cdot R_{n,i}+t\cdot\varphi(n)\cdot X_i)[\varphi(n)\cdot X_i]^{r-1}\\
&\qquad -f^{(r-1)}(\varphi(n)\cdot R_{n,i})][\varphi(n)\cdot X_i]^{r-1}|\\
&\quad\leqslant \|f^{(r-1)}(\varphi(n)\cdot R_{n,i}+t\cdot\varphi(n)\cdot X_i)\\
&\qquad -f^{(r-1)}(\varphi(n)\cdot R_{n,i})\|_{\mathscr{L}_{r-1}}\cdot\|\varphi(n)\cdot X_i\|_B^{r-1}\\
&\quad\leqslant \omega_1(f^{(r-1)},h)\cdot\left\lceil\frac{t\cdot\varphi(n)\cdot\|X_i\|_B}{h}\right\rceil\cdot(\varphi(n))^{r-1}\cdot\|X_i\|_B^{r-1}
\end{aligned}$$

by Lemma 1 and (16); i.e.,

$$\begin{aligned}
&|[f^{(r-1)}(\varphi(n)\cdot R_{n,i}+t\cdot\varphi(n)\cdot X_i)[\varphi(n)\cdot X_i]^{r-1}\\
&\qquad -f^{(r-1)}(\varphi(n)\cdot R_{n,i})[\varphi(n)\cdot X_i]^{r-1}|\\
&\quad\leqslant(\varphi(n))^{r-1}\cdot\|X_i\|_B^{r-1}\cdot\omega_1(f^{(r-1)},h)\cdot\left\lceil\frac{t\cdot\varphi(n)\cdot\|X_i\|_B}{h}\right\rceil,\quad h>0. \qquad (28)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&|[f^{(r-1)}(\varphi(n)\cdot R_{n,i}+t\cdot\varphi(n)\cdot Z_i)[\varphi(n)\cdot Z_i]^{r-1}\\
&\qquad -f^{(r-1)}(\varphi(n)\cdot R_{n,i})[\varphi(n)\cdot Z_i]^{r-1}|\\
&\quad\leqslant(\varphi(n))^{r-1}\cdot\|Z_i\|_B^{r-1}\cdot\omega_1(f^{(r-1)},h)\cdot\left\lceil\frac{t\cdot\varphi(n)\cdot\|Z_i\|_B}{h}\right\rceil. \qquad (29)
\end{aligned}$$

Therefore from (28) we get

$$\begin{aligned}
|I_{1i}|&\leqslant\frac{1}{(r-2)!}\cdot\int_0^1(1-t)^{r-2}\cdot|[f^{(r-1)}(\varphi(n)\\
&\qquad\cdot R_{n,i}+t\cdot\varphi(n)\cdot X_i)[\varphi(n)\cdot X_i]^{r-1}\\
&\qquad -f^{(r-1)}(\varphi(n)\cdot R_{n,i})[\varphi(n)\cdot X_i]^{r-1}]|\cdot dt\\
&\leqslant\omega_1(f^{(r-1)},h)\cdot\Bigg\{(\varphi(n))^{r-1}\cdot\|X_i\|_B^{r-1}\\
&\qquad\cdot\int_0^1\frac{(1-t)^{r-2}}{(r-2)!}\cdot\left\lceil\frac{t\cdot\varphi(n)\cdot\|X_i\|_B}{h}\right\rceil\cdot dt\\
&=\omega_1(f^{(r-1)},h)\cdot\phi_{r-1}(\varphi(n)\cdot\|X_i\|_B)\Bigg\},
\end{aligned}$$

the last by change of variable on ϕ_{r-1}, see (4).

We have proved that

$$|I_{1i}| \leqslant \omega_1(f^{(r-1)}, h) \cdot \phi_{r-1}(\varphi(n) \cdot \|X_i\|_B), \qquad i = 1, ..., n. \tag{30}$$

Similarly it is established that

$$|I_{2i}| \leqslant \omega_1(f^{(r-1)}, h) \cdot \phi_{r-1}(\varphi(n) \cdot \|Z_i\|_B), \qquad i = 1, ..., n. \tag{31}$$

Thus

$$\begin{aligned} |\mathbb{R}| &\leqslant |I_1| + |I_2| \leqslant \sum_{i=1}^{n} |I_{1i}| + \sum_{i=1}^{n} |I_{2i}| \leqslant \sum_{i=1}^{n} \omega_1(f^{(r-1)}, h) \\ &\quad \cdot \phi_{r-1}(\varphi(n) \cdot \|X_i\|_B) + \sum_{i=1}^{n} \omega_1(f^{(r-1)}, h) \cdot \phi_{r-1}(\varphi(n) \cdot \|Z_i\|_B) \\ &= \omega_1(f^{(r-1)}, h) \cdot \left[\sum_{i=1}^{n} (\phi_{r-1}(\varphi(n) \cdot \|X_i\|_B) + \phi_{r-1}(\varphi(n) \cdot \|Z_i\|_B)) \right] \\ &\leqslant \omega_1(f^{(r-1)}, h) \cdot \phi_{r-1}\left(\varphi(n) \cdot \sum_{i=1}^{n} (\|X_i\|_B + \|Z_i\|_B) \right), \end{aligned}$$

the last being true by superadditivity of ϕ_{r-1}, see Lemma 2, (4).

We have established $(r \geqslant 2)$

$$|\mathbb{R}| \leqslant \omega_1(f^{(r-1)}, h) \cdot \phi_{r-1}\left(\varphi(n) \cdot \sum_{i=1}^{n} (\|X_i\|_B + \|Z_i\|_B) \right). \tag{32}$$

Note that by φ-decomposability of Z we have that $E(\|Z_i\|_B^r) < \infty$. Also by φ-decomposability of Z, see (18), we have that

$$E(f(Z)) = E\left(f\left(\varphi(n) \cdot \sum_{i=1}^{n} Z_i \right) \right). \tag{33}$$

From p. 294 of Butzer *et al.* (1983), and assumptions of this theorem, we have

$$E[X_i^v \cdot f^{[v]}(\varphi(n) \cdot R_{n,i})] = E[Z_i^v \cdot f^{[v]}(\varphi(n) \cdot R_{n,i})]$$

for $1 \leqslant i \leqslant n$, $n \in \mathbb{N}$, $1 \leqslant j \leqslant r-1$, and $|v| = j$, and from Lemma 3 we obtain

$$E(f^{(j)}(\varphi(n) \cdot R_{n,i})[\varphi(n) \cdot X_i]^j) = E(f^{(j)}(\varphi(n) \cdot R_{n,i})[\varphi(n) \cdot Z_i]^j). \tag{34}$$

Integrating (24) against the probability measure P and taking into account (33) and (34) we get that

$$|E(f(\varphi(n) \cdot S_n)) - E(f(Z))| = |E(\mathbb{R})| \leqslant E(|\mathbb{R}|). \tag{35}$$

From (32) we find ($r \geqslant 2$)

$$|E(f(\varphi(n)\cdot S_n)) - E(f(Z))|$$

$$\leqslant \omega_1(f^{(r-1)}, h)\cdot E\left(\phi_{r-1}\left(\varphi(n)\cdot \sum_{i=1}^{n} (\|X_i\|_B + \|Z_i\|_B)\right)\right), \quad h > 0. \quad (36)$$

From inequality (5) we obtain for the right-hand side of (36), ($r \geqslant 2$) that

$$\text{R.H.S.}(36) \leqslant \omega_1(f^{(r-1)}, h)\cdot E\left\{\frac{\varphi^r(n)\cdot(\sum_{i=1}^{n}(\|X_i\|_B + \|Z_i\|_B))^r}{r!\,h}\right.$$
$$+\frac{\varphi^{r-1}(n)\cdot(\sum_{i=1}^{n}(\|X_i\|_B + \|Z_i\|_B))^{r-1}}{2\cdot(r-1)!}$$
$$\left.+\frac{h\cdot\varphi^{r-2}(n)\cdot(\sum_{i=1}^{n}(\|X_i\|_B + \|Z_i\|_B))^{r-2}}{8\cdot(r-2)!}\right\};$$

i.e., from (36), Hölder's inequality, and linearity of E we have

$$|E(f(\varphi(n)\cdot S_n)) - E(f(Z))|$$
$$\leqslant \omega_1(f^{(r-1)}, h)\cdot\left\{\frac{\varphi^r(n)}{r!\,h}\cdot E\left(\left(\sum_{i=1}^{n}(\|X_i\|_B + \|Z_i\|_B)\right)^r\right)\right.$$
$$+\frac{\varphi^{r-1}(n)}{2\cdot(r-1)!}\cdot\left(E\left(\left(\sum_{i=1}^{n}(\|X_i\|_B + \|Z_i\|_B)\right)^r\right)\right)^{((r-1)/r)}$$
$$\left.+\frac{h\cdot\varphi^{r-2}(n)}{8\cdot(r-2)!}\cdot\left(E\left(\left(\sum_{i=1}^{n}(\|X_i\|_B + \|Z_i\|_B)\right)^r\right)\right)^{((r-2)/r)}\right\}. \quad (37)$$

Choosing h as in (23) and noting that $h < \infty$, from (37) we are able to conclude (22). Q.E.D.

Theorem 1 includes the independent case, because a sequence of independent r.v.s X_i such that $E(X_i) = 0$ forms an MDS.

A special important case follows:

COROLLARY 1. *In the assumptions of Theorem* 1 *when* $r = 2$ *we have*

$$|E(f(\varphi(n)\cdot S_n)) - E(f(Z))| \leqslant \omega_1(f', h)\cdot\frac{h}{2}\cdot\left\{\varphi(n) + \sqrt{\varphi(n)} + \frac{1}{4}\right\}, \quad (38)$$

where

$$h = \left(\varphi(n)\cdot E\left(\left(\sum_{i=1}^{n}(\|X_i\|_B + \|Z_i\|_B)\right)^2\right)\right)^{1/2}. \quad (39)$$

Next we present the weak law of large numbers for MDS on B-spaces with rates when $\|(\|x\|_B)'\|_{\mathscr{L}_1} \leqslant A$, $x \in B - \{0\}$.

THEOREM 2. *Let* $(X_i, \mathscr{F}_i)_{i \in \mathbb{Z}_+}$ *be an MDS, and* $E(\|X_i\|_B^2) < \infty$, $i \in \mathbb{N}$. *Let* $f \in C_B^1$. *Then*

$$|E(f(\varphi(n) \cdot S_n)) - f(0)| \leqslant \omega_1(f', h) \cdot \frac{h}{2} \cdot \left\{\varphi(n) + \sqrt{\varphi(n)} + \frac{1}{4}\right\}, \tag{40}$$

where

$$h = \left(\varphi(n) \cdot E\left(\left(\sum_{i=1}^{n} \|X_i\|_B\right)^2\right)\right)^{1/2}. \tag{41}$$

Proof. We apply Corollary 1. Again we proceed as in the proof of Theorem 5 of Butzer *et al.* (1983). Here Z is the degenerate Gaussian limiting r.v. and Z_i, $1 \leqslant i \leqslant n$ are independent r.v.s distributed as Z. Note $EZ = 0$. When $r = 2$, equality (21) is always true. Also here $E(\|Z\|_B^s) = 0$ for all $s > 0$. Quantity h comes from (39) and the Cauchy–Schwarz's inequality. Q.E.D.

If X is a B-valued r.v. with $E(\|X\|_B^2) < \infty$ and $E(X) = 0$, the covariance functional of X is given by $R_X(f^*, g^*) := E(f^*(X) \cdot g^*(X))$, $f^*, g^* \in B^*$. Let X_R be the uniquely determined Gaussian r.v. with mean zero and covariance functional $R = R_{X_R}$. Since we are in a separable B-space, it is known that for $s \geqslant 0$ $E(\|X_R\|_B^s) < \infty$.

Next we present the central limit theorem for martingales in Banach spaces with a countable basis.

THEOREM 3. *Let* $(X_i, \mathscr{F}_i)_{i \in \mathbb{Z}_+}$ *be an MDS, R be the covariance functional of any mean zero Gaussian r.v., and* $r \in \mathbb{N} - \{1\}$. *Assume* $E(\|X_i\|_B^r) < \infty$, $i \in \mathbb{N}$, *and assume that there exists a sequence of* $a_i > 0$ *such that*

$$E(X_i^v \mid \mathscr{F}_{i-1}) = a_i^{|v|} \cdot E(X_R^v) \tag{42}$$

a.s. ($1 \leqslant |v| \leqslant r-1$, $i \in \mathbb{N}$). *Let* $f \in C_B^{r-1}$. *Set*

$$A_n := \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}. \tag{43}$$

Then

$$|E(f(A_n^{-1} \cdot S_n)) - E(f(X_R))| \leqslant \omega_1(f^{(r-1)}, h) \cdot h^{r-1} \cdot \left\{\frac{A_n^{1-r}}{r!} + \frac{A_n^{-(r-1)^2/r}}{2 \cdot (r-1)!} + \frac{A_n^{-((r^2-3r+2)/r)}}{8 \cdot (r-2)!}\right\}, \tag{44}$$

where

$$h=\left(A_n^{-1}\cdot E\left(\left(\sum_{i=1}^{n}(\|X_i\|_B+a_i\cdot\|X_R\|_B)\right)^r\right)\right)^{1/r}. \tag{45}$$

Proof. From Theorem 1 and as on p. 296 of Butzer *et al.* (1983), proof of their Theorem 3. Here we have $\varphi(n)=A_n^{-1}$, $Z=X_R$, $Z_i=a_iX_R$. Q.E.D.

Part B

Preliminaries

In Part B, again for the convenience of the reader to make comparisons, we follow exactly the same notations and terminology as in Butzer and Kirchfink (1986). Here we are concerned with the *weak invariance principle*, that is, with the convergence of the distribution of the random polygonal lines

$$S_n(\omega, t):=n^{-1/2}\cdot\left(\sum_{i=0}^{[nt]}X_i(\omega)+(nt-[nt])\cdot X_{[nt]+1}(\omega)\right) \tag{46}$$

to the Wiener-measure W_R on $C=C([0,1])$, the space of all real-valued, continuous functions on $[0,1]$ with the supremum-norm ($[a]$ denotes the largest integer $\leqslant a$). Here the random variables (r.v.s) X_i need not be independent.

The expectation of a $C([0,1])$-valued random function (r.f.) $X\colon \underline{\Omega}\to C$ on an arbitrary probability space $(\underline{\Omega}, \mathscr{A}, P)$ is given by the Bochner integral

$$E(X)=\int_{\underline{\Omega}}X(\omega)\,P(d\omega)$$

and, if $E(\|X\|_C^2)<\infty$, the covariance functional $R_X\colon C^*\times C^*\to\mathbb{R}$ by

$$R_X(f^*, g^*):=E(f^*(X)\cdot g^*(X)),\qquad \forall f^*, g^*\in C^*.$$

Here C^* denotes the topological dual of C. A stochastic process

$$\{W(t)\}_{0\leqslant t\leqslant 1}=\{W(\omega, t)\}_{0\leqslant t\leqslant 1}$$

is called a Wiener process or Brownian motion process (on $[0,1]$) given that (i) $W(\omega,0)=0$, $\forall\omega\in\underline{\Omega}$, (ii) $W(\omega,\cdot)$ is a continuous function on $[0,1]\ \forall\omega\in\underline{\Omega}$, (iii) for $0\leqslant t_1<t_2<\cdots<t_n=1$ the differences $W(t_1)$, $W(t_2)-W(t_1)$, ..., $W(t_n)-W(t_{n-1})$ are independent, Gaussian distributed r.v.s with expectation 0 and variance 1. $W(\omega,t)$ is a function of $\omega\in\underline{\Omega}$ and $t\in[0,1]$, such that $W(\cdot,t)$ is $\mathscr{A}$-measurable function $\forall t\in[0,1]$ and

$W(\omega, \cdot)$ belongs to C $\forall \omega \in \underline{\Omega}$. Hence the Wiener process is a r.f. taking values in C. Its distribution is called the Wiener-measure $W_R(t)$.

We need the following three lemmas. These are mentioned also in Butzer and Kirschfink (1986).

LEMMA 4. *Let $X_0 = 0, X_1, X_2, \ldots$ be a sequence of identically distributed r.v.s such that $E(X_1) = 0$, $\mathrm{Var}(X_1) = 1$. $S_n(\omega, t)$ is as in* (46) $\forall n \in \mathbb{N}$, $\omega \in \underline{\Omega}$, $t \in [0, 1]$. *Consider the trajectories*

$$X_{ni}(\omega, t) := \begin{cases} 0, & 0 < nt \leqslant i-1 \\ n^{1/2} \cdot \left(t - \dfrac{i-1}{n}\right) \cdot X_i(\omega), & i-1 < nt \leqslant i \\ n^{-1/2} \cdot X_i(\omega), & i < nt \leqslant n. \end{cases} \tag{47}$$

Then $(X_{ni})_{1 \leqslant i \leqslant n}$, $n \in \mathbb{N}$ is an array of r.f.s with values in C such that

$$\text{(i)} \qquad S_n(\omega, t) = \sum_{i=1}^{n} X_{ni}(\omega, t), \tag{48}$$

$$\text{(ii)} \qquad \|X_{ni}(\omega)\|_C = n^{-1/2} \cdot |X_i(\omega)|. \tag{49}$$

Here $\|\cdot\|_C$ is the supremum norm with respect to $t \in [0, 1]$.

The proof of (48) follows from Giné (1976).

LEMMA 5. *Let $W(t)$ be a r.f. with Wiener-measure $W_R(t)$ as its distribution on $[0, 1]$, which has covariance functional R. If $(a_{ni})_{1 \leqslant i \leqslant n}$, $n \in \mathbb{N}$ is a triangular array of positive real numbers, and*

$$A_n := \left(\sum_{i=1}^{n} a_{ni}^2\right)^{1/2}, \tag{50}$$

then there exist independent, Gaussian distributed r.f.s W_{ni} given by $P_{W_{ni}} := P_{a_{ni} \cdot W}$ such that

$$P_{A_n^{-1} \cdot \sum_{i=1}^{n} W_{ni}(t)} = P_{W(t)} = W_R(t). \tag{51}$$

The proof is given in Butzer and Schulz (1984).

A Feller-type condition on a triangular array of positive real numbers $(a_{ni})_{1 \leqslant i \leqslant n}$, $n \in \mathbb{N}$ says that

$$\lim_{n \to \infty} \max_{1 \leqslant i \leqslant n} \frac{a_{ni}}{A_n} = 0, \tag{52}$$

where A_n is given by (50).

Let C^r, $r \in \mathbb{N}$ the r-fold product space $C \times C \times \cdots \times C$ with norm

$$\|X\|_{C^r} := \max_{1 \leqslant i \leqslant r} \|X_i\|_C, \qquad \text{where} \quad X := (X_1, ..., X_r) \in C^r.$$

The space $L_r := L_r(C^r, \mathbb{R})$ of all real-valued, multilinear, continuous functions $g: C^r \to \mathbb{R}$ is a Banach space with norm

$$\|g\|_{L_r} := \sup_{\|X\|_{C^r} = 1} |g(X)|.$$

We consider the following classes of functions:

$$C_C := C_{C([0,1])} := \{g: C \to \mathbb{R};\ g \text{ continuous and bounded on } \mathbb{R}\},$$

$$C_C(L_r) := \{f: C \to L_r;\ f \text{ continuous and bounded on } L_r\}$$

$$C_C^r := \{f \in C_C;\ \text{Fréchet derivative } f^{(j)} \in C_C(L_j),\ 1 \leqslant j \leqslant r\}, \tag{53}$$

$$\bar{C}_C^r := \{f \in C_C^r;\ f^{(r)} \text{ is uniformly continuous and bounded on } L_r\}. \tag{54}$$

For $f \in C_C^r$, $\|f\|_\infty := \sup_{X \in C} |f(X)|$ may be infinite.

Let $(e_k)_{k \in \mathbb{N}}$ be a normalized countable basis for C. Then each $X \in C$ has a unique representation $X = \sum_{k=1}^{\infty} X^{(v)} \cdot e_k$, where $X^{(v)}$ are real components with respect to the given basis. Since $f^{(j)}(Y)$ are multilinear and continuous, we get

$$f^{(j)}(Y)[X]^j = \sum_{v_1 = 1, ..., v_j = 1}^{\infty} X^{(v_1)} \cdots X^{(v_j)} \cdot f^{(j)}(Y)(e_{v_1}, ..., e_{v_j}), \tag{55}$$

where $v_k \in \mathbb{N}$, $1 \leqslant k \leqslant j$, $[X]^j = (X, ..., X) \in C^j$. To shorten (55) further, for every j-tuple $v = (v_1, ..., v_j) \in \mathbb{N}^j$ we introduce

$$|v| := j,\ X^v := \prod_{k=1}^{j} X^{(v_k)},\ f^{[v]}(\cdot) := f^{[j]}(\cdot)(e_{v_1}, ..., e_{v_j}): C \to \mathbb{R}. \tag{56}$$

Therefore

$$f^{(j)}(Y)[X]^j = \sum_{|v| = j} X^v \cdot f^{[v]}(Y), \qquad (X, Y \in C). \tag{57}$$

We need also Taylor's formula for $X, Y \in C([0, 1])$, $f \in C_C^r$, $r \in \mathbb{N}$

$$f(X+Y) = f(Y) + \sum_{j=1}^{r} \frac{f^{(j)}(Y)}{j!}[X]^j + \frac{1}{(r-1)!} \cdot \int_0^1 (1-t)^{r-1} \cdot \{f^{(r)}(Y + t \cdot X)[X]^r - f^{(r)}(Y)[X]^r\}\, dt. \tag{58}$$

Smoothness of $: C \to \mathbb{R}$ is estimated through $(r \in \mathbb{N})$

$$\omega_1(f^{(r-1)}, h) := \sup_{\substack{X, Y \in C \\ \|X-Y\|_C \leq h}} \|f^{(r-1)}(X) - f^{(r-1)}(Y)\|_{L_{r-1}}, \tag{59}$$

where $h > 0$.

We assume that

$$\omega_1(f^{(r-1)}, h) \leq w, \qquad w > 0. \tag{60}$$

Let $X_0 \in C$ be such that $f^{(r-1)}(X_0) = 0$. From Lemma 1, Part A, we find

$$\|f^{(r-1)}(X)\|_{L_{r-1}} \leq \omega_1(f^{(r-1)}, h) \cdot \left\lceil \frac{\|X - X_0\|_C}{h} \right\rceil, \qquad \forall X \in C. \tag{61}$$

LEMMA 6. *Let X, Y be r.f.s with values in C such that $E(\|X\|_C^j) < \infty$ for $j \in \mathbb{N}$. Let $f \in C_C^j$. Then*

$$E(f^{(j)}(Y)[X]^j) = \sum_{|v| = j} E(X^v \cdot f^{[v]}(Y)). \tag{62}$$

We would like to mention that if X is a normally distributed r.v. and $r > 0$, then $E(|X|^r) < \infty$. Also, if W is a Brownian motion and

$$\|W\|_C := \|W(\omega)\|_C := \sup_{t \in [0,1]} |W(t, \omega)|, \qquad r > 0,$$

then it is well known that

$$l_r := E(\|W\|_C^r) < \infty. \tag{63}$$

Main Results

The next result deals with the rate of approximation of the random polygonal lines S_n, see (46) to the Wiener-measure W_R. The following invariance principle is considered to be a central limit theorem (CLT) for C-valued r.f.s $X_{n,i}(\omega, t)$, $1 \leq i \leq n$, $n \in \mathbb{N}$ defined in Lemma 4, (47), so many people mention the invariance principle as a functional CLT.

THEOREM 4. *Let $(X_i)_{i \in \mathbb{Z}_+}$ be a sequence of identically distributed real r.v.s with mean 0 and variance 1, and let $(a_{ni})_{1 \leq i \leq n} > 0$, $n \in \mathbb{N}$ and $A_n := (\sum_{i=1}^n a_{ni}^2)^{1/2}$. Assume that*

$$\zeta_r := E(|X_1|^r) < \infty \tag{64}$$

for $r \in \mathbb{N} - \{1\}$, and for the r.f.s $(X_{ni})_{1 \leq i \leq n}$, $n \in \mathbb{N}$ defined in (47) *we assume*

$$E(X_{ni}^v \mid \mathcal{A}_{ni}) = a_{ni}^j \cdot A_n^{-j} \cdot E(W^v) \qquad a.s. \tag{65}$$

for all $|\nu| = j$, $1 \leqslant j \leqslant r-1$, $1 \leqslant i \leqslant n$, *and* $n \in \mathbb{N}$. *Here*

$$\mathscr{A}_{ni} := \mathscr{A}(X_1, \ldots, X_{i-1}, W_{n,i+1}, \ldots, W_{nn}) \tag{66}$$

(*the sub-σ-algebra generated by* $\{X_1, \ldots, X_{i-1}, W_{n,i+1}, \ldots, W_{nn}\}$). *Let* $f \in C_C^{r-1}$. *Then* P_{S_n} *converges weakly to the Wiener-measure* W_R, *with rates given by*

$$\begin{aligned} &|E(f(S_n(t))) - E(f(W(t)))| \\ &\quad \leqslant \omega_1(f^{(r-1)}, h) \\ &\quad\quad \cdot \left\{\frac{1}{r!} + \left(\frac{n^{((3-r)/2)} \cdot \zeta_r^{((r-1)/r)} + A_n^{1-r} \cdot (\sum_{i=1}^n a_{ni}^{r-1}) \cdot l_r^{((r-1)/r)}}{2 \cdot (r-1)!}\right)\right. \\ &\quad\quad \left. + \frac{h}{8(r-2)!} \cdot \left\{n^{((4-r)/2)} \cdot \zeta_r^{((r-2)/r)} + A_n^{2-r}\left(\sum_{i=1}^n a_{ni}^{r-2}\right) \cdot l_r^{((r-2)/r)}\right\}\right\}. \end{aligned} \tag{67}$$

Here

$$h := h_n := n^{(2-r)/2} \cdot \zeta_r + A_n^{-r} \cdot \left(\sum_{i=1}^n a_{ni}^r\right) \cdot l_r < \infty. \tag{68}$$

Proof. We proceed as in the proof of Theorem 1 in Butzer and Kirschfink (1986). Note that $f(S_n)$ and $f(W_R)$ are real integrable r.v.s for each $f \in C_C$. Consider the r.f.

$$R_{ni} := \sum_{k=1}^{i-1} X_{nk} + A_n^{-1} \cdot \sum_{k=i+1}^{n} W_{nk}, \qquad (1 \leqslant i \leqslant n; n \in \mathbb{N}). \tag{69}$$

By Lemma 5, we have by summing up and by double application of Taylor's formula (58) the following equality

$$\begin{aligned} f(S_n) - f(W) = \sum_{i=1}^n \sum_{j=1}^{r-1} \frac{1}{j!} \cdot \{&f^{(j)}(R_{ni})[X_{ni}]^j \\ &- f^{(j)}(R_{ni})[A_n^{-1} \cdot W_{ni}]^j\} + \mathbb{R}^*, \end{aligned} \tag{70}$$

where

$$\mathbb{R}^* := \sum_{i=1}^n I_i, \tag{71}$$

with

$$I_i=\frac{1}{(r-2)!}\cdot\int_0^1 (1-t)^{r-2}\cdot\{\{f^{(r-1)}(R_{ni}+t\cdot X_{ni})[X_{ni}]^{r-1}$$
$$-f^{(r-1)}(R_{ni})[X_{ni}]^{r-1}\}$$
$$-\{f^{(r-1)}(R_{ni}+t\cdot A_n^{-1}\cdot W_{ni})[A_n^{-1}\cdot W_{ni}]^{r-1}$$
$$-f^{(r-1)}(R_{ni})[A_n^{-1}\cdot W_{ni}]^{r-1}\}\}\cdot dt. \tag{72}$$

We observe the following

$$|I_i|\leqslant\frac{1}{(r-2)!}\cdot\int_0^1 (1-t)^{r-2}\cdot|\{f^{(r-1)}(R_{ni}+t\cdot X_{ni})$$
$$-f^{(r-1)}(R_{ni})\}[X_{ni}]^{r-1}|\cdot dt$$
$$+\frac{1}{(r-2)!}\cdot\int_0^1 (1-t)^{r-2}\cdot|\{f^{(r-1)}(R_{ni}+t\cdot A_n^{-1}\cdot W_{ni})$$
$$-f^{(r-1)}(R_{ni})\}[A_n^{-1}\cdot W_{ni}]^{r-1}\}|\,dt\}$$
$$\leqslant\left\{\frac{1}{(r-2)!}\cdot\int_0^1 (1-t)^{r-2}\right.$$
$$\cdot\|f^{(r-1)}(R_{ni}+t\cdot X_{ni})-f^{(r-1)}(R_{ni})\|_{L_{r-1}}\cdot\|X_{ni}\|_C^{r-1}\cdot dt$$
$$+\frac{1}{(r-2)!}\cdot\int_0^1 (1-t)^{r-2}\cdot\|f^{(r-1)}(R_{ni}+t\cdot A_n^{-1}\cdot W_{ni})$$
$$\left.-f^{(r-1)}(R_{ni})\|_{L_{r-1}}\cdot\|A_n^{-1}\cdot W_{ni}\|_C^{r-1}\,dt\right\}$$
$$\leqslant\frac{1}{(r-2)!}\cdot\int_0^1 (1-t)^{r-2}\cdot\omega_1(f^{(r-1)},h)\cdot\left\lceil\frac{t\cdot\|X_{ni}\|_C}{h}\right\rceil$$
$$\cdot\|X_{ni}\|_C^{r-1}\cdot dt+\frac{1}{(r-2)!}\cdot\int_0^1$$
$$\cdot(1-t)^{r-2}\cdot\omega_1(f^{(r-1)},h)\cdot\left\lceil\frac{t\cdot A_n^{-1}\cdot\|W_{ni}\|_C}{h}\right\rceil$$
$$\left.\cdot(A_n^{-1})^{r-1}\cdot\|W_{ni}\|_C^{r-1}\cdot dt\right\}$$

the last inequality holds by Lemma 1)

$$=\omega_1(f^{(r-1)},h)\cdot\{\phi_{r-1}(\|X_{ni}\|_C)+\phi_{r-1}(A_n^{-1}\cdot\|W_{ni}\|_C))\},$$

by change of variables and the definition of ϕ_{r-1}, see (4).

We have found that

$$|I_i| \leqslant \omega_1(f^{(r-1)}, h) \cdot \{\phi_{r-1}(\|X_{ni}\|_C) + \phi_{r-1}(A_n^{-1} \cdot \|W_{ni}\|_C))\}, \; i = 1, \ldots, n. \tag{73}$$

Thus

$$|\mathbb{R}^*| \leqslant \omega_1(f^{(r-1)}, h) \cdot \left\{ \sum_{i=1}^{n} (\phi_{r-1}(\|X_{ni}\|_C) + \phi_{r-1}(A_n^{-1} \cdot \|W_{ni}\|_C)) \right\}. \tag{74}$$

From p. 65 of Butzer and Kirschfink (1986), and Lemma 6, we get that

$$E\{f^{(j)}(R_{ni})[X_{ni}]^j - f^{(j)}(R_{ni})[A_n^{-1} \cdot W_{ni}]^j\} = 0, \tag{75}$$

for $1 \leqslant i \leqslant n$, $n \in \mathbb{N}$ and $1 \leqslant j \leqslant r-1$. Integrating (70) against the probability measure P and taking into account (75) we find that

$$|E(f(S_n)) - E(f(W))| = |E(\mathbb{R}^*)| \leqslant E(|\mathbb{R}^*|). \tag{76}$$

From (74) now we obtain ($r \geqslant 2$)

$$\begin{aligned} |E(f(S_n(t))) - E(f(W(t)))| \leqslant \omega_1(f^{(r-1)}, h) \cdot \Bigg\{ \sum_{i=1}^{n} & (E(\phi_{r-1}(\|X_{ni}\|_C)) \\ & + E(\phi_{r-1}(A_n^{-1} \cdot \|W_{ni}\|_C))) \Bigg\}, \quad h > 0. \end{aligned} \tag{77}$$

From inequality (5) we have for the right-hand side of (77) ($r \geqslant 2$) that

$$\begin{aligned} \mathrm{RHS}(77) \leqslant \omega_1(f^{(r-1)}, h) \cdot \Bigg\{ \sum_{i=1}^{n} \Bigg[& E\left(\frac{\|X_{ni}\|_C^r}{r!\,h} + \frac{\|X_{ni}\|_C^{r-1}}{2 \cdot (r-1)!} + \frac{h \cdot \|X_{ni}\|_C^{r-2}}{8 \cdot (r-2)!} \right) \\ & + E\left(\frac{A_n^{-r} \cdot \|W_{ni}\|_C^r}{r!\,h} + \frac{A_n^{1-r} \cdot \|W_{ni}\|_C^{r-1}}{2 \cdot (r-1)!} + \frac{h \cdot A_n^{2-r} \cdot \|W_{ni}\|_C^{r-2}}{8 \cdot (r-2)!} \right) \Bigg] \Bigg\} \end{aligned}$$

(by Hölder's inequality and linearity of E)

$$\begin{aligned} &\leqslant \omega_1(f^{(r-1)}, h) \cdot \Bigg\{ \frac{\sum_{i=1}^{n} (E(\|X_{ni}\|_C^r) + A_n^{-r} \cdot E(\|W_{ni}\|_C^r))}{r!\,h} \\ &\quad + \frac{\sum_{i=1}^{n} ((E(\|X_{ni}\|_C^r))^{((r-1)/r)} + A_n^{1-r} \cdot (E(\|W_{ni}\|_C^r))^{((r-1)/r)})}{2 \cdot (r-1)!} \\ &\quad + \frac{h}{8 \cdot (r-2)!} \cdot \Bigg[\sum_{i=1}^{n} ((E(\|X_{ni}\|_C^r))^{((r-2)/r)} \\ &\quad + A_n^{2-r} \cdot (E(\|W_{ni}\|_C^r))^{((r-2)/r)}) \Bigg] \Bigg\} =: J. \end{aligned}$$

From (49) and (64) we have

$$E(\|X_{ni}\|_C^r) = E(\|X_{ni}(\omega)\|_C^r) = \frac{E(|X_i(\omega)|^r)}{n^{r/2}} = \frac{E(|X_i|^r)}{n^{r/2}} < \infty,$$

thus

$$E(\|X_{ni}\|_C^r) = \frac{E(|X_1|^r)}{n^{r/2}} = n^{-r/2} \cdot \zeta_r;$$

i.e.,

$$E(\|X_{ni}\|_C^r) = n^{-r/2} \cdot \zeta_r. \tag{78}$$

Also from Lemma 5 and (63) we find

$$E(\|W_{ni}\|_C^r) = E(\|a_{ni} \cdot W\|_C^r) = a_{ni}^r \cdot E(\|W\|_C^r) = a_{ni}^r \cdot l_r < \infty;$$

i.e.,

$$E(\|W_{ni}\|_C^r) = a_{ni}^r \cdot l_r. \tag{79}$$

Therefore from (77)–(79) we get that

$$\begin{aligned}
&|E(f(S_n(t))) - E(f(W(t)))| \\
&\quad \leqslant J = \omega_1(f^{(r-1)}, h) \cdot \Bigg\{ \frac{\sum_{i=1}^n (n^{-r/2} \cdot \zeta_r + A_n^{-r} \cdot a_{ni}^r \cdot l_r)}{r!\, h} \\
&\quad + \frac{\sum_{i=1}^n (n^{(1-r)/2} \cdot \zeta_r^{((r-1)/r)} + A_n^{1-r} \cdot a_{ni}^{r-1} \cdot l_r^{((r-1)/r)})}{2 \cdot (r-1)!} \\
&\quad + \frac{h}{8(r-2)!} \cdot \Bigg[\sum_{i=1}^n (n^{((2-r)/2)} \cdot \zeta_r^{((r-2)/r)} \\
&\quad + A_n^{2-r} \cdot a_{ni}^{r-2} \cdot l_r^{((r-2)/r)}) \Bigg] \Bigg\}.
\end{aligned}$$

That is,

$$\begin{aligned}
&|E(f(S_n(t))) - E(f(W(t)))| \\
&\quad \leqslant \omega_1(f^{(r-1)}, h) \cdot \Bigg\{ \frac{(n^{((2-r)/2)} \cdot \zeta_r + A_n^{-r} \cdot (\sum_{i=1}^n a_{ni}^r) \cdot l_r)}{r!\, h} \\
&\quad + \frac{(n^{((3-r)/2)} \cdot \zeta_r^{((r-1)/r)} + A_n^{1-r}(\sum_{i=1}^n a_{ni}^{r-1}) \cdot l_r^{((r-1)/r)})}{2 \cdot (r-1)!} \\
&\quad + \frac{h}{8 \cdot (r-2)!} \cdot \Bigg[n^{((4-r)/2)} \cdot \zeta_r^{((r-2)/r)} + A_n^{2-r} \\
&\quad \cdot \left(\sum_{i=1}^n a_{ni}^{r-2} \right) \cdot l_r^{((r-2)/r)} \Bigg] \Bigg\}.
\end{aligned} \tag{80}$$

Choosing h as in (68), inequality (67) is clearly established by inequality (80). It is also clear that this $h < \infty$. Q.E.D.

Remark 1. The right-hand side of (67) is of practical interest only if it converges to zero as $n \to \infty$. For this we need to assume $r \geqslant 5$,

$$\lim_{n \to \infty} \left(A_n^{-r} \cdot \left(\sum_{i=1}^{n} a_{ni}^r \right) \right) = 0,$$

and

$$A_n^{1-r} \cdot \left(\sum_{i=1}^{n} a_{ni}^{r-1} \right), \qquad A_n^{2-r} \cdot \left(\sum_{i=1}^{n} a_{ni}^{r-2} \right)$$

are at least bounded as $n \to \infty$.

For instance, one could assume that

$$\left(\frac{\max_{1 \leqslant i \leqslant n} a_{ni}}{A_n} \right) \leqslant c \cdot \frac{1}{\sqrt{n}}, \qquad n \in \mathbb{N}, \tag{81}$$

where c is a constant, compare with (52). Then

$$\left(\frac{\max_{1 \leqslant i \leqslant n} a_{ni}}{A_n} \right)^{r-j} \leqslant c^{r-j} \cdot \frac{1}{n^{(r-j)/2}}, \qquad j = 0, 1, 2; \qquad r \geqslant 5,$$

and

$$\frac{\sum_{i=1}^{n} a_{ni}^{r-j}}{A_n^{r-j}} \leqslant \frac{n \cdot (\max_{1 \leqslant i \leqslant n} a_{ni})^{r-j}}{A_n^{r-j}}$$

$$\leqslant n \cdot c^{r-j} \cdot \frac{1}{n^{(r-j)/2}} = c^{r-j} \cdot \frac{1}{n^{((r-j)/2)-1}}.$$

Thus

$$\frac{\sum_{i=1}^{n} a_{ni}^{r-j}}{A_n^{r-j}} \leqslant c^{r-j} \cdot \frac{1}{n^{(r-j-2)/2}} \to 0 \quad j = 0, 1, 2; \quad r \geqslant 5, \qquad \text{as} \quad n \to \infty. \tag{82}$$

In this case as $n \to \infty$, $h = h_n \to 0$ (see (68)) and $\omega_1(f^{(r-1)}, h) \to 0$, and the quantity within the braces on the right side of (67) converges to $1/r!$. Therefore (RHS)(67) converges to zero as $n \to \infty$. Consequently, from (67) we find that $E(f(S_n(t))) \to E(f(W(t)))$, as $n \to \infty$.

In Theorem 4 we made no assumptions for the dependence or independence of the r.v.s X_i. This was made possible by assumption (65), which also makes applicable Theorem 4, in particular to a CLT to be presented next. There (65) becomes more specific. For this we need

DEFINITION 3 (Butzer and Kirschfink (1986)). Let $(X_i)_{i\in\mathbb{R}}$ a sequence of real r.v.s.

(i) This sequence is said to be "dependent from below," if for each $1 \leqslant i \leqslant n$ and $n \in \mathbb{N}$

$$\begin{aligned} &E[X_i \mid \mathscr{A}(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)] \\ &\qquad = E[X_i \mid \mathscr{A}(X_1, ..., X_{i-1})] \qquad \text{a.s.} \end{aligned} \tag{83}$$

(ii) This sequence is "dependent from above," if

$$\begin{aligned} &E[X_i \mid \mathscr{A}(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)] \\ &\qquad = E[X_i \mid \mathscr{A}(X_{i+1}, ..., X_n)] \qquad \text{a.s.} \end{aligned} \tag{84}$$

Let X^* be a standard normally distributed r.v.

THEOREM 5. *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of identically distributed, possibly dependent, real r.v.s for which $E(X_1)=0$, $\operatorname{Var}(X_1)=1$, and $\zeta_r := E(|X_1|^r) < \infty$, for $r \in \mathbb{N} - \{1\}$.*

(i) *Assume that $(X_i)_{i\in\mathbb{N}}$ is dependent from below and that*

$$E(X_i^j \mid \mathscr{E}_{i-1}) = E(X^{*j}) \qquad a.s. \quad (1 \leqslant j \leqslant r-1, i \in \mathbb{N}), \tag{85}$$

where

$$\mathscr{E}_{i-1} := \mathscr{A}(X_1, ..., X_{i-1}).$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ for which $f^{(j)}$ is bounded and continuous on $\mathbb{R}$ for all $0 \leqslant j \leqslant r-1$. Then

$$\begin{aligned} &\left| E\left(f\left(\frac{\sum_{i=1}^n X_i}{\sqrt{n}}\right)\right) - E(f(X^*)) \right| \\ &\quad \leqslant \omega_1(f^{(r-1)}, h) \cdot \left\{ \frac{1}{r!} + \frac{n^{((3-r)/2)} \cdot (\zeta_r^{((r-1)/r)} + l_r^{((r-1)/r)})}{2 \cdot (r-1)!} \right. \\ &\quad \left. + \frac{h}{8 \cdot (r-2)!} \cdot n^{((4-r)/2)} \cdot (\zeta_r^{((r-2)/r)} + l_r^{((r-2)/r)}) \right\}, \end{aligned} \tag{86}$$

where

$$l_r = E(|X^*|^r) < \infty$$

and

$$h := h_n = n^{((2-r)/2)} \cdot (\zeta_r + l_r).$$

(ii) *Assume that the sequence* $(X_i)_{i\in\mathbb{N}}$ *is dependent from above and that*

$$E(X_i^j \mid F_{i+1}^*) = E(X^{*j}) \qquad a.s. \quad (1 \leqslant j \leqslant r-1, i \in \mathbb{N}), \tag{87}$$

where

$$F_{i+1}^* := \mathscr{A}(X_{i+1}, ..., X_n)$$

and f *as in part* (i). *Then* (86) *is true again. The RHS*(86) *converges to zero as* $n \to \infty$, *when* $r \geqslant 5$.

Proof. From our Theorem 4, and Theorem 3 of Butzer and Kirschfink (1986), it is the case that $t=1$. Here the r.f.s X_{ni}, $1 \leqslant i \leqslant n$, $n \in \mathbb{N}$ become real r.v.s $X_{ni}(1, \omega) = n^{-1/2} \cdot X_i(\omega)$. The r.f. has the Wiener-measure as its distribution collapses to the standard normal distribution P_{X^*}. Also, here $a_{ni} := 1$, $1 \leqslant i \leqslant n$; $n \in \mathbb{N}$ and $A_n = n^{1/2}$. Note that

$$A_n^{-r} \cdot \left(\sum_{i=1}^{n} a_{ni}^r \right) = n^{((2-r)/2)}, \qquad A_n^{1-r} \cdot \left(\sum_{i=1}^{n} a_{ni}^{r-1} \right) = n^{((3-r)/2)},$$

and

$$A_n^{2-r} \cdot \left(\sum_{i=1}^{n} a_{ni}^{r-2} \right) = n^{((4-r)/2)}.$$

Take into account also the proof of Theorem 4 on p. 72 of Butzer and Kirschfink (1986). Q.E.D.

If $(X_i)_{i\in\mathbb{N}}$ is a sequence of real r.v.s on $(\underline{\Omega}, \mathscr{A}, P)$, and $(\mathscr{F}_i)_{i\in\mathbb{Z}_+}$ is a monotone increasing sequence of sub-σ-algebras of $\mathscr{A}$ such that $\mathscr{F}_i := \mathscr{A}(X_1, ..., X_i)$, $i \in \mathbb{N}$ and $\mathscr{F}_0 := \{\phi, \underline{\Omega}\}$, $X_0 = 0$, then $(X_i, \mathscr{F}_i)_{i\in\mathbb{Z}_+}$ is called a martingale difference sequence (MDS) iff

$$E(X_i \mid \mathscr{F}_{i-1}) = 0, \qquad \text{a.s.} \quad (i \in \mathbb{N}). \tag{88}$$

Butzer and Kirschfink (1986), Lemma 8, p. 73, prove that for the case of MDS F_i is dependent from below.

Now comes the CLT for MDS with rates

THEOREM 6. *Let* $(X_i \mid \mathscr{F}_i)_{i\in\mathbb{Z}_+}$ *be an MDS such that* X_i *are identically distributed,* $\mathrm{Var}(X_1) = 1$, $\zeta_r := E(|X_1|^r) < \infty$, *for* $r \in \mathbb{N} - \{1\}$. *Assume that*

$$E(X_i^j \mid \mathscr{F}_{i-1}) = E(X^{*j}) \qquad a.s. \quad (1 \leqslant j \leqslant r-1, i \in \mathbb{N}). \tag{89}$$

Let f *be as in Theorem 5. Then inequality* (86) *is true.*

Proof. From Theorem 5(i). Note that $E(X_i) = E(X_i \mid \mathscr{F}_{i-1}) = 0$ a.s. all $i \in \mathbb{N}$, by Lemma 4 of Butzer and Kirschfink (1986) and (1988). Q.E.D.

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REFERENCES

1. ANASTASSIOU, G. A. (1986). Korovkin type inequalities in real normed vector spaces. *Approx. Theory Appl.* **2**, No. 2, 39–53.
2. ANASTASSIOU, G. A. (1986). Multi-dimensional quantitative results for probability measures approximating the unit measure. *Approx. Theory Appl.* **2**, No. 4, 93–103.
3. BUTZER, P. L., HAHN, L., AND ROECKERATH, M. TH. (1983). Central limit theorem and weak law of large numbers with rates for martingales in Banach spaces. *J. Multivariate Anal.* **13** 287–301.
4. BUTZER, P. L., AND KIRSCHFINK, H. J. W. (1986). Donsker's weak invariance principle with rates for $C[0, 1]$-valued, dependent random functions. *Approx. Theory Appl.* **2**, No. 4, 55–77.
5. BUTZER, P. L., AND SCHULZ, D. (1984). The weak invariance principle with rates for $C[0, 1]$-valued random functions. In *Anniversary Volume on Approximation Theory and Functional Analysis* (P. L. Butzer, R. L. Stens, and B. Sz.-Nagy, Eds.). ISNM 65, pp. 567–584. Birkhaüser, Basel/Boston/Stuttgart.
6. GINÉ, E. (1976). Some remarks on the central limit theorem in $C(S)$. In *International Conference on Probability in Banach Spaces, First, Oberwolfach, July 20–26, 1975. Probability in Banach Spaces* (A. Beck, Ed.), pp. 101–106. Lecture Notes in Math., Vol. 526, Springer-Verlag, Berlin.
7. LAHA, R. G., AND ROHATGI, V. K. (1979). *Probability Theory*. Wiley, New York.

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