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Functional Limit Theorems for Row and Column Exchangeable Arrays

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Sufficient conditions are given to illustrate how the various possible separately exchangeable continuous processes on $[0, 1]^2$ may arise as the weak limit of the partial sum processes of a sequence of row and column exchangeable arrays.

0. INTRODUCTION

This work was motivated by an elegant result of Kallenberg [8], which completely characterizes all continuous and separately exchangeable planar processes. Any continuous weak limit of a sequence of partial sum processes based on row and column exchangeable (RCE) arrays will be separately exchangeable and examples of two such limits are given in Ivanoff and Weber [5].

Here we explore how each of the various terms in Kallenberg's representation arise as limits of the partial sum processes of sequences of finite RCE arrays. We will do this by focussing on each of the terms in his representation in turn. We will provide a set of sufficient conditions that ensure

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convergence to the appropriate limit and also indicate how combinations of the terms can arise. Proceeding in this way we provide a set of new limit results for sequences of finite RCE arrays.

An array $\{Y_{ij}: 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r\}$ is a RCE array if

$$\{Y_{ij}\} \stackrel{D}{=} \{Y_{p(i),\, q(j)}\},$$

where p and q are arbitrary permutations of $\{1, ..., m\}$ and $\{1, ..., r\}$, respectively. We consider the process X defined on $[0, 1]^2$ by

$$X(s, t) = \sum_{j=1}^{[rs]} \sum_{i=1}^{[mt]} Y_{ij} \tag{0.1}$$

with $X(s, 0) = X(0, t) = 0$, $\forall (s, t) \in [0, 1]^2$. X is an element of the function space $D([0, 1]^2)$. In what follows, it will always be assumed that $D([0, 1]^2)$ is endowed with the usual Skorokhod topology. (For details, see Straf [11].) In this paper we shall only consider arrays of the form $Y_{ij} = y_{\sigma(i),\, \pi(j)}$, where $\{y_{ij}: 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r\}$ is an array of constants, and σ and π are independent permutations uniformly distributed on $\{1, ..., m\}$ and $\{1, ..., r\}$, respectively. Limit theorems for arrays with random entries follow quite readily from those for arrays with non-random entries using Lemma 1.1 of Kallenberg [7] (see, for example, the proof of Theorem 2.3 in [5]). As all finite subarrays of infinite RCE arrays are themselves RCE, in principle, our results are applicable to the case of infinite arrays. However, not all finite RCE arrays are embeddable into infinite arrays. For an extensive discussion of exchangeability see Aldous [1].

Limit theory for RCE arrays has a direct application in the area of generalized U-statistics based on two independent samples. Specifically, given two independent sets of independent and identically distributed random variables $(\xi_1, ..., \xi_r\}$ and $\{\eta_1, ..., \eta_m\}$ and a real-valued function, h, then if we set $Y_{ij} = h(\xi_i, \eta_j)$, $(rm)^{-1} X(1, 1)$ is the generalized U-statistic of degree $(1, 1)$ with kernel h. The weak convergence of non-degenerate, generalized U-statistics based on independent observations was the subject of Sen [10]. Neuhaus [9] considers the weak convergence of degenerate, generalized U-statistics of order $(2, 2)$. If the ξ_i's and η_j's are not independent observations but are values obtained by sampling without replacement from two finite populations, then the array $\{h(\xi_i, \eta_j)\}$ is still RCE. Thus, by studying processes based on finite RCE arrays rather than restricting attention to the case of infinite arrays, where de Finetti-type representations exist, we are able to obtain limit results for generalized finite population U-statistics as obvious corollaries, as discussed in detail in [5].

1. Preliminaries

Assume that $\{Y_{ij}^{(n)}: 1 \leqslant i \leqslant m_n,\ 1 \leqslant j \leqslant r_n\}$ is a sequence of RCE arrays with $\{Y_{ij}^{(n)}\} \stackrel{D}{=} \{y_{\sigma_n(i),\pi_n(j)}^{(n)}\}$, where $\{y_{ij}^{(n)}: 1 \leqslant i \leqslant m_n,\ 1 \leqslant j \leqslant r_n\}$ is a sequence of arrays of constants. We shall assume $n \leqslant m_n \leqslant r_n$, $\forall n$. Define $X_n(s, t)$ as in (0.1), replacing Y_{ij} with $Y_{ij}^{(n)}$ and (r, m) with (r_n, m_n). We begin by considering the class of possible continuous weak limits of the sequence (X_n). Clearly any weak limit is separately exchangeable (cf. [8]).

We recall that a Brownian sheet on $\mathbf{R}_+^2$ is a continuous, mean zero Gaussian process $W_2(\cdot, \cdot)$ with covariance function $EW_2(s, t)\, W_2(s', t') = (s \wedge s')(t \wedge t')$. Using W_2, the following Gaussian processes may be constructed:

$$B^0(s, t) = W_2(s, t) - stW_2(1, 1), \qquad (s, t) \in [0, 1]^2,$$
$$Z(s, t) = W_2(s, t) - sW_2(1, t), \qquad s \in [0, 1],\ t \in \mathbf{R}_+,$$
$$B_2(s, t) = Z(s, t) - tZ(s, 1), \qquad (s, t) \in [0, 1]^2.$$

B^0 is a Brownian bridge, Z is known as a Kiefer process, and we refer to B_2 as a "tucked-in Brownian sheet." The covariance functions are

$$EB^0(s, t)\, B^0(s', t') = (s \wedge s')(t \wedge t') - ss'tt', \qquad (s, t), (s', t') \in [0, 1]^2,$$
$$EZ(s, t)\, Z(s', t') = (s \wedge s' - ss')(t \wedge t'), \qquad s, s' \in [0, 1],\ t, t' \in \mathbf{R}_+,$$
$$EB_2(s, t)\, B_2(s', t') = (s \wedge s' - ss')(t \wedge t' - tt'), \qquad (s, t), (s', t') \in [0, 1]^2.$$

All of these processes are clearly continuous and separately exchangeable on $[0, 1]^2$. Kallenberg refers to W_2, Z, and B_2 as "Brownian sails."

We denote the usual Brownian motion on $\mathbf{R}_+$ by W_1. The Brownian bridge on $[0, 1]$ may be constructed as

$$B_1(s) = W_1(s) - sW_1(1), \qquad s \in [0, 1].$$

In [8, Theorem 6.1], Kallenberg completely characterizes continuous separately exchangeable processes. Here we are concerned only with processes defined on $[0, 1]^2$, which comprise the possible limits of the sequence (X_n).

Theorem 1.1. [8, Theorem 6.1]. *A process X on $[0, 1]^2$ is continuous and separately exchangeable iff, a.s.*

$$X(s, t) = \rho st + \sigma A(s, t) + \sum_{j=1}^{\infty} (\alpha_j B_j(s)\, C_j(t) + \beta_j B_j(s)\, t + \gamma_j C_j(t) s), \qquad (s, t) \in [0, 1]^2, \tag{1.1}$$

for some random variables ρ, σ, α_j, β_j, γ_j ($j \in \mathbf{N}$) with $\sum_{j=1}^{\infty} (\alpha_j^2 + \beta_j^2 + \gamma_j^2) < \infty$ a.s., some independent Brownian sail A and some independent sequences (B_j) and (C_j) of i.i.d. Brownian motions or bridges on $[0, 1]$.

We can write (1.1) as

$$X(s, t) = \rho st + \sigma A(s, t) + \sum_{j=1}^{\infty} \alpha_j B_j(s)\, C_j(t) + btB(s) + csC(t), \tag{1.2}$$

where A is a "tucked-in" Brownian sheet and all of the one-dimensional processes are Brownian bridges on $[0, 1]$.

In what follows, we shall see how the various one- and two-dimensional processes in (1.1) arise as weak limits of the partial sum processes defined on RCE arrays $\{Y_{ij}^{(n)}\}$. We can write

$$\begin{aligned} X_n(s, t) = \sum_{j=1}^{[r_n s]} \sum_{i=1}^{[m_n t]} [\, & \bar{Y}_{..}^{(n)} + (Y_{ij}^{(n)} - \bar{Y}_{i.}^{(n)} - \bar{Y}_{.j}^{(n)} + \bar{Y}_{..}^{(n)}) \\ & + (\bar{Y}_{.j}^{(n)} - \bar{Y}_{..}^{(n)}) + (\bar{Y}_{i.}^{(n)} - \bar{Y}_{..}^{(n)})] \\ \approx\ & Y_{..}^{(n)} st + W_n(s, t) + tV_n(s) + sU_n(t), \end{aligned} \tag{1.3}$$

where $\bar{Y}_{..}^{(n)} = (m_n r_n)^{-1} \sum_i \sum_j Y_{ij}^{(n)}$, $\bar{Y}_{i.}^{(n)} = r_n^{-1} \sum_j Y_{ij}^{(n)} = Y_{i.}^{(n)}/r_n$, $\bar{Y}_{.j} = m_n^{-1} \sum_i Y_{ij}^{(n)} = Y_{.j}^{(n)}/m_n$, and U_n is the (centred) row sum process, V_n is the column sum process, and W_n is associated with an array with 0 row and column sums. Heuristically, comparing (1.2) and (1.3), we see that ρ arises as a limit in distribution of the sums of a sequence of RCE arrays with random entries. As mentioned previously, we shall not consider this case here. For arrays with non-random entries, the terms bB and cC are the weak limits of (V_n) and (U_n), respectively. The tucked-in Brownian sheet A and product term $\sum \alpha_j B_j C_j$ (with 0 margins) are possible weak limits of the sequence (W_n) and arise if the sequence of sums of squares of array entries has a non-zero limit. In fact, both the sheet and the product terms have the same covariance structure, although clearly the product term is not a sail in Kallenberg's sense. It is the product term which is completely new to our limit theorems here, and in order to understand how it arises, a more detailed analysis of the arrays of constants $\{y_{ij}^{(n)}: 1 \leqslant i \leqslant m_n, 1 \leqslant j \leqslant r_n\}$ and the corresponding RCE arrays $\{Y_{ij}^{(n)}: 1 \leqslant i \leqslant m_n, 1 \leqslant j \leqslant r_n\}$ is required.

Suppressing dependence on n in what follows, suppose that $m \leqslant r$. Consider the array $\{y_{ij}\}$ as an $m \times r$ matrix D. DD' is a non-negative definite $m \times m$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0$ and the corresponding orthogonal, normalized eigenvectors $\mathbf{u}_1, ..., \mathbf{u}_m$. Let $U = (\mathbf{u}_1, ..., \mathbf{u}_m)$. Then

$$U'DD'U = \operatorname{diag}(\lambda_1, ..., \lambda_m),$$

and so the columns of $D'U$ are orthogonal. As in [4, p. 64], there exists an $r \times r$ orthogonal matrix V such that $U'DV = M$, where

$$M = \begin{bmatrix} \sqrt{\lambda_1} & & 0 & \vdots & \\ & \ddots & & \vdots & \\ & & \ddots & \vdots & 0 \\ 0 & & \sqrt{\lambda_m} & \vdots & \end{bmatrix}$$

is an $m \times r$ matrix. (If $\lambda_k \neq 0$, then $\mathbf{v}_k = D'\mathbf{u}_k/\sqrt{\lambda_k}$, where $\mathbf{v}_k$ is the kth column of V.) Finally, we have $D = UMV' = \sum_{k=1}^{m} \sqrt{\lambda_k}\mathbf{u}_k\mathbf{v}'_k$.

Thus, we have the following spectral decompositions:

$$y_{ij} = \sum_{k=1}^{m} \sqrt{\lambda_k} u_{ki} v_{kj}, \tag{1.4}$$

$$Y_{ij} = \sum_{k=1}^{m} \sqrt{\lambda_k} u_{k\sigma(i)} v_{k\pi(j)}. \tag{1.5}$$

Note that:

$$\sum_i \sum_j y_{ij}^2 = \sum_k \lambda_k,$$

$$\sum_i \sum_l \left(\sum_j y_{ij} y_{lj}\right)^2 = \sum_j \sum_k \left(\sum_i y_{ij} y_{ik}\right)^2 = \sum_k \lambda_k^2,$$

$$\sum_i \sum_j y_{ij} = \sum_k \sqrt{\lambda_k} u_{k.} v_{k.}, \left(u_{k.} = \sum_i u_{ki}, v_{k.} = \sum_j v_{kj}\right),$$

$$\sum_i \left(\sum_j y_{ij}\right)^2 = \sum_k \lambda_k v_{k.}^2, \sum_j \left(\sum_i y_{ij}\right)^2 = \sum_k \lambda_k u_{k.}^2.$$

In some instances, we shall be considering the special case of zero row and column sums; i.e.,

$$\sum_{i=1}^{n} y_{ij} = 0, \qquad j = 1, ..., r, \tag{1.6}$$

$$\sum_{j=1}^{r} y_{ij} = 0, \qquad i = 1, ..., m. \tag{1.7}$$

From (1.6), we have that $D'\mathbf{1} = 0$, and so we can set $\lambda_m = 0$, and $\mathbf{u}_m = \mathbf{1}/\sqrt{m}$. By orthogonality, the entries of each of the other eigenvectors sum to 0. Also, in this case, we may set $\mathbf{v}_m = \mathbf{1}/\sqrt{r}$, and $\mathbf{v}'_k\mathbf{1} = 0$, $k = 1, ..., m-1$.

Given a sequence of arrays of constants $\{y_{ij}^{(n)}\}$, the matrices $D^{(n)}$, $U^{(n)}$, $V^{(n)}$, $M^{(n)}$ (with entries $d_{ij}^{(n)}$, etc.) are defined in the obvious way from $\{y_{ij}^{(n)}\}$. Let $\lambda_1^{(n)} \geqslant \cdots \geqslant \lambda_{m_n}^{(n)}$ be the (ordered) eigenvalues of $D^{(n)}D^{(n)'}$. In the theorems which follow, we shall see that the product term in (1.2) arises only if the sequence of eigenvalues in the spectral decomposition (1.4) converges to a non-zero limit.

In Section 2, we state the various limit theorems for sequences of RCE arrays. First we give results for arrays with asymptotically zero row and column sums and mention how these can be extended to arrays where only one margin is asymptotically zero. Finally we give new results for the non-zero marginal sums case. Some examples are given in Section 3 and sketches of proofs are given in Section 4. Details of proofs, additional examples, and comments on the necessity of the various conditions may be found in [6].

2. Summary of Results

Using the notation introduced in Section 1, we begin by assuming that all row and column sums of $D^{(n)}$ are asymptotically equal to 0.

THEOREM 2.1. *Assume that the following conditions are satisfied*:

(i) $\sum_{i=1}^{m_n} \sum_{j=1}^{r_n} y_{ij}^{(n)} \to 0.$

(ii) $\sum_{i=1}^{m_n} \sum_{j=1}^{r_n} y_{ij}^{(n)2} = \sum_{k=1}^{m_n} \lambda_k^{(n)} \to 1$ *as* $n \to \infty$.

(iii) $(\sum_{i=1}^{m_n} (\sum_{j=1}^{r_n} y_{ij}^{(n)})^2) \to 0$, $(\sum_{j=1}^{r_n} (\sum_{i=1}^{m_n} y_{ij}^{(n)})^2) \to 0$, *as* $n \to \infty$.

(iv) $\sum_{i=1}^{m_n} (\sum_{j=1}^{r_n} y_{ij}^{(n)2})^2 = O(m_n^{-1})$; $\sum_{j=1}^{r_n} (\sum_{i=1}^{m_n} y_{ij}^{(n)2})^2 = O(r_n^{-1})$.

(v) $\sum_i \sum_l \sum_j \sum_h y_{ij}^{(n)} y_{ih}^{(n)} y_{lj}^{(n)} y_{lh}^{(n)} = \sum_{k=1}^{m_n} \lambda_k^{(n)2} \to 0$ *as* $n \to \infty$.

Then as $n \to \infty$, X_n *converges in distribution in* $D([0, 1]^2)$ *to a "tucked-in" Brownian sheet* A.

Comment. If one or both limits in (iii) above are allowed to be equal to 1, then the limiting sail is a Kiefer process or the Brownian bridge. (See [5, Theorem 2.1].)

Note that given condition (ii), condition (v) of Theorem 2.1 is equivalent to $\sup_{1 \leqslant k \leqslant m_n} \lambda_k^{(n)} \to 0$. We now consider a case in which (v) is not satisfied.

THEOREM 2.2. *Assume that conditions* (i)–(iv) *of Theorem* 2.1 *are satisfied, as well as*:

(v) $\sup_{i \leqslant m_n} |u_{ki}^{(n)}| \to 0$ *as* $n \to \infty$, *and* $\sup_{j \leqslant r_n} |v_{kj}^{(n)}| \to 0$ *as* $n \to \infty$, *for each* k.

(vi) *There exists a non-negative sequence* (λ_k) *such that* $\sum_{k=1}^{m_n} |\lambda_k^{(n)} - \lambda_k| \to 0$ *as* $n \to \infty$.

Then there exist independent sequences $(B_1, B_2, ...)$ *and* $(C_1, C_2, ...)$ *of independent Brownian bridges on* $[0, 1]$ *such that the process* $Q(s, t) := \sum_k \sqrt{\lambda_k}\, B_k(t)\, C_k(s)$ *is well-defined and* X_n *converges in distribution to* Q *in* $D([0, 1]^2)$.

Comment. Here and in the sequel, we say that an infinite series of processes is "well-defined" if it converges almost surely uniformly on bounded sets (see [8]).

Conditions (ii) and (vi) above imply that $\sum \lambda_k = 1$. If $\lambda_k^{(n)} \to \lambda_k\ \forall k$, and $\sum \lambda_k < 1$, then we obtain a combination of the limiting processes in the previous two theorems.

THEOREM 2.3. *Assume that conditions* (i)–(iv) *of Theorem* 2.1 *and* (v) *of Theorem* 2.2 *are satisfied, as well as*:

(vi) *There exists a non-negative sequence* (λ_k) *such that* $\lambda_k^{(n)} \to \lambda_k$ *as* $n \to \infty$, *for each* k.

Then there exist independent sequence $(B_1, B_2, ...)$ *and* $(C_1, C_2, ...)$ *of independent Brownian bridges on* $[0, 1]$ *and an independent "tucked-in Brownian sheet"* A *on* $[0, 1]^2$ *such that the process*

$$P(s, t) := \sigma A(s, t) + \sum_k \sqrt{\lambda_k}\, B_k(t)\, C_k(s) \qquad \left(\text{where } \sigma^2 = 1 - \sum_k \lambda_k\right),$$

is well-defined and X_n *converges in distribution to* P *in* $D([0, 1]^2)$.

In all the previous theorems, it was assumed that $\sum_i \sum_j y_{ij}^{(n)^2} = \sum_k \lambda_k^{(n)} \to 1$. The case in which this sum converges to 0 was considered in Theorem 2.8 of [5]. We next consider a generalization of Theorem 2.2 and Theorem 2.8 of [5].

THEOREM 2.4. *Assume that the vectors* $\mathbf{u}_k^{(n)}$ *and* $\mathbf{v}_k^{(n)}$ *may be defined in such a way that conditions* (i), (ii), *and* (iv) *of Theorem* 2.1 *and* (v) *of Theorem* 2.2 *are satisfied, as well as*:

(iii) (a) $\sum_i (\sum_j y_{ij}^{(n)})^2 = \sum_k \lambda_k^{(n)} v_{k.}^{(n)^2} \to \sigma_1^2$ *as* $n \to \infty$.

(b) $\sum_j (\sum_i y_{ij}^{(n)})^2 = \sum_k \lambda_{k.}^{(n)^2} u_{k.}^{(n)^2} \to \sigma_2^2$ *as* $n \to \infty$.

(vi) *There exists a non-negative sequence* (λ_k), *and sequences* (β_k), (γ_k) *such that*:

(a) $\sum_k |\lambda_k^{(n)} - \lambda_k| \to 0$

(b) $\sum_k (\sqrt{\lambda_k^{(n)}} v_{k.}^{(n)} - \beta_k)^2 \to 0$

(c) $\sum_k (\sqrt{\lambda_k^{(n)}} u_{k.}^{(n)} - \gamma_k)^2 \to 0$, *as* $n \to \infty$.

(vii) $\lim_{n\to\infty} (u_{k.}^{(n)})^2/m_n = 0$ *or* 1, $\lim_{n\to\infty} (v_{k.}^{(n)})^2/r_n = 0$ *or* 1 $\forall k$.

Then there exist independent sequences $(B_1, B_2, \ldots)$ *and* $(C_1, C_2, \ldots)$ *of independent processes, where* B_k *(resp.* C_k*) is a Brownian bridge on* $[0, 1]$ *if* $(u_{k.}^{(n)})^2/m_n \to 0$ *(resp.* $(v_{k.}^{(n)})^2/r_n \to 0$*), and the* 0 *process if* $(u_{k.}^{(n)})^2/m_n \to 1$ *(resp.* $(v_{k.}^{(n)})^2/r_n \to 1$*), such that*

$$Q(s, t) := \sum_k \sqrt{\lambda_k}\, B_k(t)\, C_k(s) + s \sum_k \beta_k B_k(t) + t \sum_k \gamma_k C_k(s)$$

is well-defined and X_n *converges in distribution to* Q *in* $D([0, 1]^2)$.

The following theorem contains all the terms of Kallenberg's representation. Although the preceding theorems can be considered as special cases, they are used in the proof. Also, the conditions below may be simplified considerably in the less general settings.

THEOREM 2.5. *Assume that conditions* (i), (ii), *and* (iv) *of Theorem* 2.1 *and* (iii) *and* (v) *of Theorem* 2.4 *are satisfied, as well as*

(vi) *There exist a non-decreasing sequence of natural numbers* $(N(n))$, *a non-negative, non-increasing sequence* (λ_k), *and sequences* (β_k), (γ_k) *such that*

(a) $\sum_{k \leq N(n)} \sqrt{\lambda_k^{(n)}} u_{k.}^{(n)} v_{k.}^{(n)} \to 0$,

(b) $\sum_{k \leq N(n)} |\lambda_k^{(n)} - \lambda_k| \to 0$; $\sup_{k > N(n)} \lambda_k^{(n)} \to 0$; $\lambda_k = 0\ \forall k > N$,

(c) $\sum_{k \leq N(n)} (\sqrt{\lambda_k^{(n)}} v_{k.}^{(n)} - \beta_k)^2 \to 0$; $\sum_{k=1}^{N} \beta_k^2 = \sigma_1^2 - \kappa_1$; $\beta_k = 0\ \forall k > N$,

(d) $\sum_{k \leq N(n)} (\sqrt{\lambda_k^{(n)}} u_{k.}^{(n)} - \gamma_k)^2 \to 0$; $\sum_{k=1}^{N} \gamma_k^2 = \sigma_2^2 - \kappa_2$; $\gamma_k = 0\ \forall k > N$,

(e) $\sup_i |\sum_{k > N(n)} \sqrt{\lambda_k^{(n)}} u_{ki} v_{k.}| \to 0$; $\sup_j |\sum_{k > N(n)} \sqrt{\lambda_k^{(n)}} u_{k.} v_{kj}| \to 0$,

as $n \to \infty$, *where* $N = \lim_{n\to\infty} N(n)$, $\sigma^2 = 1 - \sum_{k=1}^{\infty} \lambda_k$, $(\sigma^2 \leq \sigma_1^2 \wedge \sigma_2^2)$, *and* κ_i *is equal to either* σ^2 *or* 0, $i = 1, 2$.

(vii) $\lim_{n\to\infty} (u_{k.}^{(n)})^2/m_n = 0$ *or* 1, $\lim_{n\to\infty} (v_{k.}^{(n)})^2/r_n = 0$ *or* 1 $\forall k \leq N$.

Then there exists a Brownian sail A *and independent sequences* $(B_1, B_2, \ldots)$ *and* $(C_1, C_2, \ldots)$ *of independent processes such that*

$$Q(s, t) := \sigma A(s, t) + \sum_k \sqrt{\lambda_k}\, B_k(t)\, C_k(s) + s \sum_k \beta_k B_k(t) + t \sum_k \gamma_k C_k(s)$$

is well-defined, and X_n converges in distribution to Q in $D([0,1]^2)$, where

(I) *A is a Brownian bridge on $[0,1]^2$ if $\kappa_1=\kappa_2=\sigma^2$ and the "tucked-in" sheet if $\kappa_1=\kappa_2=0$. If $\kappa_1=0$, $\kappa_2=\sigma^2$, then A is the Kiefer process; if $\kappa_1=\sigma^2$, $\kappa_2=0$, the process $A'(s,t)=A(t,s)$ is a Kiefer process.*

(II) *B_k (resp. C_k) is a Brownian bridge on $[0,1]$ if $(u_{k.}^{(n)})^2/m_n\to 0$ (resp. $(v_{k.}^{(n)})^2/r_n\to 0$), and the 0 process if $(u_{k.}^{(n)})^2/m_n\to 1$ (resp. $(v_{k.}^{(n)})^2/r_n\to 1$).*

3. Examples

In the examples which follow, we shall be making use of matrices of the form

$$K_n=[k_{ij}^{(n)}],\qquad 1\leqslant i,j\leqslant 2^n,$$

where $k_{ij}^{(n)}=(-1)^{i+j}/2^n$, $1\leqslant i,j\leqslant 2^n$. We note that K_n is idempotent of rank 1; the maximal eigenvalue of K_n is $\xi_1^{(n)}=1$ and all other eigenvalues are 0.

The corresponding normalized orthogonal eigenvectors are $\mathbf{w}_1^{(n)},\dots,\mathbf{w}_{2^n}^{(n)}$, where

$$w_{1h}^{(n)}=\frac{(-1)^{h+1}}{2^{n/2}},\qquad h=1,\dots,2^n$$

$$w_{2^nh}^{(n)}=\frac{1}{2^{n/2}},\qquad h=1,\dots,2^n,$$

and the entries of the other eigenvectors may all be set equal to $\pm 2^{-n/2}$.

Example 1. Let

$$D_1^{(n)}=2^{-n/2}\begin{bmatrix} K_n & & 0\\ & \ddots & \\ 0 & & K_n\end{bmatrix},$$

so K_n appears 2^n times down the diagonal. It is easy to check that the conditions of Theorem 2.1 are satisfied for this sequence of arrays. Now let $m_n=r_n=2^{2n+1}$, and

$$D^{(n)}=\frac{1}{\sqrt{2}}\begin{bmatrix} K_{2n} & 0\\ 0 & D_1^{(n)}\end{bmatrix}.$$

The conditions of Theorem 2.3 may be verified, with $\sigma^2=\frac{1}{2}$.

EXAMPLE 2. Let $m_n = r_n = 2^n(2^n+1)$, and

$$D^{(n)} = \begin{bmatrix} K_{2n} & 0 \\ 0 & H^{(n)} \end{bmatrix},$$

where $H^{(n)}$ is of dimension $2^n \times 2^n$ and $(H^{(n)})_{ij} = ((-1)^i + (-1)^j)/2^{3n/2}$, $1 \leqslant i, j \leqslant 2^n$. In this case, the conditions of Theorem 2.4 are satisfied with $\sigma_1^2 = \sigma_2^2 = 1$; $\lambda_1 = 1$, $\lambda_k = 0$, $k > 1$; $\beta_2 = 1$, $\beta_k = 0$, $k \neq 2$; $\gamma_3 = 1$, $\gamma_k = 0$, $k \neq 3$.

EXAMPLE 3. Let $m_n = r_n = 2^n(2^{n+1}+1)$ and

$$D^{(n)} = \frac{1}{\sqrt{2}} \begin{bmatrix} K_{2n} & 0 & 0 \\ 0 & H^{(n)} & 0 \\ 0 & 0 & D_2^{(n)} \end{bmatrix},$$

where K_{2n} and $H^{(n)}$ are as in Example 2 and $D_2^{(n)}$ is as in the example for Theorem 2.1 of [5], but of dimension $2^{2n} \times 2^{2n}$. For $D_2^{(n)}$, $\sigma^2 = \frac{1}{2}$, $\sigma_1^2 = \sigma_2^2 = 1$, $\kappa_1 = \kappa_2 = \frac{1}{2}$. It is easy to verify that the conditions of Theorem 2.5 hold, with $\beta_2 = 2^{-1/2}$; $\beta_k = 0$, $k \neq 2$; $\gamma_3 = 2^{-1/2}$; $\gamma_k = 0$, $k \neq 3$. Also, $\lambda_1 = \frac{1}{2}$; $\lambda_k = 0$, $k > 1$.

4. PROOFS OF THEOREMS

Henceforth, we shall use the notation $s' = s/(1+s)$, $t' = t/(1+t)$, $u' = u/(1+u)$, etc.

Proof of Theorem 2.1. The techniques used here are similar to those used in the proof of Theorem 2.1 of [5]. The proof of tightness is identical and a similar time change technique is used. In particular, if

$$Z_n(s, t) = (1+s)(1+t)\, X_n(s', t'),$$

exactly the same arguments as in the proof of Theorem 2.1 in [5] show that $Z_n \xrightarrow{D} W_2$. A reverse time change and the fact that $B_2(s, t) \stackrel{D}{=} (1-s)(1-t)\, W_2(s/(1-s), t/(1-t))$ [8, Lemma 2.8] show that $X_n \xrightarrow{D} A$. ∎

Proof of Theorem 2.2. Without loss of generality, we may assume zero row and column sums, in which case this is a corollary of Theorem 2.4. ∎

Proof of Theorem 2.3. Without loss of generality, we may assume $\sum_i \sum_j y_{ij}^{(n)} = 0$, $\sum_i \sum_j y_{ij}^{(n)2} = 1$, $\sum_i y_{ij}^{(n)} = \sum_j y_{ij}^{(n)} = 0\ \forall i, j$. The process P is well-defined by Lemma 6.2 of [8], since $\sum_k \lambda_k \leqslant \lim_{n \to \infty} \sum_{k=1}^{m_n} \lambda_k^{(n)} = 1$.

By Lemma 4.3 of [3], for each n there exist two sequences $(\delta_k^{(n)})_k$ and $(\psi_k^{(n)})_k$ such that $\sqrt{\lambda_k^{(n)}} = \sqrt{\delta_k^{(n)}} + \sqrt{\psi_k^{(n)}}$, and

$$\sum_k \left(\sqrt{\delta_k^{(n)}} - \sqrt{\lambda_k}\right)^2 \to 0 \qquad \text{as} \quad n \to \infty, \tag{4.1}$$

$$\sup_k |\psi_k^{(n)}| \to 0 \qquad \text{as} \quad n \to \infty, \tag{4.2}$$

$$\sum_{k=1}^{m_n} \psi_k^{(n)} \to 1 - \sum_{k=1}^{\infty} \lambda_k = \sigma^2 \qquad \text{as} \quad n \to \infty, \tag{4.3}$$

$$\psi_k^{(n)} \delta_k^{(n)} = 0 \qquad \forall n, k. \tag{4.4}$$

(Note that since the arrays have 0 row and column sums, $\lambda_{m_n}^{(n)} = 0 = \delta_{m_n}^{(n)} = \psi_{m_n}^{(n)}$. Also, if $k \neq m_n$, $\sum_i u_{ki}^{(n)} = \sum_j v_{kj}^{(n)} = 0$.)

Write $X_n(s, t) = F_n(s, t) + A_n(s, t)$, where

$$F_n(s, t) = \sum_{i=1}^{[m_n t]} \sum_{j=1}^{[r_n s]} \left(\sum_{k=1}^{m_n} \sqrt{\delta_k^{(n)}}\, U_{ki}^{(n)} V_{kj}^{(n)} \right) = \sum_{i=1}^{[m_n t]} \sum_{j=1}^{[r_n s]} f_{\sigma(i)\pi(j)}^{(n)},$$

$$A_n(s, t) = \sum_{i=1}^{[m_n t]} \sum_{j=1}^{[r_n s]} \left(\sum_{k=1}^{m_n} \sqrt{\psi_k^{(n)}}\, U_{ki}^{(n)} V_{kj}^{(n)} \right) = \sum_{i=1}^{[m_n t]} \sum_{j=1}^{[r_n s]} a_{\sigma(i)\pi(j)}^{(n)}.$$

It may be shown that $(a_{ij}^{(n)})$ satisfies the conditions of Theorem 2.1 (with suitable renorming), and that $(f_{ij}^{(n)})$ satisfies the conditions of Theorem 2.2. Finally, to show that A_n and F_n have independent limits, we must consider the sequences $(\delta_k^{(n)})$ and $(\psi_k^{(n)})$. There are two possibilities (cf. [3, Proof of Lemma 4.3]):

(I) There exists $N < \infty$ such that for all n,

$$\delta_k^{(n)} = \begin{cases} \lambda_k^{(n)} & \text{if} \quad k \leqslant N, \\ 0 & \text{if} \quad k > N \end{cases}$$

$$\psi_k^{(n)} = \begin{cases} 0 & \text{if} \quad k \leqslant N \\ \lambda_k^{(n)} & \text{if} \quad k > N, \end{cases}$$

or

(II) There exists a sequence $N(n) \to \infty$ as $n \to \infty$ such that

$$\delta_k^{(n)} = \begin{cases} \lambda_k^{(n)} & \text{if} \quad k \leqslant N(n) \\ 0 & \text{if} \quad k > N(n) \end{cases}$$

$$\psi_k^{(n)} = \begin{cases} 0 & \text{if} \quad k \leqslant N(n) \\ \lambda_k^{(n)} & \text{if} \quad k > N(n). \end{cases}$$

In the first case, $F_n(\cdot, \cdot) \xrightarrow{D} \sum_{k=1}^{N} \sqrt{\lambda_k}\, B_k(\cdot)\, C_k(\cdot)$, and it is enough to show that A is independent of $\sigma(B_1, ..., B_N, C_1, ..., C_N)$. In the second case, it must be shown that A is independent of $\sigma(B_1, ..., C_1, ...)$.

Let $B_k^{(n)}(t) = \sum_{i=1}^{[m_n t]} U_{ki}^{(n)}$ and $C_k^{(n)}(s) = \sum_{j=1}^{[r_n s]} V_{kj}^{(n)}$. We have $F_n(s, t) = \sum_{k=1}^{N(n)} \sqrt{\lambda_k^{(n)}}\; B_k^{(n)}(t)\, C_k^{(n)}(s)$, and $A_n(s, t) = \sum_{k=N(n)+1}^{\infty} \sqrt{\lambda_k^{(n)}}\; B_k^{(n)}(t)\, C_k^{(n)}(s)$, where $N(n) = N$ in case I.

Using techniques similar to those in the proof of Theorem 2.1 of [5], it may be shown for each k such that $k \leqslant N(n)$ for all n sufficiently large, and for each t fixed that $A'(\cdot, t)$ and $C'_k(\cdot)$ are orthogonal Gaussian martingales, independent of $\sigma(B'_1, B'_2, ...)$, where A', B'_k, C'_k are the time-changed processes defined by

$$A'(s, t) = (1+s)(1+t)\, A(s', t') \tag{4.5}$$

$$B'_k(t) = (1+t)\, B_k(t') \tag{4.6}$$

$$C'_k(s) = (1+s)\, C_k(s'). \tag{4.7}$$

Since continuous, orthogonal Gaussian martingales are independent, an application of the Cramer–Wold device proves independence of A', $(B'_1, B'_2, ...)$, and $(C'_k : k \leqslant \liminf N(n))$, and hence of the corresponding original processes as well. This completes the proof of independence of A and F, in both cases I and II. ∎

Proof of Theorem 2.4. The process Q is well-defined by Lemma 6.2 of [8], since $\sum_k \lambda_k = 1$, $\sum_k \beta_k^2 = \sigma_1^2$, and $\sum_k \gamma_k{}^2 = \sigma_2{}^2$. We shall assume again without loss of generality $\sum_{i=1}^{m_n} \sum_{j=1}^{r_n} y_{ij}^{(n)} = 0$ and $\sum_{i=1}^{m_n} \sum_{j=1}^{r_n} y_{ij}^{(n)2} = 1$. Then we may express X_n as

$$X_n(s, t) = C_n(s, t) + \frac{[r_n s]}{r_n}\, D_n(t) + \frac{[m_n t]}{m_n}\, E_n(s),$$

where

$$\begin{aligned} C_n(s, t) &= \sum_{i=1}^{[m_n t]} \sum_{j=1}^{[r_n s]} (Y_{ij}^{(n)} - \bar{Y}_{i.}^{(n)} - \bar{Y}_{.j}^{(n)}) \\ &= \sum_{k=1}^{[m_n t]} \sqrt{\lambda_k^{(n)}} \left(\sum_{i=1}^{[m_n t]} (U_{ki}^{(n)} - \bar{U}_k^{(n)}) \right) \left(\sum_{j=1}^{[r_n s]} (V_{kj}^{(n)} - \bar{V}_k^{(n)}) \right), \\ D_n(t) &= \sum_{i=1}^{[m_n t]} Y_{i.}^{(n)} \\ &= \sum_{k=1}^{m_n} \sqrt{\lambda_k^{(n)}}\, V_{k.}^{(n)} \sum_{i=1}^{[m_n t]} (U_{ki}^{(n)} - \bar{U}_k^{(n)}), \end{aligned}$$

$$E_n(t) = \sum_{j=1}^{[r_n s]} Y_{\cdot j}^{(n)}$$
$$= \sum_{k=1}^{m_n} \sqrt{\lambda_k^{(n)}}\, U_{k\cdot}^{(n)} \sum_{j=1}^{[r_n s]} (V_{kj}^{(n)} - \bar{V}_k^{(n)}),$$

and $\bar{U}_k^{(n)} = U_{k\cdot}^{(n)}/m_n$, $\bar{V}_k^{(n)} = V_{k\cdot}^{(n)}/r_n$. Note that $\bar{U}_k^{(n)}$ and $\bar{V}_k^{(n)}$ are non-random.

We first observe that (v) and (vii) imply that the conditions of Theorem 24.2 of [2] are satisfied by the exchangeable sequence $(U_{ki}^{(n)} - \bar{U}_k^{(n)}$: $i = 1, ..., m_n)$, for each k. Thus,

$$B_k^{(n)}(\cdot) = \sum_{i=1}^{[m_n \cdot]} (U_{ki}^{(n)} - \bar{U}_k^{(n)}) \xrightarrow{D} B_k(\cdot), \qquad k = 1, 2, ...,$$

where for each k, B_k is a Brownian bridge on $[0, 1]$ if $(u_{k\cdot}^{(n)})^2/m_n \to 0$, and the 0 process if $(u_{k\cdot}^{(n)})^2/m_n \to 1$. The proof that the limiting processes (B_k) are independent is a straightforward calculation using the Cramer–Wold device and the asymptotic orthogonality of the vectors $\mathbf{U}_k^{(n)} - \bar{U}_k^{(n)}\mathbf{1}$, $k = 1, ..., m_n$. The processes $(C_k^{(n)})$ behave analogously and the limiting sequence (C_k) is independent of the sequence (B_k) since the processes $(B_k^{(n)})$ and $(C_k^{(n)})$ are defined using independent permutations.

Fix $N < \infty$ and consider the truncated sum

$$X_n^N(s, t) = \sum_{k=1}^{N} \sqrt{\lambda_k^{(n)}} \sum_{i=1}^{[m_n t]} U_{ki}^{(n)} \sum_{j=1}^{[r_n s]} V_{kj}^{(n)}$$
$$= C_n^N(s, t) + \frac{[r_n s]}{r_n} D_n^N(t) + \frac{[m_n t]}{m_n} E_n^N(s).$$

It is clear from the discussion above that for fixed N, $X_n^N(\cdot, \cdot) \xrightarrow{D} Q^N(\cdot, \cdot)$ in $D([0, 1]^2)$, where

$$Q^N(s, t) := \sum_{k=1}^{N} \sqrt{\lambda_k}\, B_k(t)\, C_k(s) + s \sum_{k=1}^{N} \beta_k B_k(t) + t \sum_{k=1}^{N} \gamma_k C_k(s).$$

We now use a technique adapted from that of Neuhaus [9, p. 433]. Denote by $L(Z_1, Z_2)$ the Lévy–Prohorov distance between the distributions of $D([0, 1]^2)$-valued random elements Z_1, Z_2. Then

$$L(X_n, Q) \leqslant L(X_n, X_n^N) + L(X_n^N, Q^N) + L(Q^N, Q).$$

We have seen that $L(Q^N, Q) \to 0$ for any sequence $N \to \infty$, and that for N fixed $L(X_n^N, Q^N) \to 0$ as $n \to \infty$. It is easy to show that there exists a sequence $(N(n))$ such that $N(n) \to \infty$ as $n \to \infty$ and $L(X_n^{N(n)}, Q^{N(n)}) \to 0$.

Since $(Q^{N(n)})$ is tight, it follows that $(X_n^{N(n)})$ is tight. We shall show (a) that (X_n) is also tight, and (b) that for each (s, t)

$$|X_n(s, t) - X_n^{N(n)}(s, t)| \xrightarrow{D} 0$$

for any sequence $(N(n))$ such that $N(n) \to \infty$ as $n \to \infty$. Thus, since $(X_n - X_n^{N(n)})$ is tight, we have $L(X_n, X_n^{N(n)}) \to 0$.

Thus, for the sequence $N(n)$ defined above,

$$\begin{aligned} L(X_n, Q) \leqslant L(X_n, X_n^{N(n)}) + L(X_n^{N(n)}, Q^{N(n)}) \\ + L(Q^{N(n)}, Q) \to 0 \qquad \text{as} \quad n \to \infty. \end{aligned}$$

It remains to prove (a) and (b).

Proof of (a). By considering the proof of tightness in Theorem 2.1 of [5], it is enough to show that $\sup_i |\sum_j y_{ij}^{(n)}| \to 0$ and $\sup_j |\sum_i y_{ij}^{(n)}| \to 0$. This is done by proving that

$$D_n \xrightarrow{D} \sum_k \beta_k B_k \tag{4.8}$$

and

$$E_n \xrightarrow{D} \sum_k \gamma_k C_k, \tag{4.9}$$

in $D[0, 1]$. Now, $D_n(t) = \sum_{i=1}^{[m_n t]} Y_{i.}^{(n)} = \sum_{i=1}^{[m_n t]} y_{\sigma(i).}^{(n)}$, and the limit is continuous, so by Theorem 2.2 of [7], necessarily $\sup_i |y_{i.}^{(n)}| = \sup_i |\sum_j y_{ij}^{(n)}| \xrightarrow{P} 0$. Analogously, (4.9) implies that $\sup_j |\sum_i y_{ij}^{(n)}| \xrightarrow{P} 0$. Thus it is sufficient to prove (4.8) as the proof of (4.9) is analogous.

Let $D^N = \sum_{k=1}^{N} \beta_k B_k$. We have

$$L(D_n, D) \leqslant L(D_n, D_n^N) + L(D_n^N, D^N) + L(D^N, D).$$

We have that $L(D^N, D) \to 0$ for any sequence (N) such that $N \to \infty$. For N fixed, $L(D_n^N, D^N) \to 0$ as $n \to \infty$, and so there exists a sequence $(N(n))$ such that $N(n) \to \infty$ as $n \to \infty$, and $L(D_n^{N(n)}, D^{N(n)}) \to 0$. Finally, it is enough to show that $L(D_n, D_n^{N(n)}) \to 0$ for any sequence $(N(n))$ such that $N(n) \to \infty$. In this case, $L(D_n, D) \to 0$.

Consider $L(D_n, D_n^N)$

$$L(D_n, D_n^N) \leqslant \inf\{\eta > 0 : P(\sup_t |\delta_n^N(t)| \geqslant \eta) \leqslant \eta\} \qquad \text{(cf. (2.33) of [9])},$$

where $\delta_n^N(t) = \sum_{i=1}^{[m_n t]} l_{\sigma_n(i)}^{(n)}$ and $l_i^{(n)} = \sum_{k=N+1}^{m_n} \sqrt{\lambda_k^{(n)}}\, v_{k.}^{(n)} (u_{ki}^{(n)} - \bar{u}_k^{(n)})$.

It can be shown that $\sum_{i=1}^{m_n} l_i^{(n)2} \to 0$, so Theorem 24.2 of [2] (with appropriate renormalization) (or Theorem 2.2 of [7]) may be applied to the sequence of exchangeable r.v.'s $(l^{(n)}_{\sigma_n(1)}, ..., l^{(n)}_{\sigma_n(m_n)})$ to show that $\delta^N_n \xrightarrow{D} 0$ in $D([0, 1])$. Thus, $\sup_t |\delta^N_n(t)| \xrightarrow{P} 0$ and $L(D_n, D^N_n) \to 0$, as required.

Proof of (b). It must be shown that

$$\begin{aligned}(X_n - X^N_n)(s, t) = &\sum_{k=N+1}^{m_n} \sqrt{\lambda^{(n)}_k} \left(\sum_{i=1}^{[m_n t]} U^{(n)}_{ki} - \bar{U}^{(n)}_k\right)\left(\sum_{j=1}^{[r_n s]} V^{(n)}_{kj} - \bar{V}^{(n)}_k\right)\\ &+ \frac{[r_n s]}{r_n} \sum_{k=N+1}^{m_n} \sqrt{\lambda^{(n)}_k}\, V^{(n)}_{k.} \sum_{i=1}^{[m_n t]} (U^{(n)}_{ki} - \bar{U}^{(n)}_k)\\ &+ \frac{[m_n t]}{m_n} \sum_{k=N+1}^{m_n} \sqrt{\lambda^{(n)}_k}\, U^{(n)}_{k.} \sum_{j=1}^{[r_n s]} (V^{(n)}_{kj} - \bar{V}^{(n)}_j)\\ &\xrightarrow{P} 0 \end{aligned} \tag{4.10}$$

for any sequence $(N(n))$ such that $N(n) \to \infty$ as $n \to \infty$.

The last two terms on the right-hand side of (4.10) converge in probability to 0 whenever $N(n) \to \infty$ by the proofs of (4.8) and (4.9), respectively. The first term on the right-hand side of (4.10) can be written as

$$\sum_{i=1}^{[m_n t]} \sum_{j=1}^{[r_n s]} q^{(n)}_{\sigma_n(i)\pi_n(j)},$$

where $q^{(n)}_{ij} = \sum_{k=N+1}^{m_n} \sqrt{\lambda^{(n)}_k}(u^{(n)}_{ki} - \bar{u}^{(n)}_k)(v^{(n)}_{kj} - \bar{v}^{(n)}_k)$. (*Note.* $\sum_i \sum_j q^{(n)}_{ij} = \sum_i q^{(n)}_{ij} = \sum_j q^{(n)}_{ij} = 0$.) A straightforward calculation shows that

$$E\left\{\left[\sum_{i=1}^{[m_n t]} \sum_{j=1}^{[r_n s]} q^{(n)}_{\sigma_n(i)\pi_n(j)}\right]^2\right\} \approx \sum_{i=1}^{m_n} \sum_{j=1}^{r_n} q^{(n)^2}_{ij}(st(1-s)(1-t)).$$

But

$$\begin{aligned}\sum_{i=1}^{m_n} \sum_{j=1}^{r_n} q^{(n)^2}_{ij} = &\sum_{k=N+1}^{m_n} \lambda^{(n)}_k \sum_i (u^{(n)}_{ki} - \bar{u}^{(n)}_k)^2 \sum_j (v^{(n)}_{kj} - \bar{v}^{(n)}_k)^2\\ &+ \sum_{k>N} \sum_{\substack{k'>N\\ k\neq k'}} \sum_i \sum_j \sqrt{\lambda^{(n)}_k} \sqrt{\lambda^{(n)}_{k'}}(u^{(n)}_{ki} - \bar{u}^{(n)}_k)(u^{(n)}_{k'i} - \bar{u}^{(n)}_{k'})(v^{(n)}_{kj} - \bar{v}^{(n)}_k)\\ &\times (v^{(n)}_{k'j} - \bar{v}^{(n)}_{k'})\\ = &\sum_{k=N+1}^{m_n} \lambda^{(n)}_k(1 - m_n\bar{u}^{(n)^2}_k)(1 - r_n\bar{v}^{(n)^2}_k)\\ &+ \left(\sum_{k=N+1}^{m_n} \sqrt{\lambda^{(n)}_k} \sqrt{m_n}\bar{u}^{(n)}_k \sqrt{r_n}\bar{v}^{(n)}_k\right)^2 - \sum_{k=N+1}^{m_n} \lambda^{(n)}_k m_n \bar{u}^{(n)^2}_k r_n \bar{v}^{(n)^2}_k\\ \leqslant &\sum_{k=N+1}^{m_n} \lambda^{(n)}_k + \left(\sum_{k=N+1}^{m_n} \sqrt{\lambda^{(n)}_k} \sqrt{m_n}\bar{u}^{(n)}_k \sqrt{r_n}\bar{v}^{(n)}_k\right)^2 \end{aligned}$$

since $m_n \bar{u}_k^{(n)^2} \leqslant 1$ and $r_n \bar{v}_k^{(n)^2} \leqslant 1$. Now, $\sum_{k=N+1}^{m_n} \lambda_k^{(n)} \to 0$ as $N, n \to \infty$, by (ii) and (vi) (a) of Theorem 2.4. Also,

$$\begin{aligned}
\left(\sum_{k=N+1}^{m_n} \sqrt{\lambda_k^{(n)}} \sqrt{m_n} \bar{u}_k^{(n)} \sqrt{r_n} \bar{v}_k^{(n)} \right)^2 &= (m_n r_n)^{-1} \left(\sum_{k=N+1}^{m_n} \sqrt{\lambda_k^{(n)}} u_{k.}^{(n)} v_{k.}^{(n)} \right)^2 \\
&\leqslant (m_n r_n)^{-1} \sum_{k=N+1}^{m_n} \lambda_k^{(n)} u_{k.}^{(n)^2} \sum_{k'=N+1}^{m_n} v_{k'.}^{(n)^2} \\
&\leqslant \sum_{k=N+1}^{m_n} \lambda_k^{(n)} u_{k.}^{(n)^2} \quad (\text{since } \bar{v}_{k'.}^{(n)^2} \leqslant r_n^{-1}) \\
&\to 0 \quad \text{by (iii) and (vi)(c) of Theorem 2.4.}
\end{aligned}$$

This completes the proof of (4.10). ▮

Proof of Theorem 2.5. This theorem is a corollary of Theorems 2.1, 2.4, and Theorem 2.1 of [5], using the same method of proof as that of Theorem 2.3. ▮

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