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The Law of the Iterated Logarithm and Central Limit Theorem for L-Statistics

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The Chung–Smirnov law of the iterated logarithm and the Finkelstein functional law of the iterated logarithm for empirical processes are used to establish new results on the central limit theorem, the law of the iterated logarithm, and the strong law of large numbers for L-statistics with certain bounded and smooth weight functions. These results are used to obtain necessary and sufficient conditions for almost sure convergence and for convergence in distribution of some well-known L-statistics and U-statistics, including Gini's mean difference statistic. A law of the logarithm for weighted sums of order statistics is also presented. © 2001 Academic Press

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1. INTRODUCTION

Throughout this article, $\{X, X_n; n \geqslant 1\}$ will denote a sequence of independent identically distributed (i.i.d.) real random variables with common distribution function F given by $F(x) = P(X \leqslant x)$, $x \in \mathscr{R}$, the real line. For each positive integer n, let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ be the order statistics of $X_1, X_2, ..., X_n$. Let H be a real-valued measurable function defined on $\mathscr{R}$.

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Linear combinations of order statistics (in short, L-statistics) are statistics of the form

$$L_n = \frac{1}{n} \sum_{i=1}^{n} c_{i,n} H(X_{i:n}),$$

where the weights $c_{i,n}$, $1 \leqslant i \leqslant n$, $n \geqslant 1$, are real numbers.

Much is known about the limiting behaviour of L-statistics, including asymptotic normality properties, Berry–Esséen-type bounds for normal approximation, Cramér-type large deviations, laws of the iterated logarithm, and Kolmogorov-type strong laws of large numbers under quite general conditions. See Helmers (1977), Helmers *et al.* (1988), Mason (1982), Mason and Shorack (1992), Sen (1978), Shorack (1972), Stigler (1974), van Zwet (1980), Wellner (1977a, 1977b), and, in particular, the two books by Serfling (1980) and Shorack and Wellner (1986) and references therein. However, these results all require $H(\cdot)$ to be a known function of the form $H(\cdot) = H_1(\cdot) - H_2(\cdot)$ with each H_i being increasing and left-continuous. Furthermore, $H_1(\cdot)$ and $H_2(\cdot)$ must also satisfy

$$|H_i(G(t))| \leqslant M_1 t^{-d_1} (1-t)^{-d_2}, \qquad 0 < t < 1, \tag{1.1}$$

for some fixed positive M_1, d_1, and d_2, where

$$G(t) = \inf\{s; F(s) \geqslant t\}, \qquad 0 < t < 1.$$

Some conditions on $\{c_{i,n}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ are also needed. One often chooses

$$c_{i,n} = J\left(\frac{i}{n}\right), \qquad 1 \leqslant i \leqslant n, \quad n \geqslant 1,$$

where $J(t)$ satisfies some continuity conditions on $(0, 1)$ as well as

$$|J(t)| \leqslant M_2 t^{-b_1} (1-t)^{-b_2}, \qquad 0 < t < 1, \tag{1.2}$$

for some fixed positive M_2, b_1, and b_2. To obtain central limit theorems or laws of the iterated logarithm for

$$\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(X_{i:n}), \qquad n \geqslant 1,$$

it is also assumed that

$$a = \max\{b_1 + d_1, b_2 + d_2\} < \tfrac{1}{2}.$$

(However, it is worth noting that conditions like (1.1) and (1.2) are only relevant when the weight function $J(\cdot)$ is unbounded, whereas $J(\cdot)$ will be assumed to be bounded in our results below.)

The main objective of this paper is to find necessary and sufficient conditions for the law of the iterated logarithm (LIL) and the central limit theorem (CLT) for L-statistics when the weight function $J(\cdot)$ is bounded and smooth. Our main result, Theorem 2.1 below, imposes no conditions on the function $H(\cdot)$, except that $E(|H(X)|)$ be finite; as noted above, other authors have required more stringent conditions on $H(\cdot)$, including (1.1). However, our results apply only to functions $J(\cdot)$ in a smaller class (the class of Lipschitz functions of order one) than do earlier results, which require only continuity conditions and (1.2). Nonetheless, our results can be applied in situations where earlier results cannot; Theorem 3.4, which presents limit theorems for Gini's mean difference, is a good example.

It may be useful at this point to outline the main idea behind our approach. Let $\{U, U_n; n \geqslant 1\}$ represent a sequence of i.i.d. random variables with uniform $(0, 1)$ distribution. Then it is well known that

$$\{X, X_n; n \geqslant 1\} \stackrel{d}{=} \{G(U), G(U_n); n \geqslant 1\},$$

where "$\stackrel{d}{=}$" means "equal in distribution." It now follows that

$$\{X_{i:n}; 1 \leqslant i \leqslant n, n \geqslant 1\} \stackrel{d}{=} \{G(U_{i:n}); 1 \leqslant i \leqslant n, n \geqslant 1\},$$

where $U_{i:n}$, $1 \leqslant i \leqslant n$, are the order statistics of U_i, $1 \leqslant i \leqslant n$. Note that

$$P(U_i \neq U_j \text{ for all } 1 \leqslant i < j < \infty) = 1,$$

so we have that

$$\begin{aligned}
&\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(G(U_{i:n})) \\
&\quad = \sum_{i=1}^{n} J(U_{i:n})\, H(G(U_{i:n})) + \sum_{i=1}^{n} \left(J\left(\frac{i}{n}\right) - J(U_{i:n})\right) H(G(U_{i:n})) \\
&\quad \stackrel{\text{a.s.}}{=} \sum_{i=1}^{n} J(U_i)\, H(G(U_i)) + \sum_{i=1}^{n} (J(D_n(U_{i:n})) - J(U_{i:n}))\, H(G(U_{i:n})) \\
&\quad \triangleq S_n + R_n, \qquad n \geqslant 1, \qquad (1.3)
\end{aligned}$$

where D_n is the empirical distribution function of $U_1, U_2, ..., U_n$. Clearly, classical results can be applied to S_n, which is a sum of i.i.d. random variables. We will apply some known results for empirical processes to determine the limiting behaviour of R_n. It will be seen that the limiting behaviour of $\sum_{i=1}^{n} J(\frac{i}{n})\, H(X_{i:n}),\ n \geqslant 1$, under both the law of the iterated logarithm and the central limit theorem, does not depend only on that of $\sum_{i=1}^{n} J(U_i)\, H(G(U_i)),\ n \geqslant 1$.

This approach is straightforward. The empirical process is a powerful tool which has now become a standard technique in proving limit theorems. (See, for example, Gilat and Hill (1992), Helmers *et al.* (1988), Mason and Shorack (1992), Shorack and Wellner (1986), and the references therein.) Our main result (Theorem 2.1) and its proof will appear in Section 2. In Section 3, we show how this approach can be applied to obtain necessary and sufficient conditions for either almost sure convergence or convergence in distribution of some well-known L-statistics or U-statistics.

It is very natural to consider the case of general weights $c_{i,n}$, $1 \leqslant i \leqslant n$, $n \geqslant 1$, with

$$\sup_{1 \leqslant i \leqslant n,\, n \geqslant 1} |c_{i,n}| < \infty,$$

and to ask about the limiting behaviour of

$$\sum_{i=1}^{n} c_{i,n} H(X_{i:n}), \qquad n \geqslant 1.$$

This question will be resolved in Section 4.

2. MAIN RESULTS

Let X be a real random variable with distribution function $F(x)$ and U a random variable with uniform $(0, 1)$ distribution. Let $H(\cdot)$ be a real Borel-measurable function defined on $\mathscr{R}$ with

$$E(|H(X)|) < \infty. \tag{2.1}$$

Let $J(\cdot)$ be a Lipschitz function of order one defined on $[0, 1]$; this is equivalent to the existence of an almost everywhere bounded Lebesgue derivative $J'(\cdot)$ of $J(\cdot)$. Write

$$\mu = \mu(F, J, H) = E(J(U)\, H(G(U))) = E(Z)$$

$$\text{where } Z = J(U)\, H(G(U)),$$

$$Y = -Z + \mu - \int_0^1 (I_{\{U \leqslant t\}} - t)\, J'(t)\, H(G(t))\, dt. \tag{2.2}$$

Then μ exists and is finite and Y and Z are both well-defined random variables under (2.1). Moreover,

$$\sigma^2 = \operatorname{Var}(Y) = E(Y^2). \tag{2.3}$$

To see this, note that (2.3) can hold if and only if

$$E(Y) = -E\left(\int_0^1 (I_{\{U \leqslant t\}} - t)\, J'(t)\, H(G(t))\, dt\right) = 0. \tag{2.4}$$

Since $E(I_{\{U \leqslant t\}}) = t$, (2.4) follows from an application of the Fubini theorem, subject to the existence of the integral

$$I = \int_0^1 J'(t)\, H(G(t)\, dt = \int_0^1 H(G(t)\, dJ(t) = E(J'(U)\, H(G(U))).$$

But $J(\cdot)$ has an almost everywhere bounded Lebesgue derivative $J'(\cdot)$. From this fact and the equality $E(|H(X)|) = E(|H(G(U))|) < \infty$, it follows that I exists and is finite; Clearly, $\sigma^2 < \infty$ if and only if

$$E(Z^2) < \infty. \tag{2.5}$$

Recall that a sequence of random variables $\{\xi_n; n \geqslant 1\}$ is called bounded in probability if

$$\lim_{x \to \infty} \sup_{n \geqslant 1} P(|\xi_n| \geqslant x) = 0.$$

The main result of this paper, which provides necessary and sufficient conditions for certain L-statistics with bounded and smooth weight functions to obey the LIL and CLT, will now be presented.

THEOREM 2.1. *Let $\{X, X_n; n \geqslant 1\}$ be a sequence of i.i.d. random variables. Let $H(\cdot)$ be a real Borel-measurable function defined on $\mathscr{R}$ such that* (2.1) *holds, and let $J(\cdot)$ be a Lipschitz function of order one defined on* $[0, 1]$. *Define μ and Z by* (2.2). *Then the following three statements are equivalent*:

(i) $E(Z^2) < \infty$;

(ii) $$\limsup_{n \to \infty} \frac{\left|\sum_{i=1}^n J\left(\frac{i}{n}\right) H(X_{i:n}) - n\mu\right|}{\sqrt{2n \log\log n}} < \infty \quad a.s.; \tag{2.6}$$

(iii) $$\left\{\frac{\sum_{i=1}^n J\left(\frac{i}{n}\right) H(X_{i:n}) - n\mu}{\sqrt{n}};\ n \geqslant 1\right\} \text{ is bounded in probability.} \tag{2.7}$$

Moreover, if any of (i), (ii), *or* (iii) *holds, then* σ *is finite and*

$$\limsup_{n \to \infty} (\liminf_{n \to \infty}) \frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(X_{i:n}) - n\mu}{\sqrt{2n \log \log n}} = (\overset{+}{-})\, \sigma \qquad a.s. \tag{2.8}$$

and

$$\frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(X_{i:n}) - n\mu}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2), \tag{2.9}$$

where σ^2 *is defined by* (2.3) *and* "$\xrightarrow{d}$" *means convergence in distribution.*

Remarks. (i) From our proof, it will be clear that the almost sure limit set of the sequence

$$\left\{ \frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(X_{i:n}) - n\mu}{\sqrt{2n \log \log n}} ;\, n \geqslant 1 \right\}$$

is equal to the interval $[-\sigma, \sigma]$.

(ii) The use of weight functions which are Lipschitz of order one is restrictive, and rules out many interesting L-statistics. However, Theorem 2.1 includes the classical LIL and the classical CLT as special cases (take $J(t) = 1$, $t \in [0, 1]$, and $H(x) = x$, $x \in \mathscr{R}$), where other limit theorems for L-statistics do not. Consider the following

EXAMPLE 2.1. Let $p > 1$ and $\{X, X_n; n \geqslant 1\}$ be a sequence of i.i.d. random variables with the common density function

$$f(x) = \begin{cases} \dfrac{C_p}{|x|^3 (L\,|x|)^p}, & |x| \geqslant e \\ 0, & |x| < e, \end{cases}$$

where $C_p > 0$ is a constant such that $\int_{-\infty}^{\infty} f(x)\, dx = 1$, $Lx = \log_e \max\{e, x\}$, and $L_2 x = L(Lx)$, $x \in \mathscr{R}$. Clearly,

$$E(X) = 0 \qquad \text{and} \qquad E(X^2) = \frac{2C_p}{p-1}.$$

Choose $H(x)=x$ and $J(t)\equiv 1$. One can check that no previous result can be used to determine the value of

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) X_{i:n}}{\sqrt{2n\log\log n}} = \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} X_{i:n}}{\sqrt{2n\log\log n}}$$

or the convergence in distribution of the sequence

$$\left\{ \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) X_{i:n}}{\sqrt{n}}; n\geqslant 1 \right\},$$

even though the classical LIL and classical CLT, respectively, assert that the limit superior equals $(2C_p/(p-1))^{1/2}$ almost surely and the limit distribution is $N(0, 2C_p/(p-1))$.

(iii) The following example provides a situation in which the function $H(\cdot)$ cannot be written as the difference of two increasing, left-continuous functions, so that previous results in the literature do not apply, but for which our Theorem 2.1 yields a conclusion.

Example 2.2. Let $\mathbf{W}$ be a standard Wiener process on $\mathscr{R}$ (i.e., $\{\mathbf{W}(t); t\geqslant 0\}$ and $\{\mathbf{W}(-t); t\geqslant 0\}$ are two independent copies of a Wiener process starting from 0), and let $\{X, X_n; n\geqslant 1\}$ be a sequence of i.i.d. random variables, independent of $\mathbf{W}$. For any given path of $\mathbf{W}$, consider L-statistics of the form

$$\sum_{i=1}^{n} \frac{i}{n} \mathbf{W}(X_{i:n}), \qquad n\geqslant 1.$$

As is well known, for almost every path of $\mathbf{W}$, $\mathbf{W}$ cannot be represented in the form $\mathbf{W}=\mathbf{W}_1-\mathbf{W}_2$ with $\mathbf{W}_i\uparrow$ and left-continuous. Thus the almost sure value of

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{i}{n} \mathbf{W}(X_{i:n}) - A_n}{\sqrt{2n\log\log n}}$$

and convergence in distribution of the sequence

$$\left\{ \frac{\sum_{i=1}^{n} \frac{i}{n} \mathbf{W}(X_{i:n}) - A_n}{\sqrt{n}}; n\geqslant 1 \right\},$$

for some sequence $\{A_n; n \geqslant 1\}$, cannot be obtained using any law of the iterated logarithm or central limit theorem for L-statistics established so far. However, using Theorem 2.1 above, we have

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{i}{n} \mathbf{W}(X_{i:n}) - n\mu_{\mathbf{W}}}{\sqrt{2n \log\log n}} = \sigma_{\mathbf{W}} \quad \text{a.s.}$$

and

$$\frac{\sum_{i=1}^{n} \frac{i}{n} \mathbf{W}(X_{i:n}) - n\mu_{\mathbf{W}}}{\sqrt{n}} \xrightarrow{d} N(0, \sigma_{\mathbf{W}}^2),$$

where

$$Z_{\mathbf{W}} = U\mathbf{W}(G(U)), \qquad \mu_{\mathbf{W}} = E_U(Z_{\mathbf{W}}),$$

$$Y_{\mathbf{W}} = -Z_{\mathbf{W}} + \mu_{\mathbf{W}} - \int_0^1 (I_{\{U \leqslant t\}} - t)\, \mathbf{W}(Q(t))\, dt, \qquad \sigma_{\mathbf{W}}^2 = E_{\mathbf{U}}(Y_{\mathbf{W}}^2),$$

and E_U represents the expectation with respect to U.

(iv) A LIL for weighted sums established by Tomkins (1975, 1976) implies that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) X_i}{\sqrt{2n \log\log n}} = \left(\int_0^1 J^2(t)\, dt\right)^{1/2} \quad \text{a.s.}$$

for any Lipschitz function $J(\cdot)$ of order one defined on $[0, 1]$ if $E(X) = 0$ and $E(X^2) = 1$. (In fact, Li and Tomkins (1996) showed that the converse is true if $\int_0^1 J(t)\, dt \neq 0$.) There are similarities between this result and (2.8), but note that the limiting values are not equal; that is,

$$\int_0^1 J^2(t)\, dt \neq \mathrm{Var}(Y),$$

where Y is defined as in (2.2).

(v) It will be clear from the proof of Theorem 2.1 that one can replace $J(\frac{i}{n})$, $1 \leqslant i \leqslant n$, $n \geqslant 1$ in (2.6), (2.7), (2.8), and (2.9) with $J(t_{i:n})$, $1 \leqslant i \leqslant n$, $n \geqslant 1$, for any $t_{i,n} \in [0, 1]$, $1 \leqslant i \leqslant n$, satisfying $\max_{1 \leqslant i \leqslant n} |t_{i,n} - \frac{i}{n}| = o(\frac{1}{\sqrt{n}})$ as $n \to \infty$.

(vi) An interesting open problem is to extend Theorem 2.1 to the case where $J(\cdot)$ is absolutely continuous but with a Lebesgue derivative $J'(\cdot)$ that may not be almost everywhere bounded. As noted by a referee, the assumptions (2.1) and (2.5) would have to be amended, perhaps to

$$E(|J'(U)\,H(G(U))|)<\infty \qquad \text{and} \qquad E(\{J'(U)\,H(G(U))\}^2)<\infty,$$

but the techniques used to prove Theorem 2.1 may also yield results in the case where $J'(\cdot)$ is not almost everywhere bounded.

Proof of Theorem 2.1. As mentioned in Section 1, one may set without loss of generality $X_n=G(U_n)$ and $X_{i:n}=G(U_{i:n})$ for $1\leqslant i\leqslant n$ and $n\geqslant 1$, where $\{U_n; n\geqslant 1\}$ is a sequence of i.i.d. uniform (0, 1) random variables and $U_{1:n}\leqslant U_{2:n}\leqslant \cdots \leqslant U_{n:n}$ are the order statistics of $U_1, U_2, ..., U_n$. For each $n\geqslant 1$, let $D_n(t)=n^{-1}\#\{U_i\leqslant t: 1\leqslant i\leqslant n\}$ and $Q_n(t)=\inf\{s\geqslant 0: D_n(s)\geqslant t\}$ for $0\leqslant t\leqslant 1$. Note that $Q_n(t)=U_{i:n}$ for $(i-1)/n<t\leqslant i/n$ and $i=1, 2, ..., n$. Let $\beta_n(t)=n^{1/2}(Q_n(t)-t)$ denote the empirical quantile process. Set $K(t)=H(G(t))$.

First, we will show that (i) and (ii) are equivalent and that (i) implies (2.8). To begin with, suppose that $E(|K(U)|)<\infty$ and, for some $c>0$,

$$|K(t+h)-K(t)-hK'(t)|\leqslant c\,|h|\,\eta(h)$$

uniformly over $t, h\in\mathscr{R}$, where $\eta(h)\downarrow 0$ as $|h|\downarrow 0$. For $n\geqslant 1$, define

$$\hat{T}_{n,1}=n^{-1}\sum_{i=1}^{n} J\left(\frac{i}{n}\right)K(U_{i:n})=n^{-1}\sum_{i=1}^{n} J\left(\frac{i}{n}\right)K\left(Q_n\left(\frac{i}{n}\right)\right),$$

$$\hat{T}_{n,2}=\int_0^1 J(t)\,K(Q_n(t))\,dt=\int_0^1 J(t)\,K(t+n^{-1/2}\beta_t(t))\,dt$$

$$=\sum_{i=1}^{n}\left\{\int_{(i-1)/n}^{i/n} J(t)\,dt\right\}K\left(Q_n\left(\frac{i}{n}\right)\right), \qquad T=\int_0^1 J(t)\,K(t)\,dt.$$

Under the hypotheses of the theorem, $|J(t+h)-J(t)|\leqslant b\,|h|$ uniformly for $t, t+h\in[0, 1]$, for some $b>0$. Hence

$$\left|n^{-1}J\left(\frac{i}{n}\right)-\int_{(i-1)/n}^{i/n} J(t)\,dt\right|\leqslant\int_{(i-1)/n}^{i/n}\left|J(t)-J\left(\frac{i}{n}\right)\right|dt$$

$$\leqslant bn^{-1}\int_{(i-1)/n}^{i/n} dt=\frac{b}{n^2},$$

so that, as $n \to \infty$,

$$\begin{aligned} |\hat{T}_{n,1} - \hat{T}_{n,2}| &\leqslant bn^{-2} \sum_{i=1}^{n} |K(U_{i:n})| \\ &= bn^{-2} \sum_{i=1}^{n} |K(U_n)| \\ &= (1+o(1))\, bn^{-1} E(|K(U)|) \\ &= O(n^{-1}) \qquad \text{a.s.} \end{aligned} \tag{2.10}$$

Set $\|f\| = \sup_{0 \leqslant t \leqslant 1} |f(t)|$. Since $\|\beta_n\| \leqslant (\log\log n)^{1/2}$ a.s. for all large n, it follows that

$$\begin{aligned} &|K(t + n^{-1/2}\beta_n(t)) - K(t) - n^{-1/2}\beta_n(t)\, K'(t)| \\ &\quad \leqslant cn^{-1/2} (\log\log n)^{1/2}\, \eta\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right) \\ &\quad = o\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right) \qquad \text{a.s.} \end{aligned}$$

and hence that, as $n \to \infty$,

$$\left|\left(\frac{n}{2\log\log n}\right)^{1/2} (\hat{T}_{n,2} - T) - \int_0^1 J(t)\, K'(t)\, \frac{\beta_n(t)}{\sqrt{2\log\log n}}\, dt\right| = o(1) \qquad \text{a.s.} \tag{2.11}$$

Now, by the Finkelstein (1971) functional law of the iterated logarithm,

$$\begin{aligned} &\limsup_{n \to \infty} \pm \int_0^1 J(t)\, K'(t)\, \frac{\beta_n(t)}{\sqrt{2\log\log n}}\, dt \\ &\quad = \sup_{\Phi \in \mathscr{F}} \pm \int_0^1 J(t)\, K'(t)\, \Phi(t)\, dt \qquad \text{a.s.,} \end{aligned}$$

where

$$\mathscr{F} = \left\{\Phi(t) = \int_0^t \phi(s)\, ds : \Phi(0) = \Phi(1) = 0 \text{ and } \int_0^1 \phi^2(s)\, ds \leqslant 1\right\}.$$

Integration by parts implies that, whenever $\Phi(\cdot) \in \mathscr{F}$,

$$\begin{aligned}\int_0^1 J(t)\, K'(t)\, \Phi(t)\, dt &= [J(t)\, K(t)\, \Phi(t)]_0^1 - \int_0^1 \{J(t)\, \Phi(t)\}'\, K(t)\, dt \\ &= -\int_0^1 \left\{J'(t) \int_0^t \phi(s)\, ds + J(t)\, \phi(t)\right\} K(t)\, dt \\ &\triangleq -I(\Phi). \qquad (2.12)\end{aligned}$$

It now follows from (2.10), (2.11), and (2.12) that

$$\begin{aligned}&\limsup_{n\to\infty} \pm \left(\frac{n}{2\log\log n}\right)^{1/2} \left(n^{-1} \sum_{i=1}^n J\left(\frac{i}{n}\right) K(U_{i:n}) - \int_0^1 J(t)\, K(t)\, dt\right) \\ &\quad = \sup_{\Phi \in \mathscr{F}} (\mp I(\Phi)) \qquad \text{a.s.} \qquad (2.13)\end{aligned}$$

The arguments used by Shorack and Wellner (1986) will now be used to show that (2.13) remains true even if the assumptions on $K(\cdot)$ are relaxed. In view of (2.1) and (2.5), it follows that, for any given $\varepsilon > 0$, there exists a function $K_\varepsilon(\cdot)$ such that, uniformly over $t, h \in \mathscr{R}$, $|K_\varepsilon(t+h) - K_\varepsilon(t) - hK'_\varepsilon(t)| \leqslant c_\varepsilon\, |h|\, \eta_\varepsilon(h)$, where $c_\varepsilon > 0$ is a constant and $\eta_\varepsilon(h) \downarrow 0$ as $|h| \downarrow 0$,

$$E(|K(U) - K_\varepsilon(U)|) \leqslant \varepsilon, \qquad \operatorname{Var}(Y - Y_\varepsilon) \leqslant \varepsilon, \qquad \text{and} \qquad \operatorname{Var}(Z - Z_\varepsilon) \leqslant \varepsilon,$$

where

$$Z_\varepsilon = J(U)\, K_\varepsilon(U), \qquad Y_\varepsilon = -Z_\varepsilon + \mu_\varepsilon - \int_0^1 (I_{\{U \leqslant t\}} - t)\, J'(t)\, K_\varepsilon(t)\, dt,$$

and $\mu_\varepsilon = E(Z_\varepsilon)$. Define $\hat{K}_\varepsilon(\cdot) = K(\cdot) - K_\varepsilon(\cdot)$ and $\hat{\mu}_\varepsilon = \mu - \mu_\varepsilon$. Thus, (2.13) holds when $K(\cdot)$ is replaced by $K_\varepsilon(\cdot)$. By (1.3), we have

$$\begin{aligned}\sum_{i=1}^n J\left(\frac{i}{n}\right) \hat{K}_\varepsilon(U_{i:n}) &\overset{\text{a.s.}}{=} \sum_{i=1}^n J(U_i)\, \hat{K}_\varepsilon(U_i) \\ &\quad + \sum_{i=1}^n (J(D_n(U_{i:n})) - J(U_{i:n}))\, \hat{K}_\varepsilon(U_{i:n}), \qquad n \geqslant 1.\end{aligned}$$

Now, the Hartman–Wintner–Strassen LIL implies that

$$\limsup_{n\to\infty} \frac{\left|\sum_{i=1}^n J(U_i)\, \hat{K}_\varepsilon(U_i) - n\mu_\varepsilon\right|}{\sqrt{2n \log\log n}} = \sqrt{\operatorname{Var}(Z - Z_\varepsilon)} \qquad \text{a.s.} \qquad (2.14)$$

Note that

$$\begin{aligned}
\left|\sum_{i=1}^{n} (J(D_n(U_{i:n})) - J(U_{i:n}))\, \hat{K}_\varepsilon(U_{i:n})\right| &\leqslant \max_{1\leqslant i\leqslant n} |J(D_n(U_{i:n})) - J(U_{i:n})| \sum_{i=1}^{n} |K_\varepsilon(U_{i:n}))| \\
&\leqslant b\left(\max_{1\leqslant i\leqslant n} |D_n(U_{i:n}) - U_{i:n}|\right) \sum_{i=1}^{n} |K_\varepsilon(U_i))| \\
&\leqslant b \sup_{0\leqslant t\leqslant 1} |D_n(t) - t| \sum_{i=1}^{n} |K_\varepsilon(U_i))|. \qquad (2.15)
\end{aligned}$$

So, applying the Chung–Smirnov LIL for empirical processes (see Chung (1949) and Smirnov (1944)) and the Kolmogorov strong law of large numbers (SLLN), we have

$$\limsup_{n\to\infty} \frac{\left|\sum_{i=1}^{n} (J(D_n(U_{i:n})) - J(U_{i:n}))\, \hat{K}_\varepsilon(U_{i:n})\right|}{\sqrt{2n\log\log n}} \leqslant (b/2)\, E(|K_\varepsilon(U)|) \qquad \text{a.s.} \tag{2.16}$$

Hence, by (2.14) and (2.16), for every $\varepsilon > 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\left|\sum_{i=1}^{n} J\left(\frac{i}{n}\right) \hat{K}_\varepsilon(U_{i:n}) - n\hat{\mu}_\varepsilon\right|}{\sqrt{2n\log\log n}} &\leqslant \sqrt{\operatorname{Var}(Z - Z_\varepsilon)} + (b/2)\, E(|\hat{K}_\varepsilon(U)|) \\
&\leqslant \varepsilon^{1/2} + b\varepsilon \qquad \text{a.s.} \qquad (2.17)
\end{aligned}$$

Note that

$$\int_0^1 \left\{J'(t) \int_0^t \phi(s)\, ds + J(t)\, \phi(t)\right\} K_\varepsilon(t) \to I(\Phi) \qquad \text{as} \quad \varepsilon \downarrow 0.$$

It is now a straightforward matter to show that (2.13) holds in general. Moreover, it can be shown that $\sup_{\Phi\in\mathscr{F}} I(\Phi) = \sigma$, where σ is defined by (2.3). Hence, (ii) holds if and only if σ is finite which is equivalent to (i), and (i) implies (2.8).

We now prove that (i) and (iii) are equivalent and (2.9) holds under (i). Kolmogorov (1933) proved that, for every $\lambda > 0$,

$$\lim_{n\to\infty} P(\sqrt{n} \sup_{0\leqslant t\leqslant 1} |D_n(t) - t| \geqslant \lambda) = 2 \sum_{k=0}^{\infty} \exp(-2k^2\lambda^2).$$

This fact, combined with the same argument used above, implies that

$$\left\{ \frac{\sum_{i=1}^{n} (J(D_n(U_{i:n})) - J(U_{i:n}))\, K(U_{i:n})}{\sqrt{n}} ;\, n \geqslant 1 \right\} \text{ is bounded in probability.}$$

Therefore, (2.7) holds if and only if

$$\left\{ \frac{\sum_{i=1}^{n} J(U_i)\, K(U_i) - n\mu}{\sqrt{n}} ;\, n \geqslant 1 \right\} \text{ is bounded in probability,}$$

which is, in turn, equivalent to (2.5). So (i) and (iii) are equivalent. Note that, for any $\delta > 0$,

$$\limsup_{\varepsilon \downarrow 0}\ \limsup_{n \to \infty}\ P\left(\frac{\left| \sum_{i=1}^{n} J\left(\frac{i}{n}\right) (K(U_{i:n}) - K_\varepsilon(U_{i:n})) - n(\mu - \mu_\varepsilon) \right|}{\sqrt{n}} \geqslant \delta \right) = 0 \tag{2.18}$$

and, by the same argument used above, we have

$$\frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) K_\varepsilon(U_{i:n}) - n\mu_\varepsilon}{\sqrt{n}} \xrightarrow{d} N(0, \sigma_\varepsilon^2). \tag{2.19}$$

Combining (2.18) with (2.19) yields (2.9). ∎

3. APPLICATIONS AND EXAMPLES

In this section we show how our techniques can be applied to obtain necessary and sufficient conditions for either almost sure convergence or convergence in distribution of some specific L-statistics and U-statistics.

As an application of our approach, we state the following interesting result. Its proof follows easily from the method used in (2.15) and the Chung–Smirnov LIL for empirical processes.

THEOREM 3.1. *Let* $\{X, X_n; n \geqslant 1\}$ *and* $\{U, U_n; n \geqslant 1\}$ *be two sequences of i.i.d. random variables such that* U *is uniformly distributed on* $(0, 1)$. *Let* $H(\cdot)$ *be a real Borel-measurable function defined on* $\mathscr{R}$ *and let* $J(\cdot)$ *be a Lipschitz function of order one defined on* $[0, 1]$.

(i) *If there exists a real sequence* $\{b_n; n \geqslant 1\}$ *such that*

$$b_n \to \infty \qquad \text{as} \quad n \to \infty \tag{3.1}$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} |H(X_i)|}{n\sqrt{b_n}} = 0 \qquad \text{a.s.}, \tag{3.2}$$

then, for any real sequence $\{a_n; n \geqslant 1\}$,

$$\limsup_{n\to\infty}\ (\liminf_{n\to\infty}) \frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(X_{i:n}) - a_n}{\sqrt{2n(\log\log n)\, b_n}}$$

$$= \limsup_{n\to\infty}\ (\liminf_{n\to\infty}) \frac{\sum_{i=1}^{n} J(U_i)\, H(G(U_i)) - a_n}{\sqrt{2n(\log\log n)\, b_n}} \qquad \text{a.s.} \tag{3.3}$$

(ii) *If there exist sequences* $\{b_n; n \geqslant 1\}$ *and* $\{a_n; n \geqslant 1\}$ *such that* (3.2) *holds and*

$$\frac{\sum_{i=1}^{n} J(U_i)\, H(G(U_i)) - a_n}{\sqrt{nb_n}} \xrightarrow{\mathrm{d}} \Lambda(x), \tag{3.4}$$

where $\Lambda(x)$ *is a distribution function, then*

$$\frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(X_{i:n}) - a_n}{\sqrt{nb_n}} \xrightarrow{\mathrm{d}} \Lambda(x).$$

As a corollary of Theorem 3.1, we have the following result; its proof is left to the reader.

COROLLARY 3.2. *Let* $H(\cdot)$ *and* $J(\cdot)$ *be as in Theorem* 3.1. *Let*

$$Z = J(U)\, H(G(U)).$$

If

$$E(|H(X)|) < \infty, \qquad E(Z^2) = \infty,$$

and Z is in the domain of attraction of the normal distribution, then there exist sequences $\{a_n; n \geqslant 1\}$ and $\{b_n > 0; n \geqslant 1\}$ such that

$$\frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) H(X_{i:n}) - a_n}{b_n} \xrightarrow{d} N(0, 1), \tag{3.5}$$

where b_n may be chosen as

$$b_n = \sup\left\{c : c^{-2} E(Z^2 I_{\{|Z| < c\}}) \geqslant \frac{1}{n}\right\}$$

and a_n may be taken as

$$a_n = \frac{n}{b_n} E(ZI\{|Z| < b_n\}).$$

Remark. It is interesting to note that, under the conditions of Corollary 3.2, the sequences $\{a_n; n \geqslant 1\}$ and $\{b_n; n \geqslant 1\}$ are determined by Z, whereas, by contrast, the limiting behaviour in Theorem 2.1 depends on σ, which is a function of Y, where Y is defined by (2.2).

Let $\{X, X_n; n \geqslant 1\}$ be a sequence of i.i.d. random variables. Gini's mean difference,

$$\frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} |X_i - X_j|,$$

is a well-known U-statistic for unbiased estimation of the dispersion parameter

$$\theta = E(|X_1 - X_2|);$$

see, e.g., Serfling (1980, p. 263) or Shorack and Wellner (1986, p. 676). It may be represented as an L-statistic as follows:

$$\frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} |X_i - X_j| = \frac{1}{n} \sum_{i=1}^{n} \left(4 \cdot \frac{i-1}{n-1} - 2\right) X_{i:n}. \tag{3.6}$$

Using Theorem 2.1, we can establish the following analogues of classical SLLN, LIL, and CLT for Gini's mean difference.

THEOREM 3.3. (i)

$$\limsup_{n \to \infty} \frac{2}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} |X_i - X_j| < \infty \qquad a.s. \tag{3.7}$$

if and only if

$$E(|X|) < \infty. \tag{3.8}$$

In either case,

$$\lim_{n\to\infty} \frac{2}{n(n-1)} \sum_{1\leqslant i<j\leqslant n} |X_i - X_j| = E(|X_1 - X_2|) \qquad a.s. \tag{3.9}$$

(ii) *For some θ,*

$$\limsup_{n\to\infty} \sqrt{\frac{n}{2\log\log n}} \left| \frac{2}{n(n-1)} \sum_{1\leqslant i<j\leqslant n} |X_i - X_j| - \theta \right| < \infty \qquad a.s. \tag{3.10}$$

if and only if

$$E(X^2) < \infty. \tag{3.11}$$

In either case, $\theta = E(|X_1 - X_2|)$ and

$$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \sqrt{\frac{n}{2\log\log n}} \left(\frac{2}{n(n-1)} \sum_{1\leqslant i<j\leqslant n} |X_i - X_j| - \theta \right) = (\overset{+}{-})\, \sigma \qquad a.s., \tag{3.12}$$

where

$$\sigma^2 = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |y - x|\, dF(x) \right]^2 dF(y) - \theta^2. \tag{3.13}$$

(iii) *For some θ,*

$$\left\{ \sqrt{n} \left(\frac{2}{n(n-1)} \sum_{1\leqslant i<j\leqslant n} |X_i - X_j| - \theta \right);\ n \geqslant 1 \right\} \text{ is bounded in probability} \tag{3.14}$$

if and only if (3.11) *holds. In either case, $\theta = E(|X_1 - X_2|)$ and*

$$\sqrt{n} \left(\frac{2}{n(n-1)} \sum_{1\leqslant i<j\leqslant n} |X_i - X_j| - \theta \right) \xrightarrow{d} N(0, \sigma^2), \tag{3.15}$$

where σ^2 is defined as in (3.13).

Remark. Shorack and Wellner (1986, p. 677) observed that (3.15) holds under the more stringent assumption that $E(|X|^{2+\delta}) < \infty$ for some $\delta > 0$. The "if part" of Theorem 3.3(i) is covered in Helmers *et al.* (1988).

Proof of Theorem 3.3. Take $J(t)=4t-2$, $0\leqslant t\leqslant 1$, and $H(x)=x$, $x\in\mathscr{R}$. Then (3.8) implies that

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} J(U_i)\,G(U_i)}{n}=\int_0^1 (4t-2)\,G(t)\,dt$$

$$=\int_0^1\int_0^1 |G(t)-G(s)|\,ds\,dt$$

$$=\theta \qquad \text{a.s.}$$

and

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}|X_i|}{n\sqrt{b_n}}=0 \qquad \text{a.s.,} \tag{3.16}$$

where $b_n=n/\log\log n\to\infty$ as $n\to\infty$. By Theorem 3.1, we have that

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} J\left(\frac{i}{n}\right) X_{i:n}}{n}=\theta \qquad \text{a.s.} \tag{3.17}$$

Note that (3.8) also implies that

$$\left|\frac{\sum_{i=1}^{n}\left(J\left(\frac{i-1}{n-1}\right)-J\left(\frac{i}{n}\right)\right) X_{i:n}}{n}\right|$$

$$\leqslant\frac{4}{n^2}\sum_{i=1}^{n}|X_i|\to 0 \qquad \text{a.s.} \qquad \text{as} \quad n\to\infty. \tag{3.18}$$

So (3.6), (3.17), and (3.18) together imply that (3.9) holds.

We now prove that (3.7) implies (3.8). Obviously, (3.7) implies that

$$\limsup_{n\to\infty}\frac{2}{2n(2n-1)}\sum_{i=1}^{n}|X_{2i-1}-X_{2i}|<\infty \qquad \text{a.s.}$$

Since $\{|X_{2n-1}-X_{2n}|;n\geqslant 1\}$ is a sequence of i.i.d. random variables, it follows that $E(|X_1-X_2|^{1/2})<\infty$, which is equivalent to

$$E(|X_1|^{1/2})<\infty. \tag{3.19}$$

Thus (3.18) holds. Combining (3.6) and (3.7) with (3.18), we have that

$$\limsup_{n\to\infty} \frac{\left|\sum_{i=1}^{n} J\left(\frac{i}{n}\right) X_{i:n}\right|}{n} < \infty \qquad \text{a.s.} \tag{3.20}$$

Note that (3.19) is equivalent to

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} |X_i|}{n\sqrt{n^2}} = 0 \qquad \text{a.s.}$$

By Theorem 3.1 and (3.20), then,

$$\begin{aligned}\limsup_{n\to\infty} \frac{\left|\sum_{i=1}^{n} J(U_i)\, G(U_i)\right|}{\sqrt{2n(\log\log n)\, n^2}} &= \limsup_{n\to\infty} \frac{\left|\sum_{i=1}^{n} J\left(\frac{i}{n}\right) X_{i:n}\right|}{\sqrt{2n(\log\log n)\, n^2}} \\ &= 0 \qquad \text{a.s.,}\end{aligned}$$

which implies that, for all $\varepsilon > 0$,

$$E(|J(U)\, G(U)|^{2/3-\varepsilon}) = \int_0^1 |(4t-2)\, G(t)|^{2/3-\varepsilon}\, dt < \infty. \tag{3.21}$$

It is easy to check that (3.21) is equivalent to

$$\begin{aligned}\int_0^1 |G(t)|^{2/3-\varepsilon}\, dt &= \int_{-\infty}^{\infty} |x|^{2/3-\varepsilon}\, dF(x) \\ &= E(|X|^{2/3-\varepsilon}) < \infty.\end{aligned}$$

In particular, we have that

$$E(|X|^{7/12}\, (L_2 |X|)^{7/24}) < \infty,$$

which is equivalent to (cf., e.g., Feller, 1946)

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} |X_i|}{n\sqrt{n^{10/7}/\log\log n}} = 0 \qquad \text{a.s.}$$

Using Theorem 3.1 and noting (3.20) again, we have that

$$\lim_{n\to\infty} \frac{\left|\sum_{i=1}^{n} J(U_i)\, G(U_i)\right|}{\sqrt{2n(\log\log n)\, n^{10/7}/\log\log n}} = \lim_{n\to\infty} \frac{\left|\sum_{i=1}^{n} J(U_i)\, G(U_i)\right|}{\sqrt{2n^{17/7}}} = 0 \qquad \text{a.s.,}$$

which is equivalent to

$$E(|J(U)\, G(U)|^{14/17}) < \infty$$

and hence

$$E(|X|^{14/17}) < \infty.$$

Consequently, $E(|X|^{2/3}\,(L_2|X|)^{1/3}) < \infty$. Repeating the above argument yields

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} |X_i|}{n\sqrt{n/\log\log n}} = 0 \qquad \text{a.s.}$$

Using Theorem 3.1 and noting (3.20) again, we have that

$$\lim_{n\to\infty} \frac{\left|\sum_{i=1}^{n} J(U_i)\, G(U_i)\right|}{\sqrt{2n(\log\log n)\, n/\log\log n}} = \lim_{n\to\infty} \frac{\left|\sum_{i=1}^{n} J(U_i)\, G(U_i)\right|}{\sqrt{2n}} < \infty \qquad \text{a.s.}$$

which is equivalent to

$$E(|J(U)\, G(U)|) < \infty$$

and hence (3.8) follows. The proof of (i) is therefore complete.

A similar approach can be used to show that (3.10) and (3.14) each imply that (3.11) holds and $\theta = E(|X_1 - X_2|)$. An application of Theorem 2.1 will then yield (ii) and (iii). ∎

Another important application of our approach and results relates to the expectation of order statistics. Let $\theta_k = E(X_{1:k})$, where $X_{1:k}$ is the minimum value (smallest order statistic) in a sample of size k. Then an estimate of θ_k based on a sample of size n is

$$T_n = \binom{n}{k}^{-1} \sum_{(n,k)} \min\{X_{i_1}, ..., X_{i_k}\},$$

where the sum $\sum_{(n,k)}$ is taken over all subsets $1 \leqslant i_1 < \cdots < i_k \leqslant n$ of $\{1, 2, ..., n\}$.

As noted by Lee (1990, p. 65), T_n can be expressed as an L-statistic as follows:

$$\begin{aligned}
T_n &= \binom{n}{k}^{-1} \sum_{i=1}^{n-k} \binom{n-i}{k-1} X_{i:n} \\
&= \frac{1}{\binom{n}{k}(k-1)!} \sum_{i=1}^{n} (n-i)(n-i-1)\cdots(n-i-k+2)\, X_{i:n} \\
&= \frac{1}{\binom{n}{k}(k-1)!} \left(\sum_{i=1}^{n} (n-i)^{k-1} X_{i:n} + c_1 \sum_{i=1}^{n} (n-i)^{k-2} X_{i:n} + \cdots \right. \\
&\quad \left. + c_{k-2} \sum_{i=1}^{n} (n-i)\, X_{i:n} \right),
\end{aligned}$$

where $c_1, c_2, ..., c_{k-2}$ are constants depending on k only. Note that

$$\binom{n}{k} \sim \frac{n^k}{k!} \qquad \text{as} \quad n \to \infty$$

and, for $i = 1, 2, ..., n$,

$$\frac{(n-i)^{k-j}}{(n^k/k!)(k-1)!} = \begin{cases} \dfrac{k}{n}\left(1 - \dfrac{i}{n}\right)^{k-1}, & j = 1, \\ O\left(\dfrac{1}{n^2}\right), & j \geqslant 2. \end{cases}$$

Let $J(t) = (1-t)^{k-1}$, $0 \leqslant t \leqslant 1$, and $H(x) = x$, $x \in \mathscr{R}$. Using Theorems 3.1 and 2.1, we have the following results.

THEOREM 3.4. *Let* $\{X, X_n; n \geqslant 1\}$ *be a sequence of i.i.d. random variables.*

(i) *If there exists* $p > \frac{2}{3}$ *such that*

$$E(|X|^p) < \infty, \tag{3.22}$$

then, for each integer $k \geqslant 1$, *the following three statements are equivalent*:

$$E(|\min\{X_1, X_2, ..., X_k\}|) < \infty$$

$$\limsup_{n\to\infty} \frac{k}{n} \left| \sum_{i=1}^{n} \left(1-\frac{i}{n}\right)^{k-1} X_{i:n} \right| < \infty \qquad a.s.$$

$$\limsup_{n\to\infty} |T_n| < \infty \qquad a.s.$$

In each case,

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \frac{k}{n} \sum_{i=1}^{n} \left(1-\frac{i}{n}\right)^{k-1} X_{i:n} = \theta_k \qquad a.s.$$

(ii) *Define* $\theta_k = E(X_{1:k})$. *If* $E(|X|) < \infty$, *then the following five statements are equivalent*:

$$E((1-U)^{2k-2}(G(U))^2) = \int_{-\infty}^{\infty} (1-F(x))^{2k-2}\, x^2\, dF(x) < \infty$$

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n}\right)^{1/2} |T_n - \theta_k| < \infty \qquad a.s.$$

$$\limsup_{n\to\infty} \frac{k}{\sqrt{2n\log\log n}} \left| \sum_{i=1}^{n} \left(1-\frac{i}{n}\right)^{k-1} X_{i:n} - \frac{\theta_k}{k} \right| < \infty \qquad a.s.$$

$$\{n^{1/2}(T_n - \theta_k);\, n \geqslant 1\} \text{ is bounded in probability}$$

$$\left[\frac{k}{\sqrt{n}} \left(\sum_{i=1}^{n} \left(1-\frac{i}{n}\right)^{k-1} X_{i:n} - \frac{\theta_k}{k} \right);\, n \geqslant 1 \right\} \text{ is bounded in probability.}$$

In each case,

$$\begin{aligned}
&\limsup_{n\to\infty} \left(\liminf_{n\to\infty}\right) \left(\frac{n}{2\log\log n}\right)^{1/2} (T_n - \theta_k) \\
&\quad = \limsup_{n\to\infty} \left(\liminf_{n\to\infty}\right) \frac{k}{\sqrt{2n\log\log n}} \left(\sum_{i=1}^{n} \left(1-\frac{i}{n}\right)^{k-1} X_{i:n} - \frac{\theta_k}{k} \right) \\
&\quad = \left(\frac{+}{-}\right) k\sigma_k \qquad a.s.,
\end{aligned}$$

$$n^{1/2}(T_n - \theta_k) \xrightarrow{d} N(0, k^2\sigma_k^2),$$

and

$$\frac{k}{\sqrt{n}} \left(\sum_{i=1}^{n} \left(1-\frac{i}{n}\right)^{k-1} X_{i:n} - \frac{\theta_k}{k} \right) \xrightarrow{d} N(0, k^2\sigma_k^2),$$

where $\sigma_k^2 = \mathrm{Var}(Y_k)$ *and*

$$Y_k = -(1-U)^{k-1} G(U) + \frac{\theta_k}{k} - \int_0^1 (I_{\{U \leqslant t\}} - t)(1-k)(1-t)^{k-2} G(t)\, dt$$

$$= \int_0^1 (I_{\{U \leqslant t\}} - t)(1-t)^{k-1} G(t).$$

Remarks. (i) If we replace

$$\min\{X_{i_1}, ..., X_{i_k}\}$$

with

$$\max\{X_{i_1}, ..., X_{i_k}\},$$

then, similarly,

$$\frac{1}{\binom{n}{k}} \sum_{(n,k)} \max\{X_{i_1}, ..., X_{i_k}\}$$

$$= \frac{1}{\binom{n}{k}(k-1)!} \left(\sum_{i=1}^{n} i^{k-1} X_{i:n} + d_1 \sum_{i=1}^{n} i^{k-2} X_{i:n} + \cdots + d_{k-1} \sum_{i=1}^{n} X_{i:n} \right),$$

where $d_1, d_2, ..., d_{k-1}$ are constants depending on k only. Thus, we can derive an analogue of Theorem 3.4 for the U-statistic

$$\frac{1}{\binom{n}{k}} \sum_{(n,k)} \max\{X_{i_1}, ..., X_{i_k}\}, \qquad n \geqslant k,$$

and the L-statistic

$$\frac{k}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{k-1} X_{i:n}, \qquad n \geqslant 1.$$

(ii) It follows from Theorem 3.1 that

$$\limsup_{n \to \infty} \frac{k}{n} \left| \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{k-1} X_{i:n} \right| < \infty \qquad \text{a.s.} \tag{3.23}$$

if and only if

$$E(|\max\{X_1, ..., X_k\}|) < \infty. \tag{3.24}$$

provided only that (3.22) holds with $p > 2/3$. In either case,

$$\lim_{n \to \infty} \frac{k}{n^k} \sum_{i=1}^{n} i^{k-1} X_{i:n} = E(\max\{X_1, ..., X_k\}) \qquad \text{a.s.} \tag{3.25}$$

This generalizes Theorem 1.1 of Gilat and Hill (1992), who dealt only with the case $p = 1$.

EXAMPLE. Let $\{X, X_n; n \geqslant 1\}$ be a sequence of i.i.d. random variables with common density function

$$f(x) = \frac{5}{6(-x)^{11/6}} I_{\{x \leqslant -1\}}.$$

It is easy to check that

$$E(|X|) = \infty.$$

Thus, Theorem 1.1 of Gilat and Hill (1992) cannot be applied to determine

$$\lim_{n \to \infty} \frac{k}{n^k} \sum_{i=1}^{n} i^{k-1} X_{i:n}.$$

However,

$$E(|X|^{3/4}) < \infty$$

and

$$E(|\max\{X_1, ..., X_k\}|) \leqslant E(|\max\{X_1, X_2\}|) < \infty \qquad \text{for all} \quad k \geqslant 2.$$

Consequently, (3.23), (3.24), and (3.25) imply that, for every $k \geqslant 2$,

$$\begin{aligned} \lim_{n \to \infty} \frac{k}{n^k} \sum_{i=1}^{n} i^{k-1} X_{i:n} &= E(\max\{X_1, ..., X_k\}) \\ &= \int_{-\infty}^{-1} \frac{5kx}{6(-x)^{(5k/6)+1}} \, dx \\ &= -\frac{5k}{5k-6} \qquad \text{a.s.} \end{aligned}$$

4. THE LAW OF THE LOGARITHM FOR L-STATISTICS

Let $\{X, X_n; n \geqslant 1\}$ be a sequence of i.i.d. random variables with

$$E(X) = 0 \qquad \text{and} \qquad E(X^2) = 1.$$

It is natural to try to extend our results to sums of the form $\sum_{i=1}^{n} c_{i,n} X_{i:n}$, $n \geqslant 1$, where $\{c_{i,n}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ is a uniformly bounded triangular array of real numbers. Li *et al.* (1995) showed that, for almost all such arrays, the weighted sums

$$\sum_{i=1}^{n} c_{i,n} X_i, \qquad n \geqslant 1,$$

obey what they called a Law of the Logarithm, i.e.,

$$0 < \limsup_{n \to \infty} \frac{\left| \sum_{i=1}^{n} c_{i,n} X_i \right|}{\sqrt{2n \log n}} < \infty \qquad \text{a.s.} \tag{4.1}$$

It follows that

$$\limsup_{n \to \infty} \frac{\left| \sum_{i=1}^{n} c_{i,n} X_i \right|}{\sqrt{2n \log \log n}} = \infty \qquad \text{a.s.}$$

In this section, we give a version of (4.1) for L-statistics. Before we state our result, we introduce some more notation. Let $\mathscr{I} = \{(i, n); 1 \leqslant i \leqslant n, n \geqslant 1\}$. For a given probability measure $\mathbf{v}$ on the Borel σ-field of $\mathscr{R}$, let $\mathscr{S} = \mathscr{S}(\mathbf{v})$ denote the support of $\mathbf{v}$. We will consider only those probability measures for which $\mathscr{S}$ is bounded. Let $P' = \mathbf{v}^{\mathscr{I}}$ be the product probability measure on the Borel σ-field of $\mathscr{S}^{\mathscr{I}}$.

THEOREM 4.1 (A Law of the Logarithm for L-Statistics). *Let $\{X, X_n; n \geqslant 1\}$ be a sequence of i.i.d. random variables and $H(\cdot)$ a real valued measurable function defined on $\mathscr{R}$ such that $E(H^2(X)) < \infty$. Then, for any given probability measure $\mathbf{v}$ on the Borel σ-algebra of $\mathscr{R}$ with bounded support $\mathscr{S} = \mathscr{S}(\mathbf{v})$, there exists a set $\Omega_0 \subset \mathscr{S}^{\mathscr{I}}$ such that*

$$P'(\Omega_0) = 1$$

and, for any $\{c_{i,n}; (i,n) \in \mathscr{I}\} \in \Omega_0$,

$$\limsup_{n\to\infty}\ (\liminf_{n\to\infty})\ \frac{\sum_{i=1}^{n} c_{i,n} H(X_{i:n}) - n\mu}{\sqrt{2n \log n}} = (\overset{+}{-})\, \sigma \qquad a.s.,$$

where

$$\mu = E(H(X)) \int_{\mathscr{S}} t\mathbf{v}(dt)$$

and

$$\sigma^2 = E(H^2(X)) \left(\int_{\mathscr{S}} \left(t - \int_{\mathscr{S}} s\mathbf{v}(ds) \right)^2 \mathbf{v}(dt) \right).$$

Proof. This result can be deduced using the same argument as that used in Theorem 2.4 of Li *et al.* (1995). ∎

Remark. If $E(H(X)) = 0$ and $0 < E(H^2(X)) < \infty$, then for almost all choices $\{c_{i,n}; (i,n) \in \mathscr{I}\}$ of triangular arrays of real numbers with $\sup_{(i,n)\in\mathscr{I}} |c_{i,n}| \leqslant M < \infty$ for some constant $M > 0$, the L-statistics

$$\sum_{i=1}^{n} c_{i,n} H(X_{i,n}), \qquad n \geqslant 1,$$

obey the Law of the Logarithm, i.e.,

$$0 < \limsup_{n\to\infty} \frac{\left| \sum_{i=1}^{n} c_{i,n} H(X_{i:n}) \right|}{\sqrt{2n \log n}} < \infty \qquad \text{a.s.}$$

It follows that

$$\limsup_{n\to\infty} \frac{\left| \sum_{i=1}^{n} c_{i,n} H(X_{i:n}) \right|}{\sqrt{2n \log \log n}} = \infty \qquad \text{a.s.}$$

Thus, for almost all choices $\{c_{i,n}; (i,n) \in \mathscr{I}\}$ of triangular arrays of real numbers with $\sup_{(i,n)\in\mathscr{I}} |c_{i,n}| \leqslant M < \infty$, the LIL for L-statistics

$$\sum_{i=1}^{n} c_{i,n} H(X_{i,n}), \qquad n \geqslant 1$$

fails.

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