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Note

# The distribution of the ratio $X/Y$ for all centred elliptically symmetric distributions

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**Abstract**

This note describes the relationship between ratios of random variables from centred elliptically symmetric distributions and the Cauchy distribution, with particular reference to a recent article in this journal by Nadarajah [On the ratio $X/Y$ from some elliptically symmetric distributions, J. Multivariate Anal. 97 (2006) 342–358].


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Theorem 1 (when corrected), Corollaries 1 and 5 of Nadarajah [4] all give the same result for the distribution of the ratio $Z = X/Y$ when $E = (X, Y)$ follows three particular bivariate elliptically symmetric distributions centred at zero. This is no coincidence. What is not acknowledged in that paper is that the result is a general one for any bivariate elliptically symmetric distribution centred at zero, nor that it has the simple form

$$f(z) = \frac{\sqrt{1-\rho^2}}{\pi(1+z^2-2\rho z)} \tag{1}$$

(where $\rho$ is the correlation) nor that this in turn is a general (relocated and rescaled) Cauchy density $\sigma^{-1} f_C(\sigma^{-1}(z-\mu))$ where $\mu = \rho$, $\sigma = \sqrt{1-\rho^2}$ and $f_C(z) = \{\pi(1+z^2)\}^{-1}$. (Figure 1(a) of [4] therefore plots Cauchy distributions.)

The link between ratios of (centred) elliptic random variables and the Cauchy distribution is well known. In the (centred) spherically symmetric case, it is particularly simple to explain. Let

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$S = (U, V)$ follow such a distribution. Matching [4] by reversing the usual roles of $U$ and $V$ for clarity and using polar coordinates, $U = R\sin\Theta$ and $V = R\cos\Theta$ where $\Theta$ follows a uniform distribution so that $T = U/V = \tan\Theta$ and hence $T \sim f_C$. See [2] for many further distributional relationships arising readily from simple trigonometric considerations.

In the (centred) elliptically symmetric case, make the usual transformation $E = \Sigma^{1/2}S$ where $\Sigma$ is the bivariate correlation matrix. Write $p = \sqrt{1+\rho} + \sqrt{1-\rho}$ and $q = \sqrt{1+\rho} - \sqrt{1-\rho}$. Then a little basic algebra shows that $Z = (pT + q)/(qT + p)$ and further standard manipulations show that this function of a standard Cauchy random variable follows the general Cauchy distribution (1). The result that $Z$ must follow a general Cauchy distribution (but without explicitly specifying the constants involved) is given by Arnold and Brockett [1, Theorem 2]. Also, given that $T$ is standard Cauchy, result (1) follows immediately (with constants specified) from the result on the distribution of Mobius transformations of Cauchy random variables given at (3) of [3].

The correction to Theorem 1 of [4] mentioned at the start of this note is to add the requirement that the location parameters $\alpha = \beta = 0$ (alternatively, note that $Z$ there should equal $(X - \alpha)/(Y - \beta)$). For nonzero $\alpha$ and/or $\beta$, however, the marginal distribution of $Z$, which depends on the specific elliptical distribution in question, seems to eschew elegance and leads to the rebarbative mathematics that fills much of Nadarajah's paper.

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