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# Local Power Properties of Kernel Based Goodness of Fit Tests

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If $(X_i,\ i \in \mathbb{Z})$ is a strictly stationary process with marginal density function $f$, we are interested in testing the hypothesis $H_0$: $\{f = f_0\}$, where $f_0$ is given. We consider different test statistics based on integrated quadratic forms measuring the proximity between $f_n$, a kernel estimator of $f$, and $f_0$, or between $f_n$ and its expected value computed under $H_0$. We study the asymptotic local power properties of the testing procedures under local alternatives. This study generalizes to the multidimensional case in a context of dependence the corresponding one made by P. J. Bickel and M. Rosenblatt in 1973 (*Ann. Statist.* **1**, 1071–1095). © 2001 Academic Press

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## 1. INTRODUCTION

The goodness of fit tests of $H_0 : \{f = f_0\}$ against the nonparametric alternative hypothesis $H_0^c : \{f \neq f_0\}$, where $f_0$ is a given density function and $f$ the common marginal density function of the observations, are usually performed under the assumption of independent and identically distributed observations. However, these traditional testing procedures employed on stationary dependent observations lead to invalid critical values, with for instance nonconservative testing procedures for stationary processes satisfying a positive dependence condition (cf. Moore [16] and Gleser and Moore [11]). The aim of this paper is to extend the standard approach followed by Bickel and Rosenblatt [1] to the multidimensional dependent case.



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Bickel and Rosenblatt [1] introduce tests statistics based on the $L_2$ distance between the kernel density estimator $f_n$ and either $E_0 f_n$ or $f_0$, where $E_0$ is the mathematical expectation under the null hypothesis. These statistics are

$$I_n^1(\pi) = \int_{\mathbb{R}^d} \{f_n(x) - E_0 f_n(x)\}^2 \, \pi(x) \, dx \tag{1}$$

and

$$I_n^2(\pi) = \int_{\mathbb{R}^d} \{f_n(x) - f_0(x)\}^2 \, \pi(x) \, dx, \tag{2}$$

where $f_n$ is a kernel estimator of $f$ based on the observed $d$-dimensional random variables $X_1, ..., X_n$, defined by (cf. Rosenblatt [19] and Parzen [17])

$$f_n(x) = \frac{1}{nh_n^d} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right),$$

where $K$ is a kernel, $h_n$ a bandwidth, and $\pi$ a weight function (see Rosenblatt [21], Eubank and LaRiccia [6], Fan [7], and Tenreiro [23] for goodness of fit tests based on the kernel density estimator and Csörgö [4], Fan [8], Ghosh and Ruymgaart [10], and Justel *et al.* [13] for multidimensional goodness of fit tests based on the empirical characteristic function and on the empirical distribution function).

From the work of Hall [12], we know that the asymptotic distribution of the statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ can easily be derived in a multidimensional i.i.d. framework by using degenerate U-statistics limit theorems. This approach was used by Fan [7] to generalize the results of Bickel and Rosenblatt [1] derived for $d = 1$.

In order to extend the previous results to the multidimensional dependent case and study local power properties, we consider a sequence of local alternatives, i.e., a sequence of $d$-dimensional strictly stationary processes $(X_{in},\ i \in \mathbb{Z})$ whose marginal distribution has a density function $g_n$ with respect to Lebesgue measure on $\mathbb{R}^d$ of the form $g_n = f_0 + a_n \gamma + o(a_n)\, \gamma_n$, where $\gamma$ is a bounded and integrable function providing the direction of the alternative, $(\gamma_n)$ a sequence of uniformly bounded and integrable functions and $(a_n)$ is a sequence of positive real numbers tending to zero. This sequence of processes satisfies some additional assumptions which are introduced and discussed in Section 2.

In Section 3 we establish the asymptotic normality of the two statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ under the sequence of local alternatives (and under the null hypothesis $H_0$ by taking $a_n = 0$) and we give asymptotic expansions of these statistics. These results are based on a central limit theorem for degenerate U-statistics corresponding to a geometrically $\beta$-mixing process (see Takahata and Yoshihara [22] and Tenreiro [24]).

To develop goodness of fit tests of the hypothesis $H_0$ based on $I_n^1(\pi)$ and $I_n^2(\pi)$, we consider, in Section 4, tests derived from these statistics by correcting them for their asymptotic bias. For instance, based on $I_n^1(\pi)$ (similarly to $I_n^2(\pi)$) we consider the test statistics

$$T_n^{1,1}(\pi) = nh_n^{d/2} \left\{ I_n^1(\pi) - \frac{1}{nh_n^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K^2(u)\, f_0(x)\, \pi(x+uh_n)\, dx\, du \right\}$$

and

$$T_n^{1,2}(\pi) = nh_n^{d/2} \left\{ I_n^1(\pi) - \frac{1}{nh_n^d} \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^d} K^2(u)\, \pi(X_i+uh_n)\, du \right\}.$$

Using the results of Section 3 we give asymptotic expansions of the different test statistics under local alternative and we derive associated asymptotically consistent critical regions.

Section 5 is the central part of the paper where we characterize the rates of convergence of the local alternatives such that the asymptotic probability of the critical region under local alternatives is either the same as under $H_0$ or the same as under a fixed alternative.

The aim of Section 6 is to study the local power properties and the local unbiasedness properties of the testing procedures under local alternatives and to discuss the choice of the kernel.

All proofs are gathered in appendices.

## 2. LOCAL ALTERNATIVES

To define local alternatives we consider a sequence of $d$-dimensional strictly stationary processes $(X_{in},\ i \in \mathbb{Z})$ (in the sections below we suppress the index $n$ for notational convenience).

*Assumptions on the Process* $(P)$. Denoting by $\beta_n(\cdot)$ the $\beta$-mixing coefficient of the $d$-dimensional strictly stationary process $(X_{in},\ i \in \mathbb{Z})$ defined by (cf. Volkonskiĭ and Rozanov [25])

$$\beta_n(i) = E[\sup_{A \in F_{in}^{+\infty}} |P(A \mid F_{-\infty}^{0n}) - P(A)|],$$

where $F_{in}^{+\infty}$ (resp. $F_{-\infty}^{0n}$) is the $\sigma$-algebra generated by $X_{jn}$, $j \geqslant i$ (resp. $X_{jn}$, $j \leqslant 0$), we suppose that there exist $C > 0$ and $\rho \in ]0, 1[$ such that

$$\sup_{n \in \mathbb{N}} \beta_n(i) \leqslant C\rho^i, \qquad \forall i \in \mathbb{N}.$$

In this case we say that the sequence of processes $(X_{in},\ i \in \mathbb{Z})$ is geometrically $\beta$-mixing.

Moreover, we assume that $X_{in}$ and $(X_{in}, X_{0n})$, $i \geqslant 1$, have absolutely continuous distributions with pdf $g_n$ and $g_{in}$ respectively, such that

$$\sup_{n,\, i \in \mathbb{N}} \sup_{\substack{x \in \{u \in \mathbb{R}^d \mid g_n(u) > 0\} \\ y \in \mathbb{R}}} f_{in}(y \mid x) < +\infty,$$

where $f_{in}(y \mid x) = \frac{g_{in}(y, x)}{g_n(x)}$ is the conditional density function of $X_{in}$ given $X_{0n} = x$, and for $x \in \mathbb{R}^d$,

$$g_n(x) = f_0(x) + a_n \gamma(x) + o(a_n)\, \gamma_n(x), \tag{3}$$

where $f_0$ is a bounded pdf on $\mathbb{R}^d$, $(a_n)$ is a sequence of positive real numbers tending to zero and

$$\sup_{x \in \mathbb{R}^d} |\gamma(x)| < \infty, \qquad \sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} |\gamma_n(x)| < \infty,$$

$$\int_{\mathbb{R}^d} |\gamma(x)|\, dx < \infty, \qquad \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} |\gamma_n(x)|\, dx < \infty.$$

Condition (3) gives the interpretation in terms of local alternatives since $\lim_{n \to +\infty} g_n(x) = f_0(x)$, for $x \in \mathbb{R}^d$. The associated rate of convergence is $a_n$.

At this stage it is interesting to discuss it and in particular to give its interpretation in terms of random variables. The following property can be easily derived (see Bradley [3] pp. 173–174 for the geometrical convergence of mixing coefficients).

PROPOSITION 2.1. *Let us consider a process* $(X_i,\ i \in \mathbb{Z})$ *satisfying the set of assumptions* (P) *with* $a_n = 0$, $(\delta_n)$ *a sequence of positive real numbers tending to zero and* $(Z_i,\ i \in \mathbb{Z})$ *a strictly stationary d-dimensional geometrically* $\beta$*-mixing process independent of* $(X_i,\ i \in \mathbb{Z})$. *Then the sequence of processes defined by* $X_{in} = X_i + \delta_n Z_i$, $i \in \mathbb{Z}$, *satisfies assumptions* (P).

The set of assumptions (P) with $a_n = 0$ is satisfied for i.i.d. sequences if the common marginal density function is bounded, for some stationary gaussian processes for some autoregressive moving average (ARMA) processes based on gaussian, exponential or uniform i.i.d. sequences and for some stationary Markov processes such that $\sup_{x \in \mathbb{R}^d} f_1(\cdot \mid x)$ is bounded (see Volkonskii and Rozanov [26], Rosenblatt [20], Pham and Tran [18], Bradley [3] and Mokkadem [15]). Conversely, an ARCH process defined by $X_i = \sqrt{c_0 + c_1 X_{i-i}^2}\, \varepsilon_i$, $i \in \mathbb{Z}$, where $(\varepsilon_i)$ is a standard gaussian white noise, $c_0 \geqslant 0$ and $0 < c_1 < 1$, does not satisfy the assumption on the conditional densities when $c_0 = 0$ since the conditional variance may reach values close to zero.

The form of the marginal density function $g_n$ of the process $(X_{in}, i \in \mathbb{Z})$ defined in the previous proposition and its expansion, depend on the distribution of $Z_0$. In particular, it is possible to link the rates of local alternatives in terms of density function (i.e., $a_n$) and in terms of variables (i.e., $\delta_n$).

If $Z_0$ is second order integrable, $EZ_0 \neq 0$ and if $f_0$ has partial continuous derivatives up to order two, which are bounded and integrable on $\mathbb{R}^d$, we have for $x \in \mathbb{R}^d$,

$$\begin{aligned} g_n(x) &= E[f_0(x - \delta_n Z_0)] \\ &= f_0(x) + \delta_n \gamma(x) + \delta_n^2 \gamma_n(x), \end{aligned}$$

where

$$\gamma(x) = -\sum_{i=1}^{d} E[Z_{0i}] \frac{\partial f_0}{\partial x_i}(x)$$

and

$$\gamma_n(x) = \sum_{i,\, j=1}^{d} E\left[ Z_{0i} Z_{0j} \int_0^1 \frac{\partial^2 f_0}{\partial x_i\, \partial x_j}(x - \delta_n Z_0 t)(1-t)\, dt \right].$$

If $Z_0$ is third order integrable, $EZ_0 = 0$ and if $f_0$ has partial continuous derivatives up to order three, which are bounded and integrable on $\mathbb{R}^d$, we have for $x \in \mathbb{R}^d$,

$$g_n(x) = f_0(x) + \delta_n^2 \gamma(x) + \delta_n^3 \gamma_n(x),$$

where

$$\gamma(x) = \frac{1}{2} \sum_{i,\, j=1}^{d} E[Z_{0i} Z_{0j}] \frac{\partial^2 f_0}{\partial x_i\, \partial x_j}(x)$$

and

$$\gamma_n(x) = -\frac{1}{2} \sum_{i,\, j,\, k=1}^{d} E\left[ Z_{0i} Z_{0j} Z_{0k} \int_0^1 \frac{\partial^3 f_0}{\partial x_i\, \partial x_j\, \partial x_k} (x - \delta_n Z_0 t)(1-t)^2\, dt \right].$$

In summary if $Z_0$ is zero mean, we have $a_n = \delta_n$, and otherwise $a_n = \delta_n^2$.

## 3. ASYMPTOTIC BEHAVIOUR OF $I_n^1(\pi)$ AND $I_n^2(\pi)$

In this section we study the asymptotic behaviour of the statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ defined by (1) and (2), respectively. They are related by

$$I_n^2(\pi) = I_n^1(\pi) + \int_{\mathbb{R}^d} \{E_0 f_n(x) - f_0(x)\}^2\, \pi(x)\, dx$$

$$+ 2 \int_{\mathbb{R}^d} \{f_n(x) - E_0 f_n(x)\} \{E_0 f_n(x) - f_0(x)\}\, \pi(x)\, dx.$$

In what follows, we introduce different assumptions denoted by (K), (B) or $(\pi)$ on the kernel $K$, the bandwidth $h_n$ or the weight function $\pi$.

*Assumptions on the Kernel* (K). $K$ is a bounded measurable function on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} K(u)\, du = 1$.

*Assumptions on the Bandwidth* (B). We assume that

$$h_n \to 0 \qquad \text{and} \qquad n h_n^d \to +\infty, \qquad \text{when} \quad n \to +\infty,$$

and there exists $\gamma \in\, ]0, 1[$ such that

$$\limsup_n n^{\gamma} h_n^d < +\infty.$$

The first conditions are usual in kernel estimation theory. The second one is not very restrictive and is for instance satisfied if $h_n = O(n^{-\delta})$ for $0 < \delta < 1/d$.

*Assumptions on the Weight Function* $(\pi)$. The real function $\pi$ is bounded, nonnegative and almost everywhere continuous on $\mathbb{R}^d$.

### 3.1. *Analysis of the Bias Term* $E_0 f_n(x) - f_0(x)$

Let us introduce, for $m \in \mathbb{N}$, the set $W^d(m)$ of pdf with partial continuous derivatives up to order $m$, which are bounded and integrable on $\mathbb{R}^d$, and the set $K^d(m)$ of kernels of order $m$, i.e., such that

$$\int_{\mathbb{R}^d} \|x\|^m \, |K(x)| \, dx < \infty \qquad \text{and} \qquad \int_{\mathbb{R}^d} x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} K(x) \, dx = 0,$$

for all $a_1, a_2, ..., a_d \in \mathbb{N}_0$ such that $0 < a_1 + a_2 + \cdots + a_d < m$.

If $f_0 \in W^d(m)$ and $K \in K^d(m)$ for some $m \in \mathbb{N}$, we have (see Bosq and Lecoutre [2] p. 88)

$$E_0 f_n(x) - f_0(x) = h_n^m \, \Delta_n^m f_0(x), \tag{4}$$

where

$$\Delta_n^m f_0(x) = \frac{(-1)^m}{(m-1)!} \sum_{i_1, ..., i_m = 1}^{d} \int_{\mathbb{R}^d} u_{i_1} \cdots u_{i_m} K(u)$$
$$\times \int_0^1 \frac{\partial^m f_0}{\partial x_{i_1} \cdots \partial x_{i_m}} (x - h_n u t)(1-t)^{m-1} \, dt \, du.$$

Moreover, by the dominated convergence theorem

$$\lim_{n \to +\infty} \Delta_n^m f_0(x) = \Delta^m f_0(x), \qquad x \in \mathbb{R}^d, \tag{5}$$

with

$$\Delta^m f_0(x) = \frac{(-1)^m}{m!} \sum_{i_1, ..., i_m = 1}^{d} \int_{\mathbb{R}^d} u_{i_1} \cdots u_{i_m} K(u) \, du \, \frac{\partial^m f_0}{\partial x_{i_1} \cdots \partial x_{i_m}} (x). \tag{6}$$

### 3.2. *Asymptotic Behaviour of U-Statistics*

The asymptotic random feature of the statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ essentially depends on the two following U-statistics. The first one is a second order U-statistic defined by

$$\mathscr{H}_n(\pi) = \frac{2}{n} \sum_{1 \leqslant j < i \leqslant n} \{ H_n(X_i, X_j) - E_n H_n(X_i, X_j) \},$$

where for $u, v \in \mathbb{R}^d$,

$$H_n(u, v) = \frac{1}{h_n^{3d/2}} \int_{\mathbb{R}^d} \left\{ K\left(\frac{x-u}{h_n}\right) - E_n K\left(\frac{x-X_0}{h_n}\right) \right\} \times \left\{ K\left(\frac{x-v}{h_n}\right) - E_n K\left(\frac{x-X_0}{h_n}\right) \right\} \pi(x)\, dx, \tag{7}$$

and $E_n$ denotes the mathematical expectation under the local alternative. The second U-statistic is

$$\mathscr{G}_n(\pi) = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} G_n(X_i),$$

where for $u \in \mathbb{R}^d$,

$$G_n(u) = \frac{1}{h_n^d} \int_{\mathbb{R}^d} \left\{ K\left(\frac{x-u}{h_n}\right) - E_n K\left(\frac{x-X_0}{h_n}\right) \right\} (\Delta_n^m f_0 \cdot \pi)(x)\, dx. \tag{8}$$

In order to study the asymptotic behaviour of the latter U-statistic, we need the following additional condition where $f_0 \in W^d(m)$ for some $m \in \mathbb{N}$.

*Assumption* (C). For any $i \in \mathbb{N}$, there exists $e_i \in \mathbb{R}$ such that, for any sequences $u_n$ and $v_n$ on $\mathbb{R}^d$ tending to zero, $E_n[(\Delta_n^m f_0 \cdot \pi)(X_i + u_n)(\Delta_n^m f_0 \cdot \pi)(X_0 + v_n)] \to e_i$, if $n \to +\infty$.

This condition is satisfied if for each $i \in \mathbb{N}$, and under the local alternative, the sequence of pdf $g_{in}$ converges to some pdf $g_{i\infty}$ and satisfies the dominated convergence theorem conditions. In this case, from the almost everywhere continuity of $\pi$ and the continuity of the derivatives of $f_0$, we have $e_i = E_\infty[(\Delta^m f_0 \cdot \pi)(X_i)(\Delta^m f_0 \cdot \pi)(X_0)]$, $i \in \mathbb{N}$. For independent variables the condition (C) is satisfied with $e_i = 0$ for $i \in \mathbb{N}$.

The following central limit theorem for U-statistics is proven in Appendix A.

THEOREM 3.1. *Let us assume that* (K), (B), ($\pi$), *and* (P) *are satisfied.*

(i) *The random variable $\mathscr{H}_n(\pi)$ is, under local alternative, asymptotically normal with zero mean and variance given by $2v_1^2(\pi)$ where*

$$v_1^2(\pi) = \int_{\mathbb{R}^d} f_0^2(x)\, \pi^2(x)\, dx \int_{\mathbb{R}^d} (K * \bar{K})^2\, (z)\, dz,$$

*$\bar{K}(z) = K(-z)$, and $*$ is the convolution product.*

(ii) *Moreover, if* $f_0 \in W^d(m)$ *and* $K \in K^d(m)$ *for some* $m \in \mathbb{N}$, *and the assumption* (C) *is satisfied, the bivariate random vector* $(\mathscr{H}_n(\pi), \mathscr{G}_n(\pi))$ *is, under local alternative, asymptotically normal with zero mean and diagonal covariance matrix* $\begin{bmatrix} 2v_1^2(\pi) & 0 \\ 0 & 4v_2^{*2}(\pi) \end{bmatrix}$, *where*

$$v_2^{*2}(\pi) = \operatorname{Var}_0 [(\Delta^m f_0 \cdot \pi)(X_0)] + 2 \sum_{j=1}^{+\infty} (e_j - E_0^2[(\Delta^m f_0 \cdot \pi)(X_0)]).$$

Note that the limit distribution of the degenerate U-statistic $\mathscr{H}_n(\pi)$ may a priori be a weighted sum of chi-squares or a normal distribution. The normality arises due to the presence of a kernel, $H_n(\cdot, \cdot)$, depending on $n$. In particular under $H_0$, we have

$$\begin{aligned} v_2^{*2}(\pi) &= v_2^2(\pi)(\text{say}) \\ &= \operatorname{Var}_0 [(\Delta^m f_0 \cdot \pi)(X_0)] \\ &\quad + 2 \sum_{j=1}^{+\infty} \operatorname{Cov}[(\Delta^m f_0 \cdot \pi)(X_j), (\Delta^m f_0 \cdot \pi)(X_0)]. \end{aligned}$$

### 3.3. *Asymptotic Expansions of* $I_n^1(\pi)$ *and* $I_n^2(\pi)$

The expansions of the statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ are given in the following result, which is proven in Appendix B.

THEOREM 3.2. *Let us assume that* (K), (B), $(\pi)$ *and* (P) *are satisfied.*

(i) *Under local alternative, we have*

$$\begin{aligned} I_n^1(\pi) = {} & \frac{1}{nh_n^{d/2}} \mathscr{H}_n(\pi) + a_n^2 \int_{\mathbb{R}^d} \gamma^2(x)\, \pi(x)\, dx \\ & + \frac{1}{nh_n^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K^2(u)\, g_n(x)\, \pi(x + uh_n)\, dx\, du + o_p(a_n^2) + o_p\left(\frac{1}{nh_n^{d/2}}\right). \end{aligned}$$

(ii) *Under local alternative, if* $f_0 \in W^d(m)$ *and* $K \in K^d(m)$ *for some* $m \in \mathbb{N}$, *we have*

$$\begin{aligned} I_n^2(\pi) = {} & I_n^1(\pi) + \frac{1}{\sqrt{n}\, h_n^{-m}} \mathscr{G}_n(\pi) + \int_{\mathbb{R}^d} \{E_0 f_n(x) - f_0(x)\}^2\, \pi(x)\, dx \\ & + 2h_n^m \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(u)(g_n(x) - f_0(x))(\Delta_n^m f_0 \cdot \pi)(x + uh_n)\, dx\, du. \end{aligned}$$

## 4. GOODNESS OF FIT TESTS

In what follows we consider test statistics derived from $I_n^1(\pi)$ and $I_n^2(\pi)$ by correcting them for the asymptotic bias under the null hypothesis. As we shall see, among the proposed corrections, some just correct for the bias under the null hypothesis, some others under the null and local alternatives (which is important to get asymptotic local unbiased tests). These statistics are

$$T_n^{1,1}(\pi) = nh_n^{d/2}\left\{I_n^1(\pi) - \frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)\, f_0(x)\,\pi(x+uh_n)\,dx\,du\right\},$$

$$T_n^{1,2}(\pi) = nh_n^{d/2}\left\{I_n^1(\pi) - \frac{1}{nh_n^d}\,\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^d} K^2(u)\,\pi(X_i+uh_n)\,du\right\},$$

$$T_n^{2,1}(\pi) = d(n)\left\{I_n^2(\pi) - \frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)\, f_0(x)\,\pi(x+uh_n)\,dx\,du \right.$$
$$\left. - \int_{\mathbb{R}^d}\{E_0 f_n(x) - f_0(x)\}^2\,\pi(x)\,dx\right\},$$

and

$$T_n^{2,2}(\pi) = d(n)\left\{I_n^2(\pi) - \frac{1}{nh_n^d}\,\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^d} K^2(u)\,\pi(X_i+uh_n)\,du \right.$$
$$\left. - \int_{\mathbb{R}^d}\{E_0 f_n(x) - f_0(x)\}^2\,\pi(x)\,dx\right\},$$

where

$$d(n) = \begin{cases} nh_n^{d/2}, & \text{if } \lambda \in [0, +\infty[ \\ \sqrt{n}\,h_n^{-m}, & \text{if } \lambda = +\infty \end{cases} \qquad \text{with } \lambda = \lim_{n\to+\infty} nh_n^{d+2m}. \tag{9}$$

### 4.1. *Expansions of the Test Statistics under Local Alternatives*

Under local alternatives the term

$$\frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)\, g_n(x)\,\pi(x+uh_n)\,dx\,du$$

appearing in the expansions of $I_n^1(\pi)$ and $I_n^2(\pi)$ (see Theorem 3.2) may be approximated in different ways. It is first equal to

$$\frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)\, f_0(x)\, \pi(x+uh_n)\, dx\, du$$
$$+\frac{a_n}{nh_n^d}\int_{\mathbb{R}^d}\gamma(x)\,\pi(x)\,dx\int_{\mathbb{R}^d}K^2(u)\,du+o_p\left(\frac{a_n}{nh_n^d}\right),$$

by using the definition of $g_n$, and it is also equal to

$$\frac{1}{nh_n^d}\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^d}K^2(u)\,\pi(X_i+uh_n)\,du+O_p\left(\frac{1}{nh_n^d}\frac{1}{\sqrt{n}}\right),$$

by central limit theorem.

By substituting in the expansions of Theorem 3.2, and by gathering the negligible terms, we get the expansions of the test statistics under local alternatives.

THEOREM 4.1. *Let us assume that* (K), (B), ($\pi$) *and* (P) *are satisfied.*

(i) *Under local alternative, we have*

$$T_n^{1,2}(\pi)=\mathscr{H}_n(\pi)+nh_n^{d/2}a_n^2\int_{\mathbb{R}^d}\gamma^2(x)\,\pi(x)\,dx+o_p(nh_n^{d/2}a_n^2)+o_p(1),$$

*and*

$$T_n^{1,1}(\pi)=T_n^{1,2}(\pi)+\frac{a_n}{h_n^{d/2}}\int_{\mathbb{R}^d}K^2(u)\,du\int_{\mathbb{R}^d}\gamma(x)\,\pi(x)\,dx+o\left(\frac{a_n}{h_n^{d/2}}\right)+o_p(1).$$

(ii) *Under local alternative, if* $f_0\in W^d(m)$ *and* $K\in K^d(m)$ *for some* $m\in\mathbb{N}$, *we have*

$$T_n^{2,2}(\pi)=\frac{d(n)}{nh_n^{d/2}}\mathscr{H}_n(\pi)+\frac{d(n)}{\sqrt{n}\,h_n^{-m}}\mathscr{G}_n(\pi)+d(n)\,a_n^2\int_{\mathbb{R}^d}\gamma^2(x)\,\pi(x)\,dx$$
$$+2d(n)\,h_n^m a_n\int_{\mathbb{R}^d}\Delta^m f_0(x)\,\gamma(x)\,\pi(x)\,dx+o_p(d(n)\,a_n^2)$$
$$+o(d(n)\,h_n^m a_n)+o_p(1),$$

*and*

$$T_n^{2,1}(\pi) = T_n^{2,2}(\pi) + \frac{d(n)\, a_n}{nh_n^d} \int_{\mathbb{R}^d} \gamma(x)\, \pi(x)\, dx$$

$$\times \int_{\mathbb{R}^d} K^2(u)\, du + o\left(\frac{d(n)\, a_n}{nh_n^d}\right) + o_p(1).$$

## 4.2. *The Critical Regions*

From Theorems 3.1 and 4.1 the different statistics $T_n^{j,k}(\pi)$ are asymptotically normal under the null hypothesis. Moreover for a fixed alternative hypothesis $H_1: \{f = f_1\}$, with $f_1 \neq f_0$, $T_n^{j,k}(\pi)$ converges in probability to $+\infty$, as soon as the support of the weight function contains the support of $f_0$ (see Appendix C). Therefore we have to introduce one sided critical regions based on the previous statistics.

The following result extends to the multidimensional geometrically $\beta$-mixing case Theorem 4.1 of Bickel and Rosenblatt [1]. We denote by $\phi$ the cdf of the standard normal distribution, and by $\lambda$ the parameter defined in (9).

THEOREM 4.2. *Let us assume that* (K), (B), ($\pi$) *and* (P) *are satisfied, and that the support of the weight function $\pi$ contains the support of $f_0$.*

(i) *The tests associated with the critical regions*

$$C_n^{1,k}(\pi) = \{T_n^{1,k}(\pi) \geqslant \phi^{-1}(1-\alpha)(2v_1^2(\pi))^{1/2}\},$$

*with $k = 1, 2$, are asymptotically of level $\alpha$ and consistent to test $H_0$ against $H_0^c$.*

(ii) *If $f_0 \in W^d(m)$ and $K \in K^d(m)$ for some $m \in \mathbb{N}$, the tests associated with the critical regions*

$$C_n^{2,k}(\pi) = \{T_n^{2,k}(\pi) \geqslant \phi^{-1}(1-\alpha)(2v^2{}_1(\pi) + 4\lambda v_2^2(\pi))^{1/2}\},$$

*with $k = 1, 2$ and $\lambda \in [0, +\infty[$, and*

$$C_n^{2,k}(\pi) = \{T_n^{2,k}(\pi) \geqslant 2\phi^{-1}(1-\alpha)\, |v_2(\pi)|\},$$

*with $k = 1, 2$ and $\lambda = +\infty$, are asymptotically of level $\alpha$ and consistent to test $H_0$ against $H_0^c$.*

The asymptotic variance of $\mathcal{G}_n(\pi)/2$ under $H_0$ has no simple analytic expression, but $v_2^2(\pi)$ may be consistently estimated by

$$\hat{v}_{2,n}^2(\pi) = \mathrm{Var}_0\left((\Delta^m f_0 \cdot \pi)(X_0)\right) + 2\sum_{j=1}^{m(n)} \left\{\frac{1}{n-j}\sum_{i=1}^{n-j} (\Delta^m f_0 \cdot \pi)(X_i)(\Delta^m f_0 \cdot \pi)(X_{i+j}) - \left(\frac{1}{n}\sum_{i=1}^{n} (\Delta^m f_0 \cdot \pi)(X_i)\right)^2\right\},$$

where $m(n)$ is a sequence of integers tending to $+\infty$. Under $H_0$, we have

$$\hat{v}_{2,n}^2(\pi) \xrightarrow[n\to+\infty]{p} v_2^2(\pi), \qquad \text{if} \quad \frac{m^3(n)}{n} \xrightarrow[n\to+\infty]{} 0.$$

If the tests $C_n^{2,k}(\pi)$ are used assuming that the observations are independent, it is natural to take $v_2^2(\pi) = \mathrm{Var}_0\left((\Delta^m f_0 \cdot \pi)(X_0)\right)$. However, if the independence assumption is not valid the tests $C_n^{2,k}(\pi)$, for $\lambda \in ]0, +\infty]$, are sensitive to the presence of dependence on the observations due to the miscomputed variance term. This can be avoided by using $\hat{v}_2^2(\pi)$ instead of $v_2^2(\pi)$ in the construction of $C_n^{2,k}(\pi)$.

The equivalence properties of the previous tests procedures under local alternative (and under $H_0$ by imposing $a_n = 0$) are given in the theorem below and are direct consequences of Theorems 4.1 and 4.2.

THEOREM 4.3. *Let us assume that the hypothesis* (K), (B), ($\pi$) *and* (P) *are satisfied.*

(i) *The tests $C_n^{1,1}(\pi)$ and $C_n^{1,2}(\pi)$ are asymptotically equivalent under the local alternatives with $a_n = o(h_n^{d/2})$.*

*Moreover, let us assume that $f_0 \in W^d(m)$ and $K \in K^d(m)$ for some $m \in \mathbb{N}$.*

(ii) *If $\lambda \in [0, +\infty[$ (resp. $\lambda = +\infty$) the tests $C_n^{2,1}(\pi)$ and $C_n^{2,2}(\pi)$ are asymptotically equivalent under the local alternatives with $a_n = o(h_n^{d/2})$ (resp. $a_n = o(\sqrt{nh_n^{2d+2m}})$).*

(iii) *If $\lambda = 0$ the tests $C_n^{1,1}(\pi)$, $C_n^{1,2}(\pi)$, $C_n^{2,1}(\pi)$ and $C_n^{2,2}(\pi)$ are asymptotically equivalent under the local alternatives with $a_n = o(h_n^{d/2})$.*

A number of other test statistics may have been considered by approximating differently the bias term. Among them we find those introduced by Bickel and Rosenblatt [1] which are, under some conditions on $h_n$, asymptotically equivalent to some of the statistics considered above.

## 5. SEPARATING ALTERNATIVES

Local asymptotic power analysis of testing procedures is based on the research for local alternatives providing a nondegenerate limiting power, called separating alternatives. In our framework this analysis is complicated since the appropriate rates $a_n$ of the separating alternatives depend on the direction of the alternative and on the selected bandwidth. In this section we essentially study these separating rates.

### 5.1. *Minimal and Maximal Limit Sequences*

We consider a consistent test of $H_0$ against $H_0^c$ asymptotically of level $\alpha$ defined by the critical region $C_n$ and we denote by $P_{g_n}$ the probability under the sequence of local alternative satisfying (P).

DEFINITION 5.1. The sequence $(\underline{a}_n)$ of positive real numbers, converging to zero when $n \to +\infty$, is a minimal sequence for the test $C_n$ if for any sequence of local alternatives satisfying (P) with $\gamma \neq 0$ in (3), we have

$$\lim_{n \to +\infty} P_{g_n}(C_n) = \alpha, \qquad \text{if } \frac{a_n}{\underline{a}_n} = o(1).$$

Moreover, if the minimal sequence $(\underline{a}_n^I)$ for the test $C_n$ satisfies $\underline{a}_n = O(\underline{a}_n^I)$ for any minimal sequences $(\underline{a}_n)$ for the test $C_n$, $(\underline{a}_n^I)$ is called a minimal limit sequence for $C_n$.

DEFINITION 5.2. The sequence $(\bar{a}_n)$ of positive real numbers, converging to zero when $n \to +\infty$, is a maximal sequence for the test $C_n$ if for any sequence of local alternatives satisfying (P) with $\gamma \neq 0$ in (3), we have

$$\lim_{n \to +\infty} P_{g_n}(C_n) = 1, \qquad \text{if } \frac{\bar{a}_n}{a_n} = o(1).$$

Moreover, if the maximal sequence $(\bar{a}_n^I)$ for the test $C_n$ satisfies $\bar{a}_n^I = O(\bar{a}_n)$ for any maximal sequences $(\bar{a}_n)$, for the test $C_n$, $(\bar{a}_n^I)$ is called maximal limit sequence for $C_n$.

If there both exist a minimal and a maximal limit sequence for $C_n$, such that $\underline{a}_n^I = \bar{a}_n^I = a_n^I$ (say), the faster $a_n^I$ is tending to zero, the better is the testing procedure.

Remark that for the classical tests based on the empirical distribution function the limit sequence exists and is given by $a_n^I = \frac{1}{\sqrt{n}}$ (see Milbrodt and Strasser [14]).

5.2. *Limit Sequences for* $C_n^{j,k}(\pi)$, $j, k = 1, 2$

The limit sequences depend on the testing procedure and for the tests $C_n^{j,k}(\pi)$, $j, k = 1, 2$, on the choice of the bandwidth $h_n$. These limit sequences are derived from Theorem 4.1 under the assumption that the weight function is almost everywhere strictly positive on $\mathbb{R}^d$. Then they are particularized to the case of a power function $h_n = n^{-a}$, where $a$ is strictly between 0 and $1/d$ because of assumption (B), which will allow to summarize the results by some figures. In such a case the limit sequences are also power functions $\underline{a}_n^I = n^{-\underline{b}}$ and $\bar{a}_n^I = n^{-\bar{b}}$, where $\underline{b}$ and $\bar{b}$ are the minimal and maximal limit rates respectively.

We get for the test $C_n^{1,2}(\pi)$ the limit sequence

$$a_n^I = \frac{1}{\sqrt{n}\, h_n^{d/4}},$$

and for the particular case of power functions the limit rates

$$\underline{b} = \bar{b} = \frac{1}{2} - \frac{ad}{4}.$$

For the test $C_n^{1,1}(\pi)$ the minimal and maximal limit sequences do not coincide. We get

$$\underline{a}_n^I = \frac{1}{\sqrt{n}\, h_n^{d/4}} \mathbb{1}_{\{n^{-2/3d} = O(h_n)\}} + h_n^{d/2} \mathbb{1}_{\{h_n = o(n^{-2/3d})\}},$$

and

$$\bar{a}_n^I = \frac{1}{\sqrt{n}\, h_n^{d/4}} \mathbb{1}_{\{n^{-2/3d} = O(h_n)\}} + \frac{1}{nh_n^d} \mathbb{1}_{\{h_n = o(n^{-2/3d})\}}.$$

For power functions the corresponding limit rates are

$$\underline{b} = \left(\frac{1}{2} - \frac{ad}{4}\right) \mathbb{1}_{\{a \leqslant 2/3d\}} + \frac{ad}{2} \mathbb{1}_{\{a > 2/3d\}},$$

and

$$\bar{b} = \left(\frac{1}{2} - \frac{ad}{4}\right) \mathbb{1}_{\{a \leqslant 2/3d\}} + (1 - ad) \mathbb{1}_{\{a > 2/3d\}}.$$

These rates are summarized in Figs. 1 and 2. We remark that it is possible to be very close to the parametric rate $\underline{b} = \bar{b} = \frac{1}{2}$ by choosing a large bandwidth ($a \simeq 0$).

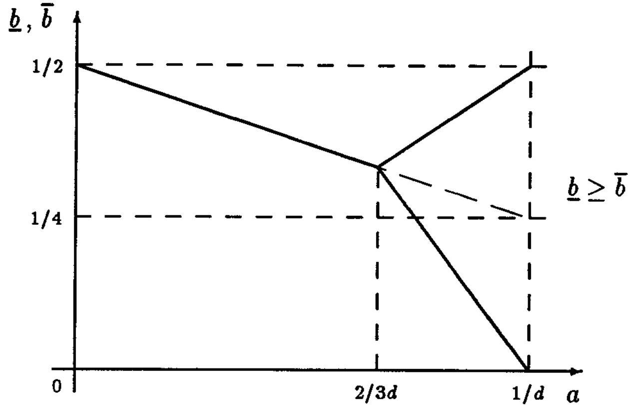


**FIG. 1.** Limit rates for $C_n^{1,1}(\pi)$.

Finally we study the asymptotic local power of the tests $C_n^{2,1}(\pi)$ and $C_n^{2,2}(\pi)$ by assuming that $\hat{v}_{2n}^2(\pi)$ is under the sequence of local alternatives a consistent estimator of the variance of $\mathcal{G}_n(\pi)/2$, i.e., $\hat{v}_{2n}^2(\pi) \xrightarrow[n\to+\infty]{p} v_2^{*2}(\pi)$. This convergence occurs if for $i \in \mathbb{N}$ we have $E_n[(\Delta^m f_0 \cdot \pi)(X_i)(\Delta^m f_0 \cdot \pi)(X_0)] \to e_i$, $n \to +\infty$, where $e_i$, $i \in \mathbb{N}$ is given in assumption (C). This condition is fulfilled under the conditions described after the introduction of assumption (C).

Using the expansion of Theorem 4.1, we get, for the test $C_n^{2,2}(\pi)$, the limit sequences

$$\underline{a}_n^I = \frac{1}{\sqrt{n}} \mathbb{1}_{\{n^{-1/(d+2m)} = O(h_n)\}} + \frac{1}{nh_n^{d/2+m}} \mathbb{1}_{\{h_n = o(n^{-1/(d+2m)}) \text{ and } n^{-1/(d/2+2m)} = O(h_n)\}} + \frac{1}{\sqrt{n}\, h_n^{d/4}} \mathbb{1}_{\{h_n = o(n^{-1/(d/2+2m)})\}},$$

and

$$\bar{a}_n^I = h_n^m \mathbb{1}_{\{n^{-1/(d/2+2m)} = O(h_n)\}} + \frac{1}{\sqrt{n}\, h_n^{d/4}} \mathbb{1}_{\{h_n = o(n^{-1/(d/2+2m)})\}}.$$

The corresponding limit rates are

$$\underline{b} = \frac{1}{2} \mathbb{1}_{\{a \leqslant 1/(d+2m)\}} + \left(1 - \frac{a}{2}(d+2m)\right) \mathbb{1}_{\{1/(d+2m) < a \leqslant 1/(d/2+2m)\}} + \left(\frac{1}{2} - \frac{ad}{4}\right) \mathbb{1}_{\{a > 1/(d/2+2m)\}},$$

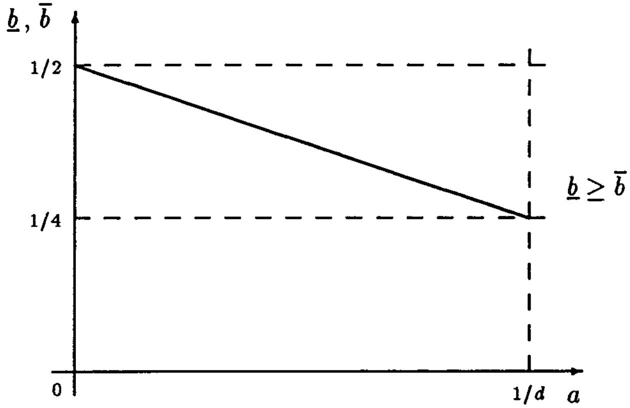


**FIG. 2.** Limit rates for $C_n^{1,2}(\pi)$.

and

$$\bar{b} = am\,\mathbb{1}_{\{a \leqslant 1/(d/2+2m)\}} + \left(\frac{1}{2} - \frac{ad}{4}\right)\mathbb{1}_{\{a > 1/(d/2+2m)\}}.$$

Similarly we get for the test $C_n^{2,1}(\pi)$:

(i) If $d < 2m$

$$\underline{a}_n^I = \frac{1}{\sqrt{n}}\,\mathbb{1}_{\{n^{-1/(d+2m)} = O(h_n)\}} + \frac{1}{nh_n^{d/2+m}}\,\mathbb{1}_{\{h_n = o(n^{-1/(d+2m)}) \text{ and } n^{-1/(d/2+2m)} = O(h_n)\}}$$

$$+ \frac{1}{\sqrt{n}\,h_n^{d/4}}\,\mathbb{1}_{\{h_n = o(n^{-1/(d/2+2m)}) \text{ and } n^{-2/3d} = O(h_n)\}} + h_n^{d/2}\,\mathbb{1}_{\{h_n = o(n^{-2/3d})\}},$$

and

$$\bar{a}_n^I = h_n^m\,\mathbb{1}_{\{n^{-1/(d/2+2m)} = O(h_n)\}} + \frac{1}{\sqrt{n}\,h_n^{d/4}}\,\mathbb{1}_{\{h_n = o(n^{-1/(d/2+2m)}) \text{ and } n^{-2/3d} = O(h_n)\}}$$

$$+ \frac{1}{nh_n^d}\,\mathbb{1}_{\{h_n = o(n^{-2/3d})\}}.$$

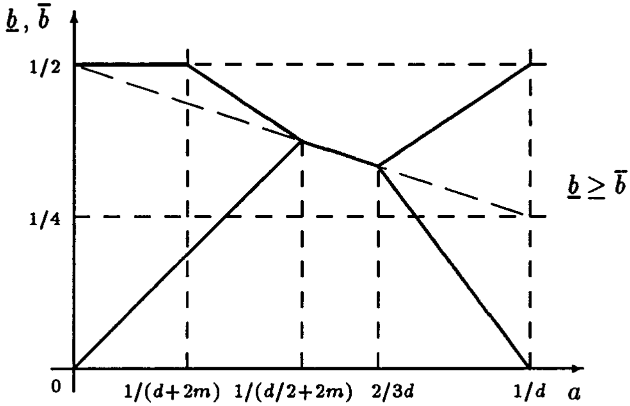


**FIG. 3.** Limit rates for the test $C_n^{2,1}(\pi)$, $d < 2m$.

The corresponding limit rates are

$$\underline{b} = \frac{1}{2}\mathbb{1}_{\{a \leqslant 1/(d+2m)\}} + \left(1 - \frac{a}{2}(d+2m)\right)\mathbb{1}_{\{1/(d+2m) < a \leqslant 1/(d/2+2m)\}}$$

$$+ \left(\frac{1}{2} - \frac{ad}{4}\right)\mathbb{1}_{\{1/(d/2+2m) < a \leqslant 2/3d\}} + \frac{ad}{2}\mathbb{1}_{\{a > 2/3d\}},$$

and

$$\bar{b} = am\,\mathbb{1}_{\{a \leqslant 1/(d/2+2m)\}} + \left(\frac{1}{2} - \frac{ad}{4}\right)\mathbb{1}_{\{1/(d/2+2m) < a \leqslant 2/3d\}} + (1 - ad)\,\mathbb{1}_{\{a > 2/3d\}}.$$

(ii) If $d \geqslant 2m$

$$\underline{a}_n^I = \frac{1}{\sqrt{n}}\mathbb{1}_{\{n^{-1/(d/2+2m)} = O(h_n)\}} + \frac{1}{nh_n^{d/2+m}}\mathbb{1}_{\{h_n = o(n^{-1/(d+2m)}) \text{ and } n^{-1/(d+m)} = O(h_n)\}}$$

$$+ h_n^{d/2}\mathbb{1}_{\{h_n = o(n^{-1/(d+m)})\}},$$

and

$$\bar{a}_n^I = h_n^m \mathbb{1}_{\{n^{-1/(d+m)} = O(h_n)\}} + \frac{1}{nh_n^d}\mathbb{1}_{\{h_n = o(n^{-1/(d+m)})\}}.$$

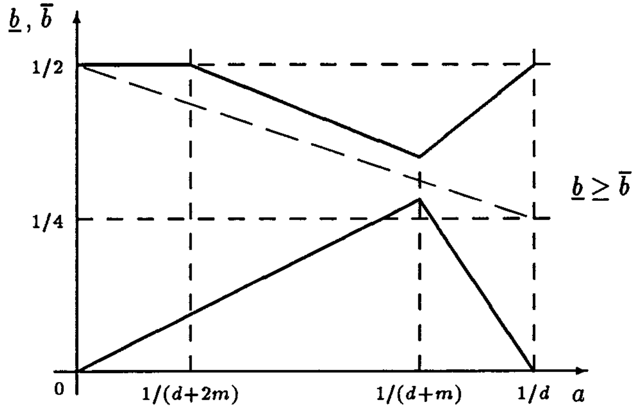


**FIG. 4.** Limit rates for the test $C_n^{2,1}(\pi)$, $d \geqslant 2m$.

The corresponding limit rates are

$$\underline{b} = \frac{1}{2} \mathbb{1}_{\{a \leqslant 1/(d+2m)\}} + \left(1 - \frac{a}{2}(d+2m)\right) \mathbb{1}_{\{1/(d+2m) < a \leqslant 1/(d+m)\}} + \frac{ad}{2} \mathbb{1}_{\{a > 1/(d+m)\}},$$

and

$$\bar{b} = am \, \mathbb{1}_{\{a \leqslant 1/(d+m)\}} + (1 - ad) \, \mathbb{1}_{\{a > 1/(d+m)\}}.$$

These rates are summarized in the Figs. 3, 4, and 5.

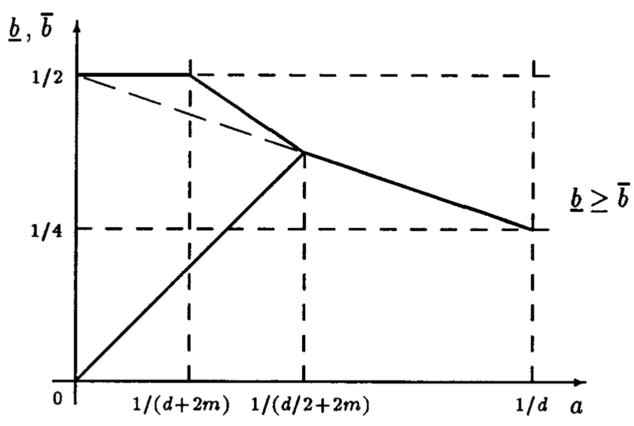


**FIG. 5.** Limit rates for the test $C_n^{2,2}(\pi)$.

## 6. LOCAL POWER ANALYSIS

The equality of the two limit sequences has interesting implications in terms of local unbiasedness properties of the testing procedures. We discuss this problem, using as a basis the different figures of the previous section. We distinguish three different cases depending on the bandwidth $h_n = n^{-c}$, $0 < c < 1/d$, used to define the goodness of fit tests. They may be described as follows: (i) in a neighbourhood of $c$ the two limit rates coincide for $c$ and on the right of $c$; (ii) in a neighbourhood of $c$ the two limit rates coincide for $c$ and on the left of $c$ but they do not coincide on the right of $c$; and (iii) in a neighbourhood of $c$ the two limit rates never coincide.

In the two first cases the asymptotic local power function of the tests $C_n^{j,k}(\pi)$ has the form

$$\lim_{n \to +\infty} P_{g_n}(C_n^{j,k}(\pi)) = \begin{cases} \alpha, & \text{if } \ a_n = o\left(\dfrac{1}{\sqrt{n}\, h_n^{d/4}}\right) \\ \beta(\gamma), & \text{if } \ a_n = \dfrac{1}{\sqrt{n}\, h_n^{d/4}} \\ 1, & \text{if } \ \dfrac{1}{\sqrt{n}\, h_n^{d/4}} = o(a_n). \end{cases}$$

In case (i), $\beta(\gamma)$ is an increasing function of $\int_{\mathbb{R}^d} \gamma^2(x)\, \pi(x)\, dx$ given by

$$\beta(\gamma) = 1 - \phi\left(\phi^{-1}(1-\alpha) - (2v_1^2(\pi))^{-1/2} \int_{\mathbb{R}^d} \gamma^2(x)\, \pi(x)\, dx\right).$$

Then, from the almost everywhere strict positivity imposed to the weight function the tests $C_n^{j,k}(\pi)$ are uniformly locally strictly unbiased (cf. Bickel and Rosenblatt [1]) since we have $\beta(\gamma) \geqslant \alpha$ and $\beta(\gamma) > \alpha$ unless $\gamma = 0$, for all bounded and integrable $\gamma$ functions.

In this case, $\beta(\gamma)$ depends on the kernel $K$ only through the asymptotic variance $v_1^2(\pi)$, i.e., through $\int_{\mathbb{R}^d} (K * \bar{K})^2\,(z)\, dz$. Therefore the power may be optimized if we choose a kernel giving the minimum of the functional $J(K) = \int_{\mathbb{R}^d} (K * \bar{K})^2\,(z)\, dz$. This problem has been solved by Ghosh and Huang [9] when $d = 1$. We can easily extend their result to the multi-dimensional case by considering $K$ a kernel in the class $\mathscr{K}^d(m, \sigma)$ of kernels on $\mathbb{R}^d$ of the form $K(x_1, ..., x_d) = \prod_{i=1}^d K_0(x_i)$ where $K_0$ is a non-negative real kernel such that $\int_{\mathbb{R}} tK_0(t)\, dt = m$ and $\int_{\mathbb{R}} (t-m)^2\, K_0(t)\, dt = \sigma^2$, with $m \in \mathbb{R}$ and $\sigma > 0$ given. The functional $J$ assumes the form

$J(K)=(\int_{\mathbb{R}} (K_0 * \bar{K}_0)^2 (t)\, dt)^d$, and from Theorem 1.1 of Ghosh and Huang [9] we deduce $\min_{K\in \mathscr{K}^d(m,\sigma)} J(K)=J(K^*)=(3\sqrt{3}\,\sigma)^{-d}$, where

$$K^*(x_1, ..., x_d)=\left(\frac{1}{2\sqrt{3}\,\sigma}\right)^d \prod_{i=1}^{d} \mathbb{1}_{\{|x_i-m|\leqslant \sigma\sqrt{3}\}}.$$

In case (ii), $\beta(\gamma)$ is given by

$$\beta(\gamma)=1-\phi\left(\phi^{-1}(1-\alpha)-(2v_1^2(\pi))^{-1/2}\right.$$
$$\left.\times\left[\int_{\mathbb{R}^d} K^2(u)\, du \int_{\mathbb{R}^d} \gamma(x)\, \pi(x)\, dx+\int_{\mathbb{R}^d} \gamma^2(x)\, \pi(x)\, dx\right]\right),$$

and there is a local bias. To get a better understanding of what happens in this case, let us consider $\gamma=\delta\gamma_0$, where $\gamma_0$ is a given function and $\delta\in\mathbb{R}$. Let us introduce the function of $\delta$ defined by $\beta^*(\delta)=\beta(\delta\gamma_0)$. The function $\beta^*(\cdot)$ is the power function corresponding to the previous alternative. For $\delta^*=-\int_{\mathbb{R}^d} K^2(u)\, du \int_{\mathbb{R}^d} \gamma_0(x)\, \pi(x)\, dx/\int_{\mathbb{R}^d} \gamma_0^2(x)\, \pi(x)\, dx$ assumed to be positive, it has the form given in Fig. 6, where it can be seen that the problem of bias is only a local one in a neighbourhood of the null hypothesis ($\delta=0$).

Note that in this case there does not exist an optimal kernel, uniformly in the alternatives.

Finally, in case (iii), i.e., when the minimal and maximal limit sequences do not coincide, the asymptotic local power of the tests $C_n^{j,k}(\pi)$ has the form

$$\lim_{n\to+\infty} P_{g_n}(C_n^{j,k}(\pi))=\begin{cases}\alpha, & \text{if } a_n=o(\bar{a}_n^I)\\ \beta(\gamma;(a_n)), & \text{if } \bar{a}_n^I=O(a_n) \quad \text{and} \quad a_n=O(\underline{a}_n^I)\\ 1, & \text{if } \underline{a}_n^I=o(a_n),\end{cases}$$

where $\beta(\gamma;(a_n))$ depends on the position of the alternative. In order to exemplify this situation we will consider the case of the test $C_n^{2,1}(\pi)$ when $0<c<1/(d+2m)$. When we consider a local alternative with $a_n=\bar{a}_n^I$, $\beta(\gamma;(a_n))$ may be either 1 or 0 depending on the either positive or negative sign of $\int_{\mathbb{R}^d} \gamma^2(x)\, \pi(x)\, dx+2\int_{\mathbb{R}^d} \Delta^m f_0(x)\, \gamma(x)\, \pi(x)\, dx$. When we consider a local alternative with $\bar{a}_n^I=o(a_n)$ and $a_n=o(\underline{a}_n^I)$, $\beta(\gamma;(a_n))$ may be either 1 or 0 depending on the either positive or negative sign of $\int_{\mathbb{R}^d} \Delta^m f_0(x)\, \gamma(x)\, \pi(x)\, dx$. Finally if we consider a local alternative with $a_n=\underline{a}_n^I$ we have

$$\beta(\gamma;(a_n))=1-\phi\left(\phi^{-1}(1-\alpha)-(2v_2^2(\pi))^{-1/2}\, 2\int_{\mathbb{R}^d} \Delta^m f_0(x)\, \gamma(x)\, \pi(x)\, dx\right),$$

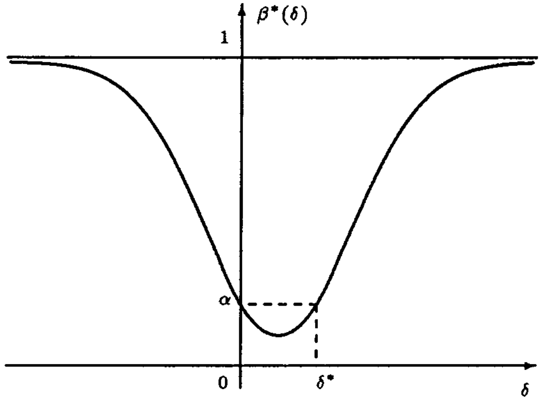


**FIG. 6.** Power function $\beta^*(\cdot)$.

which may be either larger or smaller than $\alpha$ depending on the either positive or negative sign of $\int_{\mathbb{R}^d} \Delta^m f_0(x)\, \gamma(x)\, \pi(x)\, dx$.

The lack of local unbiasedness in cases (ii) and (iii) is a feature of these kernel based testing procedures. It is a consequence of the practice of just correcting the bias of the basic statistics $I_n^1(\pi)$ and $I_n^2(\pi)$ under the null, and not under the local alternative.

## APPENDIXES

### A. *Asymptotic Normality of* $(\mathscr{H}_n(\pi), \mathscr{G}_n(\pi))$

Under $H_0$ and for $m=2$, Theorem 3.1 was derived by Tenreiro [24] as a consequence of a central limit theorem for degenerate U-statistics generated by a geometrically $\beta$-mixing process (this result follows the lines of Hall [12] and Takahata and Yoshihara [22], and is based on central limit theorem for triangular array given by Dvoretsky [5]). The introduction of local alternatives does not modify the approach (see Tenreiro [23] for $\mathscr{H}_n(\pi)$). Therefore if the hypotheses (K), (B), ($\pi$), (P) and (C) are satisfied, and under local alternative, $(\mathscr{H}_n(\pi), \mathscr{G}_n(\pi))$ is asymptotically normal with zero mean and a diagonal covariance matrix, where

$$\lim_{n \to +\infty} \mathrm{Var}_n\,(\mathscr{H}_n(\pi)) = \lim_{n \to +\infty} E_n[G_{n0}(X_0, \bar{X}_0)],$$

and

$$\lim_{n \to +\infty} \mathrm{Var}_n\,(\mathscr{G}_n(\pi)) = 4 \sum_{i=-\infty}^{+\infty} \lim_{n \to +\infty} E_n[G_n(X_i)\, G_n(X_0)],$$

where $G_{n0}(u, v) = E_n[H_n(X_0, u)\, H_n(X_0, v)]$, for $u, v \in \mathbb{R}^d$, $\bar{X}_0$ is an independent copy of $X_0$, and $H_n(\cdot, \cdot)$ and $G_n(\cdot)$ are defined by (7) and (8) respectively.

We compute in the sequel these asymptotic variances.

A.1. *Asymptotic Variance of* $\mathscr{H}_n(\pi)$. From the definition of $H_n(\cdot, \cdot)$ and assumptions (K), ($\pi$) and (P), we have, uniformly on $u, v \in \mathbb{R}^d$,

$$H_n(u, v) = \frac{1}{h_n^{3d/2}} \int_{\mathbb{R}^d} K\left(\frac{x-u}{h_n}\right) K\left(\frac{x-v}{h_n}\right) \pi(x)\, dx + O(h_n^{d/2}). \tag{16}$$

Then,

$$\begin{aligned}
&\lim_{n \to +\infty} E_n[G_{n0}(X_0, \bar{X}_0)] \\
&\quad = \lim_{n \to +\infty} \frac{1}{h_n^{3d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} K\left(\frac{x-u}{h_n}\right) \right. \\
&\qquad \left. \times K\left(\frac{x-v}{h_n}\right) \pi(x)\, dx \right)^2 g_n(u)\, g_n(v)\, du\, dv.
\end{aligned}$$

From the form of $g_n$, and the dominated convergence theorem we get

$$\lim_{n \to +\infty} \operatorname{Var}_n(\mathscr{H}_n(\pi)) = \int_{\mathbb{R}^d} f_0^2(x)\, \pi^2(x)\, dx \int_{\mathbb{R}^d} (K * \bar{K})^2\, (z)\, dz = v_1^2(\pi).$$

A.2. *Asymptotic Variance of* $\mathscr{G}_n(\pi)$. From the definition of $G_n(\cdot)$ we have, for $i = 0, 1, 2, \ldots,$

$$\begin{aligned}
&E_n[G_n(X_i)\, G_n(X_0)] \\
&\quad = E_n\left[ \frac{1}{h_n^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K\left(\frac{x-X_i}{h_n}\right) K\left(\frac{y-X_0}{h_n}\right) \right. \\
&\qquad \left. \times (\Delta_n^m f_0 \cdot \pi)(x) (\Delta_n^m f_0 \cdot \pi)(y)\, dx\, dy \right] \\
&\qquad - E_n^2\left[ \frac{1}{h_n^d} \int_{\mathbb{R}^d} K\left(\frac{x-X_0}{h_n}\right) (\Delta_n^m f_0 \cdot \pi)(x)\, dx \right] \\
&\quad = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(u)\, K(v)\, E_n[(\Delta_n^m f_0 \cdot \pi)(X_i + u h_n) \\
&\qquad \times (\Delta_n^m f_0 \cdot \pi)(X_0 + v h_n)]\, du\, dv \\
&\qquad - \left( \int_{\mathbb{R}^d} K(u)\, E_n[(\Delta_n^m f_0 \cdot \pi)(X_0 + u h_n)]\, du \right)^2.
\end{aligned}$$

Then by applying assumption (C), equality (5), the dominated convergence theorem, and by taking into account the form of $g_n$, we get for $i=1, 2, ...$

$$\lim_{n\to+\infty} E_n[G_n(X_i)\, G_n(X_0)] = e_i - E_0^2[(\Delta^m f_0 \cdot \pi)(X_0)],$$

and for $i=0$

$$\lim_{n\to+\infty} E_n[G_n(X_0)\, G_n(X_0)] = \mathrm{Var}_0\, [(\Delta^m f_0 \cdot \pi)(X_0)].$$

Finally we have

$$\begin{aligned}
&\lim_{n\to+\infty} \mathrm{Var}_n\, (\mathcal{G}_n(\pi)) \\
&\quad = 4\left[\mathrm{Var}_0\, [(\Delta^m f_0 \cdot \pi)(X_0)] + 2 \sum_{i=1}^{+\infty} (e_i - E_0^2[(\Delta^m f_0 \cdot \pi)(X_0)])\right] \\
&\quad = 4 v_2^{*2}(\pi).
\end{aligned}$$

B. *Asymptotic Expansions of* $I_n^1(\pi)$ *and* $I_n^2(\pi)$

We can first decompose $I_n^1(\pi)$ and $I_n^2(\pi)$ in the following ways

$$\begin{aligned}
I_n^1(\pi) &= \int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\}^2\, \pi(x)\, dx - E_n \int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\}^2\, \pi(x)\, dx \\
&\quad + E_n \int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\}^2\, \pi(x)\, dx \\
&\quad + 2 \int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\}\{E_n f_n(x) - E_0 f_n(x)\}\, \pi(x)\, dx \\
&\quad + \int_{\mathbb{R}^d} \{E_n f_n(x) - E_0 f_n(x)\}^2\, \pi(x)\, dx \\
&= \frac{1}{n h_n^{d/2}} A_n^1 + A_n^2 + 2A_n^3 + A_n^4 \text{ (say)}, \qquad (17)
\end{aligned}$$

and

$$I_n^2(\pi) = I_n^1(\pi) + 2\int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\}\{E_0 f_n(x) - f_0(x)\}\,\pi(x)\,dx$$

$$+ 2\int_{\mathbb{R}^d} \{E_n f_n(x) - E_0 f_n(x)\}\{E_0 f_n(x) - f_0(x)\}\,\pi(x)\,dx$$

$$+ \int_{\mathbb{R}^d} \{E_0 f_n(x) - f_0(x)\}^2\,\pi(x)\,dx$$

$$= I_n^1(\pi) + \frac{1}{\sqrt{n}\,h_n^{-m}} B_n^1 + 2B_n^2 + \int_{\mathbb{R}^d} \{E_0 f_n(x) - f_0(x)\}^2\,\pi(x)\,dx \text{ (say).} \tag{18}$$

As usual the idea is to provide expansions for each term of the decompositions.

B.1. *Expansion of* $I_n^1(\pi)$. Let us consider the decomposition (17).

(i) We get

$$A_n^1 = \frac{1}{n}\sum_{i=1}^{n}\{H_n(X_i, X_i) - E_n H_n(X_i, X_i)\}$$

$$+ \frac{2}{n}\sum_{1\leqslant j<i\leqslant n}\{H_n(X_i, X_j) - E_n H_n(X_i, X_j)\}$$

$$= \mathscr{H}_n(\pi) + o_p(1),$$

since we easily check that the second order moment of the first term is negligible using (16) and the absolute convergence of the mixing coefficients.

(ii) The second term is

$$A_n^2 = \frac{1}{nh_n^{d/2}}\left[E_n[H_n(X_0, X_0)] + \frac{2}{n}\sum_{1\leqslant j<i\leqslant n} E_n[H_n(X_i, X_j)]\right],$$

where

$$E_n[H_n(X_0, X_0)] = \frac{1}{h_n^{3d/2}}\int_{\mathbb{R}^d}\left\{E_n K^2\left(\frac{x - X_0}{h_n}\right) - E_n^2 K\left(\frac{x - X_0}{h_n}\right)\right\}\pi(x)\,dx$$

$$= \frac{1}{h_n^{d/2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)\, g_n(y)\,\pi(y + uh_n)\,dy\,du + O(h_n^{d/2}).$$

We now check that the covariance term is negligible. Firstly, from (16) and the assumptions on the pdf $g_n$ and on the cdf $f_{in}$ we conclude that, for $r>1$, there exists $C>0$ such that (cf. Tenreiro [24], pp. 206–207)

$$\max\{\max_{1\leqslant i\leqslant n} E_n^{1/r}\,|H_n(X_i, X_0)|^r, E_n^{1/r}\,|H_n(\bar{X}_0, X_0)|^r\}\leqslant C(h_n^d)^{1/r-1/2}.$$

From Lemma 1 of Yoshihara [27], we get

$$\begin{aligned}\left|\frac{1}{n}\sum_{1\leqslant j<i\leqslant n} E_n[H_n(X_i, X_j)]\right| &\leqslant \sum_{i=1}^{n-1} |E_n[H_n(X_i, X_0)]|\\ &\leqslant 4C(h_n^d)^{1/r-1/2}\sum_{i=1}^{n-1}\beta_n^{(r-1)/r}(i),\end{aligned}$$

where the RHS is negligible for $r<2$ from the assumption on the mixing coefficients.

Therefore

$$A_n^2=\frac{1}{nh_n^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(u)\, g_n(y)\,\pi(y+uh_n)\,dy\,du+O\left(\frac{1}{nh_n^{d/2}}\right).$$

(iii) The third term is

$$\begin{aligned}A_n^3&=\frac{a_n}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{h_n^d}\int_{\mathbb{R}^d}\left\{K\left(\frac{x-X_i}{h_n}\right)-E_nK\left(\frac{x-X_0}{h_n}\right)\right\} p_n(x)\,\pi(x)\,dx\\ &=\frac{a_n}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} L_n(X_i)\ \text{(say)},\end{aligned}$$

where

$$\begin{aligned}p_n(x)&=a_n^{-1}\{Ef_{nn}(x)-E_0 f_n(x)\}\\ &=\frac{1}{h_n^d}\int_{\mathbb{R}^d} K\left(\frac{x-y}{h_n}\right)(\gamma(y)+o(1)\,\gamma_n(y))\,dy,\end{aligned}$$

is uniformly bounded using the assumptions on $\gamma$ and $\gamma_n$. Then, since $L_n(\cdot)$ is uniformly bounded and the sequence of mixing coefficients is absolutely convergent, we have

$$A_n^3=O_p\left(\frac{a_n}{\sqrt{n}}\right).$$

(iv) Finally, from the dominated convergence theorem and the assumptions on $g_n$, we have, for $\gamma$ continuous,

$$
\begin{aligned}
A_n^4 &= \int_{\mathbb{R}^d} \{E_n f_n(x) - E_0 f_n(x)\}^2 \pi(x)\, dx \\
&= \int_{\mathbb{R}^d} \left\{ \frac{1}{h_n^d} \int_{\mathbb{R}^d} K\left(\frac{x-y}{h_n}\right) (g_n(y) - f_0(y))\, dy \right\}^2 \pi(x)\, dx \\
&= a_n^2 \int_{\mathbb{R}^d} \left\{ \frac{1}{h_n^d} \int_{\mathbb{R}^d} K\left(\frac{x-y}{h_n}\right) \gamma(y)\, dy \right\}^2 \pi(x)\, dx + o(a_n^2) \\
&= a_n^2 \int_{\mathbb{R}^d} \gamma^2(x)\, \pi(x)\, dx + o(a_n^2).
\end{aligned}
$$

Density arguments permit us to extend the previous expansion for $\gamma$ bounded and integrable on $\mathbb{R}^d$.

B.2. *Expansion of $I_n^2(\pi)$.* Let us consider the decomposition (18) and the equality (4).

(i) We have

$$
\begin{aligned}
2B_n^1 &= 2\sqrt{n}\, h_n^{-m} \int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\}\{E_0 f_n(x) - f_0(x)\}\, \pi(x)\, dx \\
&= 2\sqrt{n} \int_{\mathbb{R}^d} \{f_n(x) - E_n f_n(x)\} (\Delta_n^m f_0 \cdot \pi)(x)\, dx \\
&= 2\sqrt{n}\, \frac{1}{n h_n^d} \sum_{i=1}^n \int_{\mathbb{R}^d} \left\{ K\left(\frac{x-X_i}{h_n}\right) - E_n K\left(\frac{x-X_i}{h_n}\right) \right\} (\Delta_n^m f_0 \cdot \pi)(x)\, dx \\
&= \mathscr{G}_n(\pi).
\end{aligned}
$$

(ii) The same kind of arguments gives

$$
B_n^2 = h_n^m \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(u)(g_n(x) - f_0(x))(\Delta_n^m f_0 \cdot \pi)(x + u h_n)\, dx\, du.
$$

By taking into account all these expansions, we deduce the expansions of $I_n^1(\pi)$ and $I_n^2(\pi)$ given in Theorem 3.2.

*C. Consistency of the Test Procedures*

In what follows we establish the consistency of the tests based on critical regions $C_n^{j,k}(\pi)$ by proving that $T_n^{j,k}(\pi) \xrightarrow[n \to +\infty]{p} +\infty$, $j, k = 1, 2$, for a fixed alternative hypothesis $H_1$: $\{f = f_1\}$, with $f_1 \neq f_0$.

If $f_1$ is the density of the observed process $(X_i,\ i \in \mathbb{Z})$, we have

$$I_n^1(\pi) = 2 \int_{\mathbb{R}^d} \{f_n(x) - E_1 f_n(x)\}\{E_1 f_n(x) - E_0 f_n(x)\}\ \pi(x)\, dx$$

$$+ \int_{\mathbb{R}^d} \{f_n(x) - E_1 f_n(x)\}^2\, \pi(x)\, dx + \int_{\mathbb{R}^d} \{E_1 f_n(x) - E_0 f_n(x)\}^2\, \pi(x)\, dx,$$

where $E_1$ is the mathematical expectation under the hypothesis $H_1$. From points (ii) and (iv) of Appendix B, the first two terms converge to zero in probability, when $n \to +\infty$, and

$$\int_{\mathbb{R}^d} \{E_1 f_n(x) - E_0 f_n(x)\}^2\, \pi(x)\, dx = \int_{\mathbb{R}^d} \{f_1(x) - f_0(x)\}^2\, \pi(x)\, dx + o(1).$$

Therefore, for $k = 1, 2$ we have

$$\frac{T_n^{1,k}(\pi)}{nh_n^{d/2}} = I_n^1(\pi) - \frac{1}{nh_n^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K^2(u)\, f_0(x - uh_n)\, \pi(x)\, dx\, du$$

$$= \int_{\mathbb{R}^d} \{f_1(x) - f_0(x)\}^2\, \pi(x)\, dx + o_p(1).$$

If $f_0 \in W^d(m)$ and $K \in K^d(m)$, the same kind of arguments and the equality

$$\int_{\mathbb{R}^d} \{E_0 f_n(x) - f_0(x)\}^2\, \pi(x)\, dx = h_n^{2m} \int_{\mathbb{R}^d} (\Delta_n^m f_0(x))^2\, dx \qquad \text{(cf.(4))},$$

gives for $k = 1, 2$,

$$\frac{T_n^{2,k}(\pi)}{d(n)} = \int_{\mathbb{R}^d} \{f_1(x) - f_0(x)\}^2\, \pi(x)\, dx + o_p(1).$$

The conclusion follows from the convergence $nh_n^{d/2} \xrightarrow[n \to +\infty]{} +\infty$ and $d(n) \xrightarrow[n \to +\infty]{} +\infty$, and the fact that $\int_{\mathbb{R}^d} \{f_1(x) - f_0(x)\}^2\, \pi(x)\, dx > 0$, since the support of the weight function $\pi$ contains the support of $f_0$.

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