Journal of Multivariate Analysis 113 (2013) 153–160

Contents lists available at SciVerse ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Hazard rate comparison of parallel systems with heterogeneous gamma components[☆]

N. Balakrishnan[a,∗,1], Peng Zhao[b]

[a] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

[b] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

ARTICLE INFO

Article history:
Available online 12 May 2011

AMS 2000 subject classifications:
primary 60E15
secondary 60K10

Keywords:
Gamma distribution
Stochastic order
Hazard rate order
Order statistics
Parallel system

ABSTRACT

We compare the hazard rate functions of the largest order statistic arising from independent heterogeneous gamma random variables and that arising from i.i.d. gamma random variables. Specifically, let $X_1, \ldots, X_n$ be independent gamma random variables with X_i having shape parameter $0 < r \le 1$ and scale parameter λ_i, $i = 1, \ldots, n$. Denote by $Y_{n:n}$ the largest order statistic arising from i.i.d. gamma random variables $Y_1, \ldots, Y_n$ with Y_i having shape parameter r and scale parameter $\tilde{\lambda} = (\prod_{i=1}^{n} \lambda_i)^{1/n}$, the geometric mean of λ_i's. It is shown that $X_{n:n}$ is stochastically larger than $Y_{n:n}$ in terms of hazard rate order. The result derived here strengthens and generalizes some of the results known in the literature and leads to a sharp upper bound on the hazard rate function of the largest order statistic from heterogeneous gamma variables in terms of that of the largest order statistic from i.i.d. gamma variables. A numerical example is finally provided to illustrate the main result established here.

1. Introduction

Order statistics play a prominent role in statistical inference, reliability theory, life testing, operations research, and many other areas; see, for example, the two encyclopedic volumes by [2,3]. Denote by $X_{1:n} \le \cdots \le X_{n:n}$ the order statistics arising from random variables $X_1, \ldots, X_n$. Then, it is well-known that the kth order statistic $X_{k:n}$ corresponds to the lifetime of a $(n-k+1)$-out-of-n system, a very popular structure of redundancy in fault-tolerant systems in reliability theory that has been studied extensively in the literature. Series and parallel systems are the building blocks of more complex coherent systems, wherein the lifetime of a parallel system corresponds to the largest order statistic $X_{n:n}$ and the lifetime of a series system corresponds to the smallest order statistic $X_{1:n}$. Many authors have studied various aspects of order statistics when the observations are independent and identically distributed (i.i.d.). The case when observations are non-i.i.d., however, often arises naturally in different situations. Due to the complexity of the distribution theory in this case, limited work can be found in the literature; see, for example, [6,2,3], and the recent review article of Balakrishnan [1] for comprehensive discussions on the independent and non-identically distributed (i.ni.d.) case.

The exponential distribution has a nice mathematical form and the unique memoryless property and hence has widely been applied in many fields. Many papers have appeared on the stochastic comparison of order statistics arising from

☆ This work was supported by Natural Sciences and Engineering Research Council of Canada for the first author, and by National Natural Science Foundation of China (11001112), Research Fund for the Doctoral Program of Higher Education (20090211120019), and the Fundamental Research Funds for the Central Universities (lzujbky-2010-64) for the second author.

∗ Corresponding author.
E-mail addresses: bala@mcmaster.ca, bala@univmail.cis.mcmaster.ca (N. Balakrishnan), zhaop07@gmail.com (P. Zhao).

1 King Saud University, Faculty of Science (Riyadh, Saudi Arabia) and National Central University (Taiwan).

0047-259X/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2011.05.001

i.ni.d. exponential random variables including [19,20,12,7,9–11,5,13,14,17,18,22–27]. The gamma distribution has been used extensively in reliability and survival analysis due to its flexibility in shape and some nice distributional properties; for more details on this distribution, one may refer to Johnson et al. [8]. Assuming that X is a gamma random variable with shape parameter r and scale parameter λ, X has its probability density function as

$$f(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x > 0,\ r > 0,\ \lambda > 0.$$

It is an extremely flexible family of distributions with decreasing, constant, and increasing hazard rates when $0 < r < 1$, $r = 1$ and $r > 1$, respectively. This paper will focus on the largest order statistic arising from heterogeneous gamma variables, i.e., the lifetime of a parallel system with independent heterogeneous gamma components. The results established here extend the corresponding ones in the literature for the exponential case.

Let us first recall some notions of stochastic orders. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*. For two random variables X and Y with densities f_X and f_Y, and distribution functions F_X and F_Y, respectively, let $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ be the corresponding survival functions. X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $f_Y(x)/f_X(x)$ is increasing in x; X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in x; X is said to be smaller than Y in the stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\overline{F}_Y(x) \geq \overline{F}_X(x)$. It is well-known that the likelihood ratio order implies the hazard rate order which in turn implies the usual stochastic order. For a comprehensive discussion on various stochastic orderings, one may refer to Shaked and Shanthikumar [21].

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with hazard rate $\bar{\lambda} = \sum_{i=1}^n \lambda_i/n$, the arithmetic mean of $\lambda_i' s$, and denote by $Y_{n:n}$ the corresponding largest order statistic. Dykstra et al. [7] then showed that

$$X_{n:n} \geq_{\mathrm{hr}} Y_{n:n}, \tag{1}$$

which was further strengthened by Kochar and Xu [13] as

$$X_{n:n} \geq_{\mathrm{lr}} Y_{n:n}. \tag{2}$$

[10] also strengthened the result in (1), under a weaker condition, by proving that if $Z_1, \ldots, Z_n$ is a random sample of size n from an exponential distribution with hazard rate $\tilde{\lambda} = \left(\prod_{i=1}^n \lambda_i\right)^{\frac{1}{n}}$, the geometric mean of $\lambda_i' s$, then

$$X_{n:n} \geq_{\mathrm{hr}} Z_{n:n}. \tag{3}$$

Recently, Kochar and Xu [14] proved that the largest order statistic from heterogeneous exponential variables is more skewed in the sense of the convex transform order than that from homogeneous exponential variables, which is quite a general conclusion as there is no restriction on the parameters.

It is natural to ask whether and how the result in (3) can be extended from the exponential case to the gamma distribution. This paper confirms this result for the case when the shape parameter is at most 1. Specifically, let $X_1, \ldots, X_n$ be independent gamma random variables with X_i having shape parameter $0 < r \leq 1$ and scale parameter λ_i, $i = 1, \ldots, n$, and let $Z_1, \ldots, Z_n$ be a random sample of size n from a gamma distribution with shape parameter r and scale parameter $\tilde{\lambda} = \left(\prod_{i=1}^n \lambda_i\right)^{\frac{1}{n}}$. We then show that

$$X_{n:n} \geq_{\mathrm{hr}} Z_{n:n}, \tag{4}$$

thus generalizing and strengthening the corresponding result for the exponential case established earlier in the literature.

2. Main result

In this section, before presenting our main result, we first present several useful lemmas. The first one, due to Bon and Păltănea [5], plays an important role in establishing the main result which presents a sufficient condition to reach the maximum point of a symmetrical function in a compact set.

Lemma 1. *Let $\phi : \Re_+^n \to \Re_+$ be a symmetric and continuously differentiable mapping. If for any n-dimensional vector $\mathbf{y} = (y_1, \ldots, y_n) \in \Re_+^n$ with $y_p = \min y_i$ and $y_q = \max y_i$, we have*

$$(y_p - y_q)\left(\frac{\frac{\partial\phi(\mathbf{y})}{\partial y_p}}{\sum\limits_{i\neq p}^n y_i} - \frac{\frac{\partial\phi(\mathbf{y})}{\partial y_q}}{\sum\limits_{i\neq q}^n y_i}\right) < 0, \quad \textit{for } y_p \neq y_q,$$

then the following inequality holds:

$$\phi(y_1, \ldots, y_n) \leq \phi(\underbrace{\tilde{y}, \ldots, \tilde{y}}_{n}),$$

where $\tilde{y} = \left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}}$ *is the geometric mean of* $\mathbf{y} = (y_1, \ldots, y_n)$.

Lemma 2. *For* $0 < r \leq 1$ *and* $y \in \Re_+$, *the function*

$$f(x) = x + \frac{e^{-x}}{\int_0^1 u^{r-1} e^{-xu} du}$$

is increasing in $x \in \Re_+$.

Lemma 3. *For* $0 < r \leq 1$ *and* $y_i \in \Re_+ (1 \leq i \leq n)$, *we have*

$$\frac{\sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u} du}}{1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du} \geq y_p + \frac{e^{-y_p}}{\int_0^1 u^{r-1} e^{-y_p u} du},$$

where, as before, $y_p = \min y_i$.

The technical details of the proofs of Lemmas 2 and 3 are presented in the Appendix.

Theorem 1. *Let* $X_1, \ldots, X_n$ *be independent gamma random variables with* X_i *having shape parameter* $0 < r \leq 1$ *and scale parameter* λ_i, $i = 1, \ldots, n$, *and let* $Y_1, \ldots, Y_n$ *be a random sample of size n from a gamma distribution with shape parameter r and a common scale parameter* $\lambda \geq \tilde{\lambda} = \left(\prod_{i=1}^{n} \lambda_i\right)^{1/n}$. *Then,*

$$X_{n:n} \geq_{\text{hr}} Y_{n:n}.$$

Proof. Let Z_λ be the largest order statistic in a random sample of size n from a gamma distribution with common shape parameter r and scale parameter λ. Assume $\lambda < \mu$. We then have $Z_\lambda \geq_{\text{lr}} Z_\mu$ from Theorem 1.C.33 of [21] and thus $Z_\lambda \geq_{\text{hr}} Z_\mu$. Based on this fact, we only need to prove the result for the case when $\lambda = \tilde{\lambda}$. The density function of $X_{n:n}$ is

$$f_{X_{n:n}}(t) = \sum_{i=1}^{n} \frac{t^{r-1} e^{-\lambda_i t}}{\int_0^t u^{r-1} e^{-\lambda_i u} du} \prod_{j=1}^{n} \int_0^t \frac{\lambda_j^r}{\Gamma(r)} u^{r-1} e^{-\lambda_j u} du, \quad t > 0,$$

and so its hazard rate function is given by

$$\begin{aligned} r_{X_{n:n}}(t) &= \frac{f_{X_{n:n}}(t)}{\bar{F}_{X_{n:n}}(t)} \\ &= \frac{\prod_{j=1}^{n} \int_0^t \frac{\lambda_j^r}{\Gamma(r)} u^{r-1} e^{-\lambda_j u} du}{1 - \prod_{j=1}^{n} \int_0^t \frac{\lambda_j^r}{\Gamma(r)} u^{r-1} e^{-\lambda_j u} du} \sum_{i=1}^{n} \frac{t^{r-1} e^{-\lambda_i t}}{\int_0^t u^{r-1} e^{-\lambda_i u} du} \\ &= \frac{1}{t} \phi(\lambda_1 t, \ldots, \lambda_n t), \end{aligned}$$

where the symmetric function $\phi : \Re_+^n \to (0, 1)$ is defined as

$$\phi(y_1, \ldots, y_n) = \frac{\prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du}{1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du} \sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u} du}.$$

Similarly, the hazard rate function of $Y_{n:n}$ can be written as

$$r_{Y_{n:n}}(t) = \frac{\left[\int_0^t \frac{\tilde{\lambda}^r}{\Gamma(r)} u^{r-1} e^{-\tilde{\lambda} u} du\right]^n}{1 - \left[\int_0^t \frac{\tilde{\lambda}^r}{\Gamma(r)} u^{r-1} e^{-\tilde{\lambda} u} du\right]^n} \frac{n t^{r-1} e^{-\tilde{\lambda} t}}{\int_0^t u^{r-1} e^{-\tilde{\lambda} u} du} = \frac{1}{t} \phi(\underbrace{\tilde{\lambda} t, \ldots, \tilde{\lambda} t}_{n}).$$

To reach the desired result that $X_{n:n} \geq_{\text{hr}} Y_{n:n}$, it suffices to show that

$$\phi(y_1, \ldots, y_n) \leq \phi(\underbrace{\tilde{y}, \ldots, \tilde{y}}_{n}),$$

where $\tilde{y} = \left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}}$ for any vector $\mathbf{y} = (y_1, \ldots, y_n) \in \Re_+^n$. As before, let $y_p = \min y_i$ and $y_q = \max y_i$. Now, we observe

$$\begin{aligned}
\frac{\partial\phi(\mathbf{y})}{\partial y_p} &= \frac{\prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du}{1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du} \frac{e^{-y_p}\left[\int_0^1 u^r e^{-y_p u} du - \int_0^1 u^{r-1} e^{-y_p u} du\right]}{\left[\int_0^1 u^{r-1} e^{-y_p u} du\right]^2} \\
&\quad + \sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u} du} \frac{\prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du}{\left[1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du\right]^2} \frac{e^{-y_p}}{y_p \int_0^1 u^{r-1} e^{-y_p u} du} \\
&= \frac{\prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du}{1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du} \left[\frac{e^{-y_p}\left[\int_0^1 u^r e^{-y_p u} du - \int_0^1 u^{r-1} e^{-y_p u} du\right]}{\left[\int_0^1 u^{r-1} e^{-y_p u} du\right]^2} \right. \\
&\quad \left. + \sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u} du} \frac{1}{1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du} \frac{e^{-y_p}}{y_p \int_0^1 u^{r-1} e^{-y_p u} du} \right].
\end{aligned}$$

Since the function ϕ is permutation symmetric, each partial derivative has the same structure. On the other hand, by using integration by parts, we find

$$y \int_0^1 u^r e^{-yu} du = r \int_0^1 u^{r-1} e^{-yu} du - e^{-y}.$$

Thus, we have

$$\begin{aligned}
\frac{\frac{\partial\phi(\mathbf{y})}{\partial y_p}}{\sum_{i\neq p}^{n} y_i} - \frac{\frac{\partial\phi(\mathbf{y})}{\partial y_q}}{\sum_{i\neq q}^{n} y_i} &\stackrel{\text{sgn}}{=} r\left[\frac{e^{-y_p}}{\int_0^1 u^{r-1} e^{-y_p u} du} - \frac{e^{-y_q}}{\int_0^1 u^{r-1} e^{-y_q u} du}\right]\left[1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du\right] \\
&\quad + \frac{e^{-y_p}}{\int_0^1 u^{r-1} e^{-y_p u} du}\left[\sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u} du} - \left(y_p + \frac{e^{-y_p}}{\int_0^1 u^{r-1} e^{-y_p u} du}\right) \right. \\
&\quad \left. \times \left(1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du\right)\right] \\
&\quad - \frac{e^{-y_q}}{\int_0^1 u^{r-1} e^{-y_q u} du}\left[\sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u} du} - \left(y_q + \frac{e^{-y_q}}{\int_0^1 u^{r-1} e^{-y_q u} du}\right) \right. \\
&\quad \left. \times \left(1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du\right)\right] \\
&= \theta_1 + \theta_2, \quad \text{say,}
\end{aligned}$$

where '$\stackrel{\text{sgn}}{=}$' means equality of signs and

$$\theta_1 = r\left[\frac{e^{-y_p}}{\int_0^1 u^{r-1} e^{-y_p u} du} - \frac{e^{-y_q}}{\int_0^1 u^{r-1} e^{-y_q u} du}\right]\left[1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du\right]$$

and

$$\begin{aligned}
\theta_2 &= \frac{e^{-y_p}}{\int_0^1 u^{r-1} e^{-y_p u} du}\left[\sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u} du} - \left(y_p + \frac{e^{-y_p}}{\int_0^1 u^{r-1} e^{-y_p u} du}\right)\left(1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du\right)\right] \\
&\quad - \frac{e^{-y_q}}{\int_0^1 u^{r-1} e^{-y_q u} du}\left[\sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1} e^{-y_i u} du} - \left(y_q + \frac{e^{-y_q}}{\int_0^1 u^{r-1} e^{-y_q u} du}\right)\left(1 - \prod_{j=1}^{n} \int_0^1 \frac{y_j^r}{\Gamma(r)} u^{r-1} e^{-y_j u} du\right)\right].
\end{aligned}$$

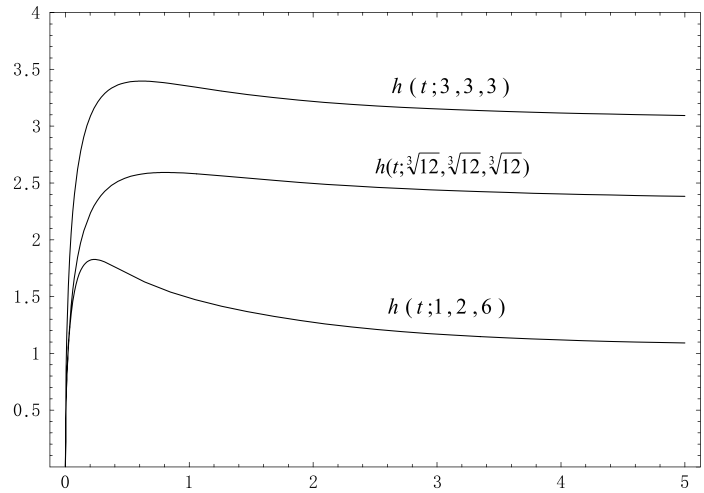

Fig. 1. Plots of the hazard rate functions when $r = 0.5$.

It is easy to see that the function

$$\frac{e^{-y}}{\int_0^1 u^{r-1}e^{-yu}du} = \frac{1}{\int_0^1 u^{r-1}e^{(1-u)y}du}$$

is decreasing in $y \in \Re_+$ which implies that $\theta_1 \geq 0$. Next, we will show that θ_2 is also nonnegative. From Lemma 3, it is known that

$$\sum_{i=1}^{n} \frac{e^{-y_i}}{\int_0^1 u^{r-1}e^{-y_i u}du} - \left(y_p + \frac{e^{-y_p}}{\int_0^1 u^{r-1}e^{-y_p u}du}\right)\left(1 - \prod_{j=1}^{n}\int_0^1 \frac{y_j^r}{\Gamma(r)}u^{r-1}e^{-y_j u}du\right) \geq 0,$$

and so we have

$$\theta_2 \geq \frac{e^{-y_q}}{\int_0^1 u^{r-1}e^{-y_q u}du}\left[\left(y_q + \frac{e^{-y_q}}{\int_0^1 u^{r-1}e^{-y_q u}du}\right) - \left(y_p + \frac{e^{-y_p}}{\int_0^1 u^{r-1}e^{-y_p u}du}\right)\right]\left[1 - \prod_{j=1}^{n}\int_0^1 \frac{y_j^r}{\Gamma(r)}u^{r-1}e^{-y_j u}du\right]$$
$$\geq 0,$$

where the last inequality is obtained by using Lemma 2. Now, we get

$$(y_p - y_q)\left(\frac{\frac{\partial\phi(\mathbf{y})}{\partial y_p}}{\sum_{i\neq p}^{n} y_i} - \frac{\frac{\partial\phi(\mathbf{y})}{\partial y_q}}{\sum_{i\neq q}^{n} y_i}\right) \leq 0,$$

and the desired result can then be derived from Lemma 1. Hence, the theorem. □

Remark 1. It is evident that the result in Theorem 1 generalizes and strengthens those in [7,10] from the exponential case to the gamma distribution. One may wonder here whether we could establish more general comparison results between two largest order statistics both of which arise from heterogeneous variables under some majorization (cf. [15]) type assumptions on the scale parameter vectors. For the special case when $n = 2$, such results have been obtained recently by Zhao and Balakrishnan [25] for the dispersive and the star orderings. Such results, however, do not hold in general for the case when $n > 2$; see, for example, [4] for a counterexample with respect to the hazard rate order.

As a direct consequence of Theorem 1, we can obtain an upper bound on the hazard rate function of $X_{n:n}$ from heterogeneous gamma variables in terms of the hazard rate function of $Y_{n:n}$ from an i.i.d. gamma sample. We now present a numerical example to illustrate this fact. Let (X_1, X_2, X_3) be a vector of independent heterogeneous gamma random variables with common shape parameter $r = 0.5$ and scale parameter vector $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 6)$, and let $h(t; 1, 2, 6)$ denote the hazard rate function of $X_{3:3}$. Let (Y_1, Y_2, Y_3) be an i.i.d. gamma random sample with common shape parameter 0.5 and scale parameter 3 (the arithmetic mean of (1, 2, 6)), and let $h(t; 3, 3, 3)$ denote the hazard rate function of $Y_{3:3}$. Let (Z_1, Z_2, Z_3) be an i.i.d. gamma random sample with common shape parameter 0.5 and scale parameter $\sqrt[3]{12}$ (the geometric mean of (1, 2, 6)), and let $h(t; \sqrt[3]{12}, \sqrt[3]{12}, \sqrt[3]{12})$ denote the hazard rate function of $Z_{3:3}$. Fig. 1 presents a plot of the hazard rate

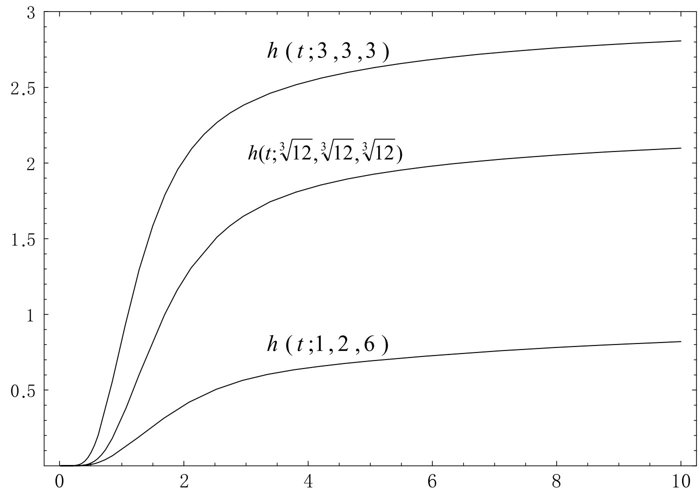

Fig. 2. Plots of the hazard rate functions when $r = 3$.

functions of these three largest order statistics, which can be seen to be in accordance with the result of Theorem 1. It can also be seen that the upper bound given by $h(t;\ \sqrt[3]{12},\ \sqrt[3]{12},\ \sqrt[3]{12})$ is better than that offered by $h(t;\ 3,\ 3,\ 3)$.

3. Concluding remarks

In this paper, we have discussed the hazard rate comparison between the largest order statistics from heterogeneous and homogeneous gamma random variables when the common shape parameter is at most 1. One natural question that arises is whether the result in Theorem 1 also holds for the case when the shape parameter is larger than 1 and it is possible that this may be true as shown in Fig. 2 (the hazard rate plots under the same setup as in Fig. 1, but the shape parameter is now 3). This remains as an open problem. Moreover, it would be interesting to see whether the result in Theorem 1 can be established for the likelihood ratio order. For the exponential case, such results have been obtained by Khaledi and Kochar [10] and Kochar and Xu [13]. We are currently working on these problems and hope to report these findings in a future paper.

Acknowledgments

The authors would like to thank the two anonymous referees for their valuable comments and suggestions, which resulted in an improvement in the presentation of this manuscript. The first author also acknowledges the support from King Saud University (Riyadh, Saudi Arabia) through project number KSU-VPP-107.

Appendix. Proofs of Lemmas 2 and 3

Proof of Lemma 2. Taking derivative with respect to x, we get

$$\begin{aligned} f'(x) &= 1 + \frac{-\mathrm{e}^{-x}\int_0^1 u^{r-1}\mathrm{e}^{-xu}\mathrm{d}u + \mathrm{e}^{-x}\int_0^1 u^r\mathrm{e}^{-xu}\mathrm{d}u}{\left[\int_0^1 u^{r-1}\mathrm{e}^{-xu}\mathrm{d}u\right]^2} \\ &= 1 - \frac{1}{\int_0^1 u^{r-1}\mathrm{e}^{x(1-u)}\mathrm{d}u} + \frac{\mathrm{e}^{-x}\int_0^1 u^r\mathrm{e}^{-xu}\mathrm{d}u}{\left[\int_0^1 u^{r-1}\mathrm{e}^{-xu}\mathrm{d}u\right]^2}. \end{aligned} \tag{5}$$

It is evident that the last term on the RHS in (5) is nonnegative, and so it is enough if we show that

$$1 - \frac{1}{\int_0^1 u^{r-1}\mathrm{e}^{x(1-u)}\mathrm{d}u}$$

is also nonnegative, which is true upon noting that

$$\begin{aligned} 1 - \frac{1}{\int_0^1 u^{r-1}\mathrm{e}^{x(1-u)}\mathrm{d}u} &\geq 1 - \frac{1}{\int_0^1 u^{r-1}\mathrm{d}u} \\ &= 1 - r \\ &\geq 0. \quad \square \end{aligned}$$

We will need the following three propositions to prove Lemma 3.

Proposition 1 ([16]). *For real numbers α_i, β_i and λ_i, $i = 1, \ldots, n$, if $\beta_i > 0$ and $\lambda_i > 0$, then*

$$\min_{1\le i\le n}\left\{\frac{\alpha_i}{\beta_i}\right\} \le \frac{\sum_{i=1}^{n}\alpha_i\lambda_i}{\sum_{i=1}^{n}\beta_i\lambda_i} \le \max_{1\le i\le n}\left\{\frac{\alpha_i}{\beta_i}\right\}.$$

Equalities hold if and only if $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are proportional.

Proposition 2. *For $x_i \in (0, 1)$, $1 \le i \le n$, the following inequality holds:*

$$1 - \prod_{i=1}^{n}(1 - x_i) \le \sum_{i=1}^{n} x_i.$$

Proof. The proof is carried out by induction. The result is trivially true for the case when $n = 1$. Now let us assume that the inequalities hold for all $m(1 \le m < n)$. We then have

$$\begin{aligned} 1 - \prod_{i=1}^{m+1}(1 - x_i) &= 1 - \prod_{i=1}^{m}(1 - x_i)(1 - x_{m+1}) \\ &= 1 - \prod_{i=1}^{m}(1 - x_i) + x_{m+1}\prod_{i=1}^{m}(1 - x_i) \\ &\le \sum_{i=1}^{m} x_i + x_{m+1} \\ &= \sum_{i=1}^{m+1} x_i, \end{aligned}$$

which is the required result. □

Proposition 3. *For $0 < r \le 1$ and $y_i \in \Re_+(1 \le i \le n)$, we have*

$$1 - \prod_{i=1}^{n}\int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u} du \le \sum_{i=1}^{n}\frac{e^{-y_i}}{y_i\int_0^1 u^{r-1}e^{-y_i u}du + e^{-y_i}}. \tag{6}$$

Proof. It can be readily seen that the left hand side of (6) is nonnegative since each term in the product is no more than 1. From Proposition 2, it follows that

$$1 - \prod_{i=1}^{n}\int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u} du \le \sum_{i=1}^{n}\left(1 - \int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u} du\right).$$

So it suffices to show that, for each $y_i \in \Re_+$,

$$1 - \int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u} du \le \frac{e^{-y_i}}{y_i\int_0^1 u^{r-1}e^{-y_i u}du + e^{-y_i}},$$

which is equivalent to showing that

$$\int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u} du + \frac{y_i^{r-1}}{\Gamma(r)} e^{-y_i} \ge 1.$$

This is seen to be true upon observing that

$$\int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u} du + \frac{y_i^{r-1}}{\Gamma(r)} e^{-y_i} \ge \int_0^1 \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u} du + \int_1^\infty \frac{y_i^r}{\Gamma(r)} u^{r-1} e^{-y_i u} du = 1. \quad \square$$

Proof of Lemma 3. From Proposition 3, we have

$$\frac{\sum_{i=1}^{n}\frac{e^{-y_i}}{\int_0^1 u^{r-1}e^{-y_i u}du}}{1-\prod_{j=1}^{n}\int_0^1\frac{y_j^r}{\Gamma(r)}u^{r-1}e^{-y_j u}du} \geq \frac{\sum_{i=1}^{n}\frac{e^{-y_i}}{\int_0^1 u^{r-1}e^{-y_i u}du}}{\sum_{i=1}^{n}\frac{e^{-y_i}}{y_i\int_0^1 u^{r-1}e^{-y_i u}du+e^{-y_i}}}. \tag{7}$$

On the other hand, upon using Proposition 1 and Lemma 2, we have

$$\frac{\sum_{i=1}^{n}\frac{e^{-y_i}}{\int_0^1 u^{r-1}e^{-y_i u}du}}{\sum_{i=1}^{n}\frac{e^{-y_i}}{y_i\int_0^1 u^{r-1}e^{-y_i u}du+e^{-y_i}}} \geq y_p + \frac{e^{-y_p}}{\int_0^1 u^{r-1}e^{-y_p u}du}. \tag{8}$$

Now, upon combining (7) and (8), we obtain the required result. □

References

[1] N. Balakrishnan, Permanents, order statistics, outliers, and robustness, Revista Matemática Complutense 20 (2007) 7–107.
[2] N. Balakrishnan, C.R. Rao (Eds.), Handbook of Statistics, in: Order Statistics: Theory and Methods, vol. 16, Elsevier, Amsterdam, 1998.
[3] N. Balakrishnan, C.R. Rao (Eds.), Handbook of Statistics, in: Order Statistics: Applications, vol. 17, Elsevier, Amsterdam, 1998.
[4] P.J. Boland, E. EL-Neweihi, F. Proschan, Schur properties of convolutions of exponential and geometric random variables, Journal of Multivariate Analysis 48 (1994) 157–167.
[5] J.L. Bon, E. Păltănea, Comparisons of order statistics in a random sequence to the same statistics with i.i.d. variables, ESAIM: Probability and Statistics 10 (2006) 1–10.
[6] H.A. David, H.N. Nagaraja, Order Statistics, 3rd ed., John Wiley & Sons, Hoboken, New Jersey, 2003.
[7] R. Dykstra, S.C. Kochar, J. Rojo, Stochastic comparisons of parallel systems of heterogeneous exponential components, Journal of Statistical Planning and Inference 65 (1997) 203–211.
[8] N.L. Johnson, S. Kotz, N. Balakrishnan, Continuous Univariate Distributions, second ed., vol. 1, John Wiley & Sons, New York, 1994.
[9] S. Khaledi, S. Farsinezhad, S.C. Kochar, Stochastic comparisons of order statistics in the scale models, Journal of Statistical Planning and Inference 141 (2011) 276–286.
[10] B. Khaledi, S.C. Kochar, Some new results on stochastic comparisons of parallel systems, Journal of Applied Probability 37 (2000) 283–291.
[11] B. Khaledi, S.C. Kochar, Stochastic orderings of order statistics of independent random variables with different scale parameters, Communications in Statistics—Theory and Methods 36 (2007) 1441–1449.
[12] S.C. Kochar, J. Rojo, Some new results on stochastic comparisons of spacings from heterogeneous exponential distributions, Journal of Multivariate Analysis 59 (1996) 272–281.
[13] S.C. Kochar, M. Xu, Stochastic comparisons of parallel systems when components have proportional hazard rates, Probability in the Engineering and Informational Sciences 21 (2007) 597–609.
[14] S.C. Kochar, M. Xu, Comparisons of parallel systems according to the convex transform order, Journal of Applied Probability 46 (2009) 342–352.
[15] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.
[16] D.S. Mitrinović, P.M. Vasić, Analytic Inequlities, Springer-Verlag, Berlin, 1970.
[17] E. Păltănea, On the comparison in hazard rate ordering of fail-safe systems, Journal of Statistical Planning and Inference 138 (2008) 1993–1997.
[18] E. Păltănea, Bounds for mixtures of order statistics from exponentials and applications, Journal of Multivariate Analysis 102 (2011) 896–907.
[19] P. Pledger, F. Proschan, Comparisons of order statistics and of spacings from heterogeneous distributions, in: J.S. Rustagi (Ed.), Optimizing Methods in Statistics, Academic Press, New York, 1971, pp. 89–113.
[20] F. Proschan, J. Sethuraman, Stochastic comparisons of order statistics from heterogeneous populations, with applications in reliability, Journal of Multivariate Analysis 6 (1976) 608–616.
[21] M. Shaked, J.G. Shanthikumar, Stochastic Orders, Springer-Verlag, New York, 2007.
[22] L. Sun, X. Zhang, Stochastic comparisons of order statistics from gamma distributions, Journal of Multivariate Analysis 93 (2005) 112–121.
[23] P. Zhao, N. Balakrishnan, Some characterization results for parallel systems with two heterogeneous exponential components, Statistics (2010) doi:10.1080/02331888.2010.485276.
[24] P. Zhao, N. Balakrishnan, MRL ordering of parallel systems with two heterogeneous components, Journal of Statistical Planning and Inference 141 (2011) 631–638.
[25] P. Zhao, N. Balakrishnan, New results on comparisons of parallel systems with heterogeneous gamma components, Statistics & Probability Letters 81 (2011) 36–44.
[26] P. Zhao, X. Li, N. Balakrishnan, Conditional ordering of k-out-of-n systems with independent but non-identical components, Journal of Applied Probability 45 (2008) 1113–1125.
[27] P. Zhao, X. Li, N. Balakrishnan, Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables, Journal of Multivariate Analysis 100 (2009) 952–962.