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Transformations with Improved Chi-Squared Approximations

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Suppose that a nonnegative statistic T is asymptotically distributed as a chi-squared distribution with f degrees of freedom, χ_f^2, as a positive number n tends to infinity. Bartlett correction $\tilde{T}$ was originally proposed so that its mean is coincident with the one of χ_f^2 up to the order $O(n^{-1})$. For log-likelihood ratio statistics, many authors have shown that the Bartlett corrections are asymptotically distributed as χ_f^2 up to $O(n^{-1})$, or with errors of terms of $O(n^{-2})$. Bartlett-type corrections are an extension of Bartlett corrections to other statistics than log-likelihood ratio statistics. These corrections have been constructed by using their asymptotic expansions up to $O(n^{-1})$. The purpose of the present paper is to propose some monotone transformations so that the first two moments of transformed statistics are coincident with the ones of χ_f^2 up to $O(n^{-1})$. It may be noted that the proposed transformations can be applied to a wide class of statistics whether their asymptotic expansions are available or not. A numerical study of some test statistics that are not a log-likelihood ratio statistic is discribed. It is shown that the proposed transformations of these statistics give a larger improvement to the chi-squared approximation than do the Bartlett corrections. Further, it is seen that the proposed approximations are comparable with the approximation based on an Edgeworth expansion.

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1. INTRODUCTION

Suppose that a nonnegative statistic T is asymptotically distributed as a chi-squared distribution χ_f^2 with f degrees of freedom, as a positive number n tends to infinity. The Bartlett correction $\tilde{T}$ was originally proposed so that its mean is coincident with the one of χ_f^2 up to the order $O(n^{-1})$. This fact can be stated more concretely as follows: Suppose that the mean of T can be expanded as

$$E(T) = f\{1 + n^{-1}c_1 + O(n^{-2})\}. \tag{1.1}$$

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Then, the Bartlett correction is defined by

$$\tilde{T} = (1 - n^{-1} c_1)\, T, \qquad (n > c_1). \tag{1.2}$$

which satisfies

$$E(\tilde{T}) = f + O(n^{-2}). \tag{1.3}$$

Further, if T is a log-likelihood ratio statistic, under an appropriate condition it is known that

$$P(\tilde{T} \leqslant x) = G_f(x) + O(n^{-2}). \tag{1.4}$$

For a general discussion of the above result, see, e.g., Lawley (1956), Barndorff-Nielsen and Cox (1984), Barndorff-Nielsen and Hall (1988), etc. The correction factor $1 - n^{-1} c_1$ for various multivariate test statistics can be found in the books on multivariate analysis, e.g., Anderson (1984), Muirhead (1982), Siotani, Hayakawa and Fujikoshi (1985).

On the other hand, there exist some test statistics such that their Bartlett corrections do work in the sense of (1.3), but do not work in the sense of (1.4). In this case, one way is to find other transformation of T satifying the property (1.4), which is called Bartlett-type correction. Especially, it is important to solve this problem by a monotone transfomation. For this study, Fujikoshi (1997), Fujisawa (1997) and Kakizawa (1996) have proposed some transformations, assuming that T can be expanded as

$$P(T \leqslant x) = G_f(x) + \frac{1}{n} \sum_{j=0}^{k} a_j G_{f+2j}(x) + O(n^{-2}), \tag{1.5}$$

where k is a positive integer. Here the coefficients a_j's do not depend on the parameter $n > 0$ and must satisfy a relation $\sum_{j=0}^{k} a_j = 0$, which can be derived by letting $x \to \infty$ in (1.5).

It may be noted that Bartlett-type correction for a statistic with (1.5) depends on the coefficients a_j, and in some cases the coefficients a_j are unknown and must be estemated in a practical use. Further, we often encounter the situations where it is difficult to obtain the coefficients a_j in an expansion (1.5), even though its existence is assured. These situations appear in treating the distributions of multivariate test statistics under nonnormality. In order to overcome these difficulties, in this paper we propose some monotone transformations so that the first two moments of transformed statistics are coincident with the ones of χ_f^2 up to $O(n^{-1})$. So, such transformations give an improvement to the chi-squared approximation than does a Bartlett correction. For this purpose, we need to obtain the second moment of T as in an expanded form,

$$E(T^2) = f(f+2)\{1 + n^{-1}c_2 + O(n^{-2})\}, \tag{1.6}$$

adding to (1.1). The proposed transformations depend on the coefficients c_1 and c_2. In general, the problem of deriving (1.1) and (1.6) is more tractable than the one of deriving (1.5). Similarly, the problem of estimating the coefficient c_1 and c_2 is simpler than the one of estimating the coefficients a_j. Further, it is seen that the proposed approximations are comparable with the approximation based on an Edgeworth expansion. Therefore, it may be noted that the proposed transformations could be applied to a wide class of statistics whether their asymptotic expansions are available or not.

The present paper is organized as in the following way. In Section 2 we introduce monotone transformations beyond Bartlett correction. In Section 3 we give some distributional properties of our transformations when T has an asymptotic expansion (1.5). In Section 4 a numerical study of some test statistics that are not a log-likelihood ratio statistic is discribed. It is shown that the proposed transformations of these statistics give a larger improvement to the chi-squared approximation than do the Bartlett corrections. Further, it is shown that the proposed approximations are comparable with the approximation based on an Edgeworth expansion.

2. NEW TRANSFORMATIONS

For a nonnegative statistic T whose asymptotic distribution is χ^2_f, we assume that the first and second moments are expanded as in (1.1) and (1.6), respectively. Then, for the Bartlett statistic $\tilde{T}$ it holds that

$$E(\tilde{T}^2) = f(f+2)\{1 + n^{-1}\tilde{c}_2 + O(n^{-2})\}, \tag{2.1}$$

where

$$\tilde{c}_2 = c_2 - 2c_1. \tag{2.2}$$

Therefore, if $\tilde{c}_2 = 0$, we have that the differences between the first two moments of the Bartlett correction and χ^2_f are $O(n^{-2})$. However, if $\tilde{c}_2 \neq 0$, in order to keep such an optimum property we need to consider other transformations beyond the Bartlett correction.

Now we consider the case $\tilde{c}_2 \neq 0$. Let us consider the following transformations $Y_i = Y_i(T)$, $i = 1, 2, 3$;

$$
\begin{aligned}
&\text{(i)} \quad \text{For } \alpha > 0 \text{ and } n\alpha + \beta > 0; \\
&\qquad Y_1 = (n\alpha + \beta) \log \left(1 + \frac{1}{n\alpha} T \right), \\
&\text{(ii)} \quad \text{For } \alpha < 0 \text{ and } n\alpha + \beta < 0; \\
&\qquad Y_2 = T + \frac{1}{n} \left(\frac{\beta}{\alpha} T - \frac{1}{2\alpha} T^2 \right), \\
&\text{(iii)} \quad \text{For any } \alpha, n \text{ and } \beta; \\
&\qquad Y_3 = (n\alpha + \beta) \left\{ 1 - \exp \left(-\frac{1}{n\alpha} T \right) \right\}.
\end{aligned}
\tag{2.3}
$$

The transformations Y_1 and Y_2 have been used by Fujikoshi (1997), relating to Bartlett-type corrections for the statistic which has an expansion (1.5) with $k = 2$. The transformation Y_3 is a modification of Y_1, based on a relation of

$$\log(1 + t) - \{1 - \exp(-t)\} = O(t^3).$$

Note that Y_1 and Y_2 are monotone increasing functions when the parameters α and β satify the parameter restrictions of (i) and (ii), respectively. On the other hand, Y_3 is always a motone function. Further, note that asymptotic expansions of Y_i's are the same up to $O(n^{-1})$, and they are given by

$$Y_i = T + \frac{1}{n} \left\{ \alpha^{-1} \beta T - \frac{1}{2} \alpha^{-1} T^2 \right\} + O(n^{-2}). \tag{2.4}$$

Using (2.4), we have for $i = 1, 2, 3$,

$$E(Y_i) = f \left[1 + \frac{1}{n} \{ c_1 + \alpha^{-1} \beta - (2\alpha)^{-1} (f + 2) \} + O(n^{-2}) \right] \tag{2.5}$$

and

$$E(Y_i^2) = f(f + 2) \left[1 + \frac{1}{n} \{ c_2 + 2\alpha^{-1} \beta - \alpha^{-1} (f + 4) \} + O(n^{-2}) \right]. \tag{2.6}$$

In order to make the terms of order n^{-1} in (2.5) and (2.6) vanish, we need to choose α and β as

$$\alpha = 2/\tilde{c}_2, \ \beta = \tfrac{1}{2}\{(f+2)\,c_2 - 2(f+4)\,c_1\}/\tilde{c}_2. \tag{2.7}$$

These results can be summarized as a theorem as follows:

THEOREM 1. *Suppose that a nonnegative random variate T has an asymptotic chi-sqared distribution with f degrees of freedom, and its first two moments are expanded as in* (1.1) *and* (1.6). *For the case $\tilde{c}_2 = c_2 - c_1 \neq 0$, let Y_i, $i = 1, 2, 3$ be the transformations* (2.3) *with the quantities α and β defined by* (2.7). *Then, it holds that Y_i's are monotone functions of T and*

$$E(Y_i) = f + O(n^{-2}), \qquad E(Y_i^2) = f(f+2) + O(n^{-2}). \tag{2.8}$$

From Theorem 1 it is suggested to use Y_1 or Y_2, depending on the signs of α and β, i.e.,

$$T_{1\cdot 2}(u) = \begin{cases} T_1(u) & \text{if} \quad \alpha > 0 \text{ and } \alpha n + \beta > 0, \\ T_2(u) & \text{if} \quad \alpha < 0 \text{ and } \alpha n + \beta < 0, \end{cases} \tag{2.9}$$

or to use Y_3 for any α and β. Note that

$$Y_1 \geqslant Y_3 \tag{2.10}$$

when $\alpha > 0$ and $n\alpha + \beta > 0$. This follows from the fact that $\log(1+t) \geqslant 1 - \exp(-t)$ for $t > 0$.

Let $t(u)$ be a function of u defined by a relation

$$P(T \leqslant t(u)) = P(\chi_f^2 \leqslant u). \tag{2.11}$$

Note that $P(T \leqslant t(u)) = P(Y_i(T) \leqslant Y_i(t(u)))$ and the distribution of $Y_i(T)$ is close to a chi-squared distribution χ_f^2 in the sense of (2.8). This suggests that an approximation $t_i(u)$ for $t(u)$ may be proposed by $Y_i(t_i(u)) = u$. Since $t_i(u)$ is an inverse function of Y_i, we can express these approximations as follows:

$$\begin{aligned} t_1(u) &= n\alpha[\exp(u/(n\alpha+\beta)) - 1], \qquad (\alpha > 0,\ n\alpha + \beta > 0), \\ t_2(u) &= n\alpha[1 + \beta/(n\alpha) - \{(1+\beta/(n\alpha))^2 - 2u/(n\alpha)\}^{1/2}], \\ &\qquad (\alpha < 0,\ n\alpha + \beta < 0), \\ t_3(u) &= -n\alpha \log\{1 - u/(n\alpha+\beta)\}, \qquad (|u/(n\alpha+\beta)| < 1). \end{aligned} \tag{2.12}$$

Here we note that the asymptotic expansions of $t_i(u)$, $i=1, 2, 3$ up to the order $O(n^{-1})$ are the same, and they are given by

$$t_i(u)=u+\frac{1}{n}\left\{\alpha^{-1}\beta u-\frac{1}{2}\alpha^{-1}u^2\right\}+O(n^{-2}).$$

The accuracy of the approximations $t_i(u)$ to the true percent point $t(u)$ of T can be evaluated by using

$$P(T\leqslant t_i(u))=P(Y_i(T)\leqslant Y_i(t(u)))=P(Y_i\leqslant u). \tag{2.13}$$

Using the same argument, we can write the approximation for $t(u)$ based on Bartlett correction $\tilde{T}$ as

$$\tilde{T}(u)=u/(1-n^{-1}c_1),\ (n>c). \tag{2.14}$$

3. FURTHER PROPERTIES

In this section we study some distributional properties of the transformed statistics $Y_i=Y_i(T)$ when a statistic T can be expanded as in (1.5), in addition to the assumptions of Theorem 1. Especially we examine how much the distributions of Y_i are simplified and close to the distribution of χ^2_f. Before we treat the distributions of Y_i, we give the α and β in (2.7) in terms of the coefficients a_j. Using (1.5) we have

$$E(T)=f+\frac{1}{n}\sum_{j=0}^{k}a_j(f+2j)+O(n^{-2})$$

and

$$E(T^2)=f(f+2)+\frac{1}{n}\sum_{j=0}^{k}a_j(f+2j)(f+2j+2)+O(n^{-2}).$$

Noting that $\sum_{j=0}^{k}a_j=0$, we obtain

$$c_1=\frac{2}{f}\sum_{j=1}^{k}ja_j, \qquad c_2=\frac{4}{f(f+2)}\sum_{j=1}^{k}j(f+j+1)\,a_j. \tag{3.1}$$

For the case $k=1$, $\tilde{c}_2=0$, and hence the Bartlett correction $\tilde{T}$ yields an improvement on approximation of the second moment as well as the first moment of χ_f^2. Further, we can get (1.4). So, we consider the case $k \geqslant 2$. Then

$$\tilde{c}_2 = \frac{4}{f(f+2)} \sum_{j=2}^{k} (j-1)\, ja_j, \tag{3.2}$$

and hence the α and β in (2.7) are given by

$$\alpha = \tfrac{1}{2} f(f+2) \Big/ \left\{ \sum_{j=2}^{k} (j-1)\, ja_j \right\}^{-1},$$

$$\beta = \tfrac{1}{2}(f+2) \sum_{j=1}^{k} (j-3)\, ja_j \Big/ \left\{ \sum_{j=2}^{k} (j-1)\, ja_j \right\}^{-1}. \tag{3.3}$$

Now we consider asymptotic expansions of the distributions of Y_1, Y_2 and Y_3 with an eror term of $O(n^{-2})$. For the purpose, from (2.4) we may deal with

$$Y = Y_2 = T + \frac{1}{n} \left(\frac{\beta}{\alpha} T - \frac{1}{2\alpha} T^2 \right).$$

The characteristic function of Y can be expanded as

$$\begin{aligned} C(t) &= E[e^{itY}] \\ &= E\left[e^{itT} \left\{ 1 + \frac{it}{n} \left(\frac{\beta}{\alpha} T - \frac{1}{2\alpha} T^2 \right) \right\} \right] + O(n^{-2}) \\ &= (1-2it)^{-f/2} \left[1 + \frac{1}{n} \sum_{j=0}^{k} a_j (1-2it)^{-j} \right] \\ &\quad + \frac{it}{n} E\left[e^{itT} \left(\frac{\beta}{\alpha} T - \frac{1}{2\alpha} T^2 \right) \right] + O(n^{-2}). \end{aligned}$$

Note that

$$E[Te^{itT}] = f(1-2it)^{-f/2-1} + O(n^{-1}),$$

$$E[T^2 e^{itT}] = f(f+2)(1-2it)^{-f2-2} + O(n^{-1}).$$

Using these results, we have

$$C(t) = (1-2it)^{-f/2} \left[1 + \frac{1}{n} \sum_{j=0}^{k} \tilde{a}_j (1-2it)^{-j} + O(n^{-2}) \right], \tag{3.4}$$

where

$$
\begin{aligned}
\tilde{a}_0 &= a_0 - \tfrac{1}{2} \sum_{j=1}^{k} j(j-3)\, a_j, \\
\tilde{a}_1 &= a_1 + \tfrac{1}{2} \sum_{j=1}^{k} j(j-3)\, a_j + \tfrac{1}{2} \sum_{j=2}^{k} (j-1)\, ja_j, \\
\tilde{a}_2 &= a_2 - \tfrac{1}{2} \sum_{j=2}^{k} (j-1)\, ja_j, \\
\tilde{a}_j &= a_j \ (j \geqslant 3).
\end{aligned} \tag{3.5}
$$

Inverting (3.4) we can obtain the following theorem:

THEOREM 2. *Suppose that a nonnegative random variate T has an asymptotic expansion* (1.5), *and its first two moments can be expanded as in* (1.1) *and* (1.6). *Assume that $k \geqslant 2$ and $\sum_{j=2}^{k} (j-1)\, ja_j > 0$. Then, neglecting the terms of $O(n^{-2})$, Y_i's have the same asymptotic expansion, which is given by*

$$P(Y_i \leqslant x) = G_f(x) + \frac{1}{n} \sum_{j=0}^{k} \tilde{a}_j G_{f+2j}(x) + O(n^{-2}), \tag{3.6}$$

where the coefficients $\tilde{a}_j$ are given by (3.5).

Theorem 2 shows that the differences between the asymptotic expansions for T and Y_i appear in only the first three coefficients a_j, $\tilde{a}_j$, $j = 0, 1, 2$. Further, we can see that the asymptotic expansions for Y_i in the cases $k = 2$ and 3 are considerably simple, and are close to the distribution of χ^2_f. In fact,

(i) The case $k = 2$; $\tilde{a}_j = 0$, $j = 0, 1, 2$, and

$$P(Y_i \leqslant x) = G_f(x) + O(n^{-2}). \tag{3.7}$$

(ii) The case $k = 3$; $\tilde{a}_0 = -a_3$, $\tilde{a}_1 = 3a_3$, $\tilde{a}_2 = -3a_3$, $\tilde{a}_3 = a_3$, and

$$
\begin{aligned}
P(Y_i \leqslant x) &= G_f(x) \\
&\quad + \frac{a_3}{n} \{ -G_f(x) + 3G_{f+2}(x) - 3G_{f+4}(x) + G_{f+6} \} + O(n^{-2}) \\
&= G_f(x) + \frac{2a_3}{nf}\, g_f(x) \left\{ -x + \frac{2}{f+2}\, x^2 \right. \\
&\quad \left. + \frac{3}{(f+2)(f+4)}\, x^3 \right\} + O(n^{-2}),
\end{aligned} \tag{3.8}
$$

where $g_f(x)$ is the probability density function of χ^2_f.

It may be noted that the transformations Y_i in (2.3) have removed the terms of $O(n^{-1})$ for a statistic which has an asymptotic expansion (1.5) with $k=2$. For $k=3$, we have a simple asymptotic expansion (3.8) for the distribution of Y_i, which becomes more close to the chi-squared distribution as a_3 becomes to zero.

4. SOME APPLICATIONS

In this section we give the transformations (2.3), and examine the accuracy of the approximations (2.11), for some statistics.

EXAMPLE 1. Consider chi-squared approximations for the distribution of $T=\chi_q^2/(\chi_n^2/n)$. It is known (see, e.g., Siotani (1956)) that T has an asymptotic expansion (1.5) with $k=2$, $f=q$ and the coefficients given by

$$a_0=-\tfrac{1}{4}q(q-2), \qquad a_1=\tfrac{1}{2}q^2, \qquad a_2=-\tfrac{1}{4}q(q+2).$$

Therefore, the quantities c_1 and c_2 in (1.1) and (1.6) are given by $c_1=2$ and $c_2=6$. The Bartlett correction is given by $\tilde{T}=(1-2n^{-1})\,T$, which gives an approximation

$$\tilde{t}(u)=u/(1-2n^{-1}) \tag{4.1}$$

for $t(u)$ in (2.11).

Since $\tilde{c}_2\neq 0$, we can define the quantities α and β in (2.7) or (3.3) as

$$\alpha=1, \qquad \beta=\tfrac{1}{2}(q-2).$$

Therefore, we can use the transformations Y_1 and Y_3 which are given by

$$\begin{aligned} Y_1&=\left\{n+\frac{1}{2}(q-2)\right\}\log\left(1+\frac{1}{n}T\right),\\ Y_3&=\left\{n+\frac{1}{2}(q-2)\right\}\left\{1-\exp\left(-\frac{1}{n}T\right)\right\}. \end{aligned} \tag{4.2}$$

Note that from Theorem 1 the Y_1 and Y_2 have an asymptotic expansion as in (3.7). The approximations (2.12) for $t(u)$ are given by

$$\begin{aligned} t_1(u)&=n[\exp\{u/(n+\tfrac{1}{2}(q-2))\}-1],\\ t_3(u)&=-n\log\{1-u/(n+\tfrac{1}{2}(q-2))\}. \end{aligned} \tag{4.3}$$

TABLE I

The exact percentile $t(u)$ and four approximations u, $\tilde{t}(u)$, $t_1(u)$, $t_3(u)$ for the upper 5% points

Parameters		Upper 5% points				
q	n	$t(u)$	u	$\tilde{t}(u)$	$t_1(u)$	$t_3(u)$
3	8	12.21	7.81	10.41	12.05	20.09
3	12	10.47	7.81	9.37	10.41	11.76
5	13	15.15	11.07	13.08	14.89	18.74
5	20	13.55	11.07	12.3	13.47	14.46

It is known (Fujikoshi and Mukaihata (1993)) that $t_1(u)$ satisfies a nice property of

$$\begin{aligned} t_1(u) &\geqslant t(u) \qquad (0<q<2),\\ t_1(u) &= t(u) \qquad (q=2),\\ t_1(u) &\leqslant t(u) \qquad (q>2). \end{aligned} \tag{4.4}$$

From Table I it is seen that the approximation $t_1(u)$ is very accurate, but the approximation $t_3(u)$ is not so good when the value of $u/(n+\frac{1}{2}(q-2))$ is near to one.

Example 2. The null distribution of Lawley–Hotelling trace criterion in a MANOVA problem is expressed as $T=n \operatorname{tr} S_h S_e^{-1}$, where S_e and S_h are independently distributed as Wishart distributions $W_p(I_p, n)$ and $W_p(I_p, q)$, respectively. It is known (see, e.g., Ito (1956), Siotani (1956)) that T has an asymptotic expansion (1.5) with $k=2$, $f=pq$ and the coefficients given by

$$a_0=\tfrac{1}{4}pq(q-p-1), \qquad a_1=-\tfrac{1}{2}pq^2, \qquad a_2=\tfrac{1}{4}pq(q+p+1).$$

This is an extension of Example 1. The quantity c_1 is given by $c_1=p+1$. The Bartlett correction is given by $\tilde{T}=\{1-(p+1)\,n^{-1}\}\,T$, which gives an approximation

$$\tilde{t}(u)=u/\{1-(p+1)\,n^{-1}\} \tag{4.5}$$

for $t(u)$ in (2.14).

Letting $\gamma=(pq+2)/(p+q+1)$, we can write the quantities α and β in (2.7) or (3.3) as

$$\alpha=\gamma, \qquad \beta=\tfrac{1}{2}\gamma(q-p-1).$$

Therefore, we can use the transformations Y_1 and Y_3 given by

$$
\begin{aligned}
Y_1 &= \gamma \left\{ n + \frac{1}{2}(q-p-1) \right\} \log \left(1 + \frac{1}{n\gamma} T \right), \\
Y_3 &= \gamma \left\{ n + \frac{1}{2}(q-p-1) \right\} \left\{ 1 - \exp\left(-\frac{1}{n\,\gamma} T \right) \right\}.
\end{aligned} \tag{4.6}
$$

Note that from Theorem 1 the Y_1 and Y_3 have an asymptotic expansion as in (3.7). The approximations (2.12) for $t(u)$ are given by

$$
\begin{aligned}
t_1(u) &= n\gamma[\exp\{u\gamma^{-1}/(n + \tfrac{1}{2}(q-p-1))\} - 1], \\
t_3(u) &= -n\gamma \log\{1 - u\gamma^{-1}/(n + \tfrac{1}{2}(q-p-1))\}.
\end{aligned} \tag{4.7}
$$

The upper 5% points of $T = n \operatorname{tr} S_h S_e^{-1}$ and their approximations are given in Table II for some values of the parameters. The exact values were extracted from Davis (1970). It is seen that both the approximations $t_1(u)$ and $t_3(u)$ are also accurate except for the small sample case. It is interesting that the approximation t_3 is more accurate in multivariate case, i.e., $p > 1$.

Example 3. Let $T = (n-q)\, s_h^2/s_e^2$ be a test statistic for testing the equality of means of q nonnormal populations $\Pi_i (i = 1, ..., q)$ with common variance. Here s_h^2 and s_e^2 are the sums of squares due to the hypothesis and the error, respectively, based on the sample of the size n_i from Π_i. Let $\rho_i = \sqrt{n_i/n}$, where n is the total sample size. Assume that $\rho_i = O(1)$ as n_j's tend to infinity. Let κ_3 and κ_4 be the third and the fourth cumulants of the standardized variate. Then, under a general condition it is known (Fujikoshi, Ohmae and Yanagihara (1999)) that the null distribution of T

TABLE II

The exact percentile $t(u)$ and four approximations u, $\tilde{t}(u)$, $t_1(u)$, $t_3(u)$ for the upper 5% points

Parameters			Upper 5% points				
p	q	n	$t(u)$	u	$\tilde{t}(u)$	$t_1(u)$	$t_3(u)$
2	3	10	21.7	11.07	15.8	17.3	23.6
2	5	20	24.1	18.31	21.5	23.6	26.1
3	4	30	26.1	21.0	30.1	25.8	26.8
3	6	40	34.0	28.9	32.1	33.2	33.9

has an asymptotic expansion (1.5) with $k=3$, $f=q-1$ and the coefficients given by

$$
\begin{aligned}
a_0 &= \tfrac{1}{4}(q-1)(q-3) - d_1\kappa_3^2 + d_2\kappa_4,\\
a_1 &= -\tfrac{1}{2}(q-1)^2 + 3d_1\kappa_3^2 - 2d_2\kappa_4,\\
a_2 &= \tfrac{1}{4}(q^2-1) - 3d_1\kappa_3^2 + d_2\kappa_4,\\
a_3 &= d_1\kappa_3^2.
\end{aligned} \tag{4.8}
$$

where

$$
\begin{aligned}
d_1 &= \frac{5}{24}\left\{\sum_{j=1}^{q}\frac{n}{n_j} - q^2\right\} + \frac{1}{12}(q-1)(q-2),\\
d_2 &= \frac{1}{8}\left\{\sum_{j=1}^{q}\frac{n}{n_j} - q^2\right\} - \frac{1}{4}(q-1).
\end{aligned} \tag{4.9}
$$

Fujikoshi, Ohmae and Yanagihara (1999) studied a numerical accuracy for approximations of the percent points and actual test sizes of T based on the limting distribution, Bartlett correction and an asymptotic expansion. We examine performance of our new transfomations under the following five nonnormal models and normal model;

(i) $X+YZ$, where X, Y, Z is independent normal distribution N(0, 1),

(ii) Student t-distribution with 5 degrees of freedom,

(iii) symmetric uniform distribution U$(-5, 5)$,

(iv) χ^2 distribution with 4 degrees of freedom,

(v) χ^2 distribution with 8 degrees of freedom,

(vi) normal distribution.

The third and the fourth cumulants κ_3 and κ_4 are given in Table III for each of the six models. The values of d_1 and d_2, which measure the degree of equalty of sample sizes, are also given in Table III.

Our new transformations were examined in terms of the actual test sizes given by

$$
\begin{aligned}
&\alpha_1 = P(T>u), && \alpha_2 = P(T>\tilde{t}(u)),\\
&\alpha_3 = P(T>t_{\mathrm{E}}(u)), && \alpha_4 = P(T>t_{1\cdot 2}(u)),\\
&\alpha_5 = P(T>t_3(u)),
\end{aligned} \tag{4.10}
$$

where $t_{\mathrm{E}}(u) = t(u) + O(n^{-2})$, an expansion formula based on the asymptotic expansion up to the order $O(n^{-1})$ and

$$t_{1\cdot 2}(u) = \begin{cases} t_1(u) & \text{if} \quad \alpha > 0 \quad \text{and} \quad \alpha n + \beta > 0, \\ t_2(u) & \text{if} \quad \alpha < 0 \quad \text{and} \quad \alpha n + \beta < 0. \end{cases} \tag{4.11}$$

Here α and β are defined by (3.3). The values of α_i's in (4.10), which are given in Table III were computed by a simulation experiment based on ten

TABLE III

The actual test sizes in the case $q = 3$

	Sample sizes					Nominal 5% test				
	n_1	n_2	n_3	d_1	d_2	α_1	α_2	α_3	α_4	α_5
	5	5	5	1/6	−1/2	8.7	6.2	6.0	5.6	5.5
Model (i)	10	10	10	1/6	−1/2	6.4	5.5	5.3	5.2	5.1
$\kappa_3 = 0$	15	15	15	1/6	−1/2	6.1	5.5	5.4	5.4	5.3
$\kappa_4 = 1.5$	5	10	15	7/12	−1/4	6.9	5.8	5.5	5.4	5.3
	5	5	20	53/48	1/16	7.1	6.0	5.5	5.4	5.3
	5	5	5	1/6	−1/2	7.9	5.6	6.3	6.3	6.1
Model (ii)	10	10	10	1/6	−1/2	6.3	5.3	5.5	5.5	5.5
$\kappa_3 = 0$	15	15	15	1/6	−1/2	6.0	5.3	5.5	5.4	5.4
$\kappa_4 = 6$	5	10	15	7/24	−1/4	6.4	5.3	5.2	5.1	5.1
	5	5	20	53/48	1/16	7.1	6.0	5.7	5.6	5.5
	5	5	5	1/6	−1/2	9.2	6.9	6.1	5.9	5.1
Model (iii)	10	10	10	1/6	−1/2	6.7	5.7	5.3	5.2	5.1
$\kappa_3 = 0$	15	15	15	1/6	−1/2	6.0	5.5	5.2	5.2	5.1
$\kappa_4 = -1.2$	5	10	15	7/12	−1/4	6.1	5.0	4.6	4.6	4.5
	5	5	20	53/48	1/16	6.1	5.0	4.6	4.6	4.5
	5	5	5	1/6	−1/4	7.9	5.7	5.9	5.5	5.4
Model (iv)	10	10	10	1/6	−1/4	6.3	5.4	5.4	5.2	5.2
$\kappa_3 = \sqrt{2}$	15	15	15	1/6	−1/4	5.7	4.8	4.8	4.8	4.7
$\kappa_4 = 3$	5	10	15	7/12	−1/2	6.4	5.4	5.4	5.2	5.1
	5	5	20	53/48	1/16	6.4	5.3	5.4	4.9	4.9
	5	5	5	1/6	−1/4	8.4	6.0	5.8	5.4	5.1
Model (v)	10	10	10	1/6	−1/4	6.2	5.2	5.1	5.0	5.0
$\kappa_3 = 1$	15	15	15	1/6	−1/4	5.7	5.1	4.9	4.8	4.8
$\kappa_4 = 1.5$	5	10	15	7/12	−1/2	6.2	5.2	5.1	4.8	4.7
	5	5	20	53/48	1/16	6.5	5.6	5.5	5.2	5.1
	5	5	5	1/6	−1/4	9.3	6.8	6.3	5.9	5.4
Model (vi)	10	10	10	1/6	−1/4	6.7	5.7	5.3	5.2	5.1
$\kappa_3 = 0$	15	15	15	1/6	−1/4	6.0	5.2	4.9	4.8	4.8
$\kappa_4 = 0$	5	10	15	7/12	−1/2	7.5	6.1	5.8	5.6	5.5
	5	5	20	53/48	1/16	7.1	5.9	5.6	5.5	5.4

thousand reprications for the case $q = 3$ and some values of (n_1, n_2, n_3). From (2.13) and (3.7) we can see that the approximations $t_{1\cdot 2}(u)$ and t_3 satisfy

$$P(T \leqslant t_i(u)) = P(T \leqslant t(u)) + O(n^{-2}), \tag{4.12}$$

for the case $\kappa_3 = 0$. Though the expansion formula (4.12) does not always hold for the case $\kappa_3 \neq 0$, Table III shows that the proposed approximations $t_{1\cdot 2}(u)$ and $t_3(u)$ are better than the approximations u and $\tilde{t}(u)$. Further, it may be noted that the proposed approximations $t_{1\cdot 2}(u)$ and $t_3(u)$ are slightly good in a comparison with the approximation $t_E(u)$ based on an asymptotic expansion, even in the case $\kappa_3 \neq 0$. When the moments are unknown, we need to estimate these parameters from the data. In this case we have seen that the proposed approximations work well as in the case when the moments are known, through a numerical study.

Through three statistics in Examples 1, 2 and 3, we have seen how much the distributions of the transformed statistics Y_i are close to the one of χ^2_f-variate, or how much the percent point approximations $t_i(u)$ are close to the percent point $t(u)$ of T. It is shown that the proposed transformations of these statistics give a larger improvement to the chi-squared approximation than do the Bartlett corrections. In Example 3 we have seen that the proposed transformations are comparable with the approximation based on an Eghworth expansion. For a choice among Y_i or t_i, we can roughly say that (1) Y_1 or $t_1(u)$ may be recommendable if $\alpha > 0$, $n\alpha + \beta > 0$, and (2) Y_3 or $t_3(u)$ may be also recommendable if $|u/(n\alpha + \beta)| < 1$ is not near to one.

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