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Asymptotic expansion of the null distribution of test statistic for linear hypothesis in nonnormal linear model ☆

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Abstract

This paper is concerned with the null distribution of test statistic T for testing a linear hypothesis in a linear model without assuming normal errors. The test statistic includes typical ANOVA test statistics. It is known that the null distribution of T converges to χ^2 when the sample size n is large under an adequate condition of the design matrix. We extend this result by obtaining an asymptotic expansion under general condition. Next, asymptotic expansions of one- and two-way test statistics are obtained by using this general one. Numerical accuracies are studied for some approximations of percent points and actual test sizes of T for two-way ANOVA test case based on the limiting distribution and an asymptotic expansion.

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1. Introduction

Consider a linear model

$$\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}.$$

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In this model, $\boldsymbol{y} = (y_1, y_2, \dots, y_n)'$ is an observable vector of random variables, $X = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n)'$ is an $n \times k$ design matrix of known fixed values with full rank k, where $\boldsymbol{x}_i$'s are $k \times 1$ vectors, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_k)'$ is an unknown parameter vector and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)'$ is an unobservable error vector of random variables and assumed that their components are i.i.d. with mean 0 and finite positive variance σ^2.

We are interested in testing for null hypothesis as

$$\mathrm{H}_0 : H\boldsymbol{\beta} = \mathbf{0},$$

where H is a known $h \times k$ matrix with rank h $(\leqslant k)$. Then a well-known test statistic is

$$T = \frac{\hat{\boldsymbol{\beta}}' H' \{H(X'X)^{-1}H'\}^{-1} H\hat{\boldsymbol{\beta}}}{\hat{\sigma}^2}, \tag{1.1}$$

which is a likelihood ratio statistic in a normal error case, where

$$\hat{\boldsymbol{\beta}} = (X'X)^{-1}X'\boldsymbol{y}, \quad \hat{\sigma}^2 = \frac{1}{n-k}\boldsymbol{y}'\{I_n - X(X'X)^{-1}X'\}\boldsymbol{y}.$$

Under normality, it is well known that the null distribution of T/h is distributed as $F_{h,n-k}$. Under nonnormality, it is known that null distribution of T converges χ_h^2 as the sample size n tends to infinity under an adequate condition of X (see [1, p. 141; 8]). In the nonnormal case, however, using this limiting distribution, we have high-risk generally, because a nominal test size based on the limiting distribution is far from the actual test size (see simulation results in [5,10] etc.). Therefore, it is necessary to improve the approximation of its distribution. It is well known to derive an asymptotic expansion in one of the important methods which makes an improved approximation. The main purpose of this paper is to obtain an asymptotic expansion of the null distribution of T up to the order n^{-1} without assuming normal errors. Using the general result we can obtain asymptotic expansions of usual ANOVA test statistics.

In order to calculate such asymptotic expansion effectively, we rewrite test statistic T in (1.1) as

$$T = \frac{\boldsymbol{z}'\Omega\boldsymbol{z}}{\hat{\sigma}^2/\sigma^2}, \tag{1.2}$$

where

$$\boldsymbol{z} = (z_1, z_2, \dots, z_k)' = (X'X)^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})/\sigma \tag{1.3}$$

and

$$\Omega = (X'X)^{-1/2}H'\{H(X'X)^{-1}H'\}^{-1}H(X'X)^{-1/2}.$$

It goes without saying that $\Omega^2 = \Omega$ and $\mathrm{rank}(\Omega) = \mathrm{tr}(\Omega) = h$. Using asymptotic expansion for the probability density function of $\boldsymbol{z}$ and the joint probability density function of $\boldsymbol{z}$ and $\hat{\sigma}^2$, we can obtain the asymptotic expansion of T.

The present paper is organized in the following way. In Section 2, we prepare the Edgeworth expansion for the density function of $\boldsymbol{z}$ and the joint density function of $\boldsymbol{z}$

and normalized $\hat{\sigma}^2$. In Section 3, we derive an expansion of the null distribution of T, by expanding the characteristic function of T. Some applications are given in Section 4. In Section 5, numerical accuracies are studied for an approximation of the percentage points and actual test sizes of two-way ANOVA test statistics with balanced replications, based on the limiting distribution and an asymptotic expansion.

2. Preliminary result

In this section, using the same notation as before and assuming, without loss of generality, that $\sigma^2 = 1$, we consider the Edgeworth expansion for the density function of $\boldsymbol{z}$ and the joint density function of $\boldsymbol{z}$ and transformed $\hat{\sigma}^2$ as

$$v = \sqrt{n}(\hat{\sigma}^2 - 1).$$

Let λ_n be the smallest eigenvalue of $X'X$, κ_j denote the jth cumulant of ε_i and $M_n = \max\{||\boldsymbol{x}_i|| : i = 1, 2, \ldots, n\}$, where $|| \cdot ||$ is the Euclidean norm. The Edgeworth expansion of the distribution of $\boldsymbol{z}$ has been given by Qumsiyeh [9] as follows.

Lemma 2.1. *Suppose that X and ε_i, $i = 1, 2, \ldots, n$, satisfy the following assumptions;*
C1. *The characteristic function $C_\varepsilon(t)$ of ε_i is integrable,*
C2. *ε_i have the fifth absolute moment, and for some integer r*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} ||\boldsymbol{x}_i||^r < \infty, \quad 1 \leqslant r \leqslant 5,$$

C3. $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$, $M_n = O(n^c)$, *for some* $c \in [0, 1/2)$.
Let $\bar{\chi}_a$, $\bar{\chi}_{ab}$, etc., be defined by

$$\bar{\chi}_{a_1 \cdots a_j} = \frac{1}{n} \sum_{i=1}^{n} \prod_{l=1}^{j} \chi_{ia_l}, \tag{2.1}$$

where

$$\sqrt{n}(X'X)^{-1/2}\boldsymbol{x}_i = (\chi_{i1}, \chi_{i2}, \ldots, \chi_{ik})'.$$

Then the probability density function of $\boldsymbol{z}$ can be expanded as

$$\phi_k(\boldsymbol{z})\left[1 + \frac{\kappa_3}{6\sqrt{n}} \sum_{a,b,c}^{k} \bar{\chi}_{abc} H_{abc}(\boldsymbol{z}) + \frac{\kappa_4}{24n} \sum_{a,b,c,d}^{k} \bar{\chi}_{abcd} H_{abcd}(\boldsymbol{z}) \right.$$
$$\left. + \frac{\kappa_3^2}{72n} \sum_{a,b,c,d,e,f}^{k} \bar{\chi}_{abc}\bar{\chi}_{def} H_{abcdef}(\boldsymbol{z})\right] + O(n^{-3/2}), \tag{2.2}$$

where $\phi_k(\boldsymbol{z})$ is the probability density function of $N_k(\boldsymbol{0}, I_k)$ given by $\phi_k(\boldsymbol{z}) = (2\pi)^{-k/2}\exp(-\boldsymbol{z}'\boldsymbol{z}/2)$ and functions $H_a(\boldsymbol{z})$, $H_{ab}(\boldsymbol{z})$, etc. are the Hermite polynomials *defined by*

$$H_{a_1\cdots a_j}(\boldsymbol{z}) = (-1)^j \frac{\partial^j}{\partial z_1 \cdots \partial z_j} \phi_k(\boldsymbol{z}),$$

for example

$$H_{abc}(\boldsymbol{z}) = z_a z_b z_c - \sum_{[3]} z_a \delta_{bc},$$

$$H_{abcd}(\boldsymbol{z}) = z_a z_b z_c z_d - \sum_{[6]} z_a z_b \delta_{cd} + \sum_{[3]} \delta_{ab}\delta_{cd},$$

$$H_{abcdef}(\boldsymbol{z}) = z_a z_b z_c z_d z_e z_f - \sum_{[15]} z_a z_b z_c z_d \delta_{ef} + \sum_{[45]} z_a z_b \delta_{cd}\delta_{ef} - \sum_{[15]} \delta_{ab}\delta_{cd}\delta_{ef}.$$

Here $\sum_{[j]}$ *means the sum of all j possible combinations, for example* $\sum_{[3]} \delta_{ab}\delta_{cd} = \delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$, $\sum_{a_1,\dots,a_j}^{k}$ *means* $\sum_{a_1=1}^{k} \cdots \sum_{a_j=1}^{k}$ *and* δ_{ab} *is the Kronecker delta, i.e.,* $\delta_{aa} = 1$ *and* $\delta_{ab} = 0$ *for* $a \neq b$.

If $\boldsymbol{z}$ is a sum of independent identical random vectors, we get a valid Edgeworth expansion of $\boldsymbol{z}$ up to order n^{-1} under the existence of fourth absolute moment of ε_i (see [4, p. 188]). However, the $\boldsymbol{z}$ in (1.2) is a sum of independent random vectors but not identical ones as

$$\boldsymbol{z} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \sqrt{n} (X'X)^{-1/2} \boldsymbol{x}_i.$$

In this case, we need to assume a stronger moment condition such as the fifth absolute moment, in order to get the Edgeworth expansion up to order n^{-1}.

Let

$$\tilde{v} = \sqrt{n}(\tilde{\sigma}^2 - 1),$$

where $\tilde{\sigma}^2 = n^{-1}\sum_{i=1}^{n} \varepsilon_i^2$. We can easily obtain an asymptotic expansion of the joint characteristic function of $(\boldsymbol{z}, \tilde{v})$, because $(\boldsymbol{z}', \tilde{v})'$ is a sum of independent random vectors as

$$\begin{pmatrix} \boldsymbol{z} \\ \tilde{v} \end{pmatrix} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \boldsymbol{s}_i,$$

where

$$\boldsymbol{s}_i = \begin{pmatrix} \varepsilon_i \sqrt{n} (X'X)^{-1/2} \boldsymbol{x}_i \\ \varepsilon_i^2 - 1 \end{pmatrix}.$$

Further, noting that $v = \tilde{v} + n^{-1/2}(k - \boldsymbol{z}'\boldsymbol{z}) + n^{-1}k\tilde{v} + O_p(n^{-3/2})$, we can obtain an asymptotic expansion of the joint characteristic function of $(\boldsymbol{z}, v)$ as in Lemma 2.2.

Lemma 2.2. *Suppose that X and ε_i, $i = 1, 2, \ldots, n$, satisfy* C3 *and* C4. *ε_i have the eighth absolute moment, and for some integer r*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} ||\boldsymbol{x}_i||^r < \infty, \quad 1 \leqslant r \leqslant 4.$$

Then characteristic function of $(\boldsymbol{z}, v)$ can be expanded as

$$\begin{aligned} C_{(\boldsymbol{z},v)}(\boldsymbol{t}_1, t_2) = {} & \exp\{-\tfrac{1}{2}(\boldsymbol{t}_1'\boldsymbol{t}_1 + 2\kappa_3 t_2 \bar{\boldsymbol{\chi}}_1'\boldsymbol{t}_1 + \kappa_4 t_2^2)\} \\ & \times \left[1 + \frac{i^3}{6\sqrt{n}}\left\{\kappa_3 \sum_{a,b,c}^{k} \bar{\chi}_{abc} t_a^{(1)} t_b^{(1)} t_c^{(1)} \right.\right. \\ & + 3\kappa_4 t_2 \boldsymbol{t}_1'\boldsymbol{t}_1 + 3(\kappa_5 + 4\kappa_3) t_2^2 \bar{\boldsymbol{\chi}}_1'\boldsymbol{t}_1 \\ & \left.\left. + (\kappa_6 + 12\kappa_4 + 2\kappa_3^2(5 - 3\bar{\boldsymbol{\chi}}_1'\bar{\boldsymbol{\chi}}_1) + 8) t_2^3\right\}\right] + O(n^{-1}), \end{aligned} \tag{2.3}$$

where $\boldsymbol{t}_1 = (t_1^{(1)}, t_2^{(1)}, \ldots, t_k^{(1)})'$ and

$$\bar{\boldsymbol{\chi}}_1 = (\bar{\chi}_1, \bar{\chi}_2, \ldots, \bar{\chi}_k)' = \sqrt{n}(X'X)^{-1/2}X'\mathbf{1}_n.$$

Here $\mathbf{1}_n$ is an n-dimensional vector, all of whose elements are 1.

From Lemma 2.2, the limiting distribution of $(\boldsymbol{z}, v)$ is normal with mean $\mathbf{0}$ and covariance matrix

$$\Sigma = \begin{pmatrix} I_k & \kappa_3\bar{\boldsymbol{\chi}}_1 \\ \kappa_3\bar{\boldsymbol{\chi}}_1' & \kappa_4 + 2 \end{pmatrix}.$$

Let us decompose the limiting probability density of $(\boldsymbol{z}, v)$ into the marginal probability density of $\boldsymbol{z}$ and conditional probability density of v given $\boldsymbol{z}$ as

$$\begin{aligned} g(\boldsymbol{z}, v) &= \left(\frac{1}{\sqrt{2\pi}}\right)^k \exp\left(-\frac{1}{2}\boldsymbol{z}'\boldsymbol{z}\right)\frac{1}{\sqrt{2\pi}\zeta}\exp\left\{-\frac{1}{2\zeta^2}(v - \kappa_3\bar{\boldsymbol{\chi}}_1'\boldsymbol{z})^2\right\} \\ &= \phi_k(\boldsymbol{z})\phi(v; \kappa_3\bar{\boldsymbol{\chi}}_1'\boldsymbol{z}, \zeta^2), \end{aligned} \tag{2.4}$$

where $\zeta^2 = \det(\Sigma) = \kappa_4 + 2 - \kappa_3^2\bar{\boldsymbol{\chi}}_1'\bar{\boldsymbol{\chi}}_1 > 0$. Based on expression (2.4), we will obtain inversion of each term in the expanded expression (2.3). Inversion of each term can be evaluated by using an inversion formula

$$\begin{aligned} & (2\pi)^{-(k+1)}\int_{\mathscr{R}^k}\int_{\mathscr{R}} \exp\{-i\boldsymbol{t}_1'\boldsymbol{z} - it_2 v\}\left\{\prod_{j=1}^{k}(-it_j^{(1)})^{\alpha_j}\right\}(-it_2)^a \\ & \quad \times \exp\left\{-\frac{i^2}{2}(\boldsymbol{t}_1'\boldsymbol{t}_1 + 2\kappa_3 t_2\bar{\boldsymbol{\chi}}_1'\boldsymbol{t}_1 + \kappa_4 t_2^2)\right\} d\boldsymbol{t}_1\, dt_2 \\ & \quad = \left\{\prod_{j=1}^{k}\left(\frac{\partial}{\partial z_j}\right)^{\alpha_j}\right\}\left(\frac{\partial}{\partial v}\right)^a \phi_k(\boldsymbol{z})\phi(v; \kappa_3\bar{\boldsymbol{\chi}}_1'\boldsymbol{z}, \zeta^2). \end{aligned}$$

This yields a useful expression for the joint density function of $(\boldsymbol{z}, v)$ and the conditional density function of v given $\boldsymbol{z}$, which are given in Lemma 2.3.

Lemma 2.3. *Under the same conditions as in Lemma* 2.2 *and assumption* C5, *the joint characteristic function* $C_{(\varepsilon,\varepsilon^2)}(t_1, t_2)$ *of* ε_i *and* ε_i^2 *is integrable, it holds that*

(i) *the joint probability density function of* $(\boldsymbol{z}, v)$ *can be expanded as*

$$f(\boldsymbol{z}, v) = g(\boldsymbol{z}, v)\left\{1 - \frac{1}{6\sqrt{n}} g_{(1)}(\boldsymbol{z}, v)\right\} + O(n^{-1}),$$

(ii) *the conditional probability density function of* v *given* $\boldsymbol{z}$ *can be expanded as*

$$\phi(v; \kappa_3 \bar{\boldsymbol{\chi}}_1' \boldsymbol{z}, \zeta^2)\left[1 - \frac{1}{6\sqrt{n}}\left\{g_{(1)}(\boldsymbol{z}, v) + \kappa_3 \sum_{a,b,c}^{k} \bar{\chi}_{abc} H_{abc}(\boldsymbol{z})\right\}\right] + O(n^{-1}),$$

where

$$\begin{aligned} g_{(1)}(\boldsymbol{z}, v) = &\kappa_3 \sum_{a,b,c}^{k} \bar{\chi}_{abc} g_{abc}^{(1)}(\boldsymbol{z}, v) + 3\kappa_4 \sum_{a=1}^{k} g_a^{(2)}(\boldsymbol{z}, v) \\ &+ 3(\kappa_5 + 4\kappa_3) \sum_{a=1}^{k} \bar{\chi}_a g_a^{(3)}(\boldsymbol{z}, v) \\ &+ \{\kappa_6 - 12\kappa_4 + 2\kappa_3^2(5 - 3\bar{\boldsymbol{\chi}}_1'\bar{\boldsymbol{\chi}}_1) + 8\} g^{(4)}(\boldsymbol{z}, v) \end{aligned}$$

and

$$\begin{aligned} g_{abc}^{(1)}(\boldsymbol{z}, v) = &\sum_{[3]} (\delta_{ab} + \zeta^{-2}\kappa_3^2 \bar{\chi}_a \bar{\chi}_b)(z_c - \zeta^{-1}\kappa_3 \bar{\chi}_c) \\ &+ (-z_a + \zeta^{-1}\kappa_3 \bar{\chi}_a w)(-z_b + \zeta^{-1}\kappa_3 \bar{\chi}_b w)(-z_c + \zeta^{-1}\kappa_3 \bar{\chi}_c w), \end{aligned}$$

$$\begin{aligned} g_a^{(2)}(\boldsymbol{z}, v) = &2\zeta^{-2}\kappa_3 \bar{\chi}_a(-z_a + \zeta^{-1}\kappa_3 \bar{\chi}_a w) \\ &- \zeta^{-1} w\{-1 - \zeta^{-2}\kappa_3^2 \bar{\chi}_a^2 + (-z_a + \zeta^{-1}\kappa_3 \bar{\chi}_a w)^2\}, \end{aligned}$$

$$g_a^{(3)}(\boldsymbol{z}, v) = -2\zeta^{-3}\kappa_3 \bar{\chi}_a w + \zeta^{-2}(-1 + w^2)(-z_a + \zeta^{-1}\kappa_3 \bar{\chi}_a w),$$

$$g^{(4)}(\boldsymbol{z}, v) = \zeta^{-3}(3w - w^3),$$

$$w = \zeta^{-1}(v - \kappa_3 \bar{\boldsymbol{\chi}}_1' \boldsymbol{z}).$$

As for the lead on an asymptotic expansion of the density function of a variable, see, e.g., [2, p. 131; 7, p. 39].

3. Asymptotic expansion of T

3.1. The characteristic function of T

In this section, we derive an asymptotic expansion of the null distribution of T up to the order n^{-1}. Without loss of generality, we may replace $\boldsymbol{\varepsilon}$ by $\boldsymbol{\varepsilon}/\sigma$, which has $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $\mathrm{Var}(\boldsymbol{\varepsilon}) = I_n$, in the expressions of T since T is invariant under the transformation from $\boldsymbol{y}$ to $\boldsymbol{y}/\sigma$. Let the jth cumulant of $\boldsymbol{\varepsilon}/\sigma$ be denoted by κ_j.

Suppose that X and $\varepsilon_i, i = 1, \ldots, n$ satisfy C3 and

C6. The joint characteristic function $C_{(\varepsilon,\varepsilon^2)}(t_1, t_2)$ of ε_i and ε_i^2 satisfies the Cramér's condition, i.e.,

$$\limsup_{||\boldsymbol{t}|| \to \infty} |C_{(\varepsilon,\varepsilon^2)}(t_1, t_2)| < 1, \quad \boldsymbol{t} = (t_1, t_2)'.$$

C7. ε_i have the tenth absolute moment, and for some integer r,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} ||\boldsymbol{x}_i||^r < \infty, \quad 1 \leqslant r \leqslant 5.$$

Note that T is a smooth function of $z_1, z_2, \ldots, z_k$ and v. So, from a general result (see, e.g., [3]) on asymptotic expansions it can be shown that T has a valid expansion up to the order n^{-1} under the assumptions C3, C6 and C7. In the following, we will find an asymptotic expansion of the characteristic function of T up to the order n^{-1}, which may be inverted formally. We can expand T as

$$T = T_0 + \frac{1}{\sqrt{n}} T_1 + \frac{1}{n} T_2 + O_p(n^{-3/2}), \tag{3.1}$$

where

$$T_0 = \boldsymbol{z}' \Omega \boldsymbol{z}, \quad T_1 = -v T_0, \quad T_2 = v^2 T_0.$$

From (3.1), we can write the characteristic function of T as

$$C_T(t) = C_0(t) + \frac{1}{\sqrt{n}} C_1(t) + \frac{1}{n} C_2(t) + o(n^{-1}), \tag{3.2}$$

where

$$C_0(t) = E[\exp(itT_0)], \quad C_1(t) = E[itT_1 \exp(itT_0)],$$

$$C_2(t) = E[\{itT_2 + \tfrac{1}{2}(itT_1)^2\} \exp(itT_0)].$$

For an evaluation of each term in (3.2), we will use an asymptotic expansion of the joint density function of $\boldsymbol{z}$ and v. This requires assumptions C1 and C5 as Lemmas 2.1 and 2.3, which are stronger than the Cramér's condition. However, it may be noted that we need not assume the strong assumptions in final result due to the uniqueness of the characteristic function.

3.2. Computation of $C_0(t)$

Using Lemma 2.1, we obtain

$$C_0(t) = \int_{\mathscr{R}^k} e^{itT_0} \phi_k(\boldsymbol{z}) \left\{ 1 + \frac{1}{\sqrt{n}} Q_1(\boldsymbol{z}) + \frac{1}{n} Q_2(\boldsymbol{z}) \right\} d\boldsymbol{z} + O(n^{-3/2}),$$

where

$$Q_1(\boldsymbol{z}) = \frac{1}{6}\kappa_3 \sum_{a,b,c}^{k} \bar{\chi}_{abc} H_{abc}(\boldsymbol{z}),$$

$$Q_2(\boldsymbol{z}) = \frac{1}{24}\kappa_4 \sum_{a,b,c,d}^{k} \bar{\chi}_{abcd} H_{abcd}(\boldsymbol{z}) + \frac{1}{72}\kappa_3^2 \sum_{a,b,c,d,e,f}^{k} \bar{\chi}_{abc}\bar{\chi}_{def} H_{abcdef}(\boldsymbol{z}).$$

Note that

$$itT_0 - \tfrac{1}{2}\boldsymbol{z}'\boldsymbol{z} = -\tfrac{1}{2}(\Gamma^{-1/2}\boldsymbol{z})'(\Gamma^{-1/2}\boldsymbol{z}),$$

where $\Gamma = (1-2it)^{-1}\Omega + I_k - \Omega$. Considering the transformation $\boldsymbol{z}^* = \{(1-2it)^{1/2} \Omega + I_k - \Omega\}\boldsymbol{z} = \Gamma^{-1/2}\boldsymbol{z}$, we have

$$C_0(t) = \varphi^{h/2} E_{\boldsymbol{z}^*}\left[1 + \frac{1}{\sqrt{n}} Q_1(\Gamma^{1/2}\boldsymbol{z}^*) + \frac{1}{n} Q_2(\Gamma^{1/2}\boldsymbol{z}^*) \right] + O(n^{-3/2}),$$

where $\varphi = (1-2it)^{-1}$ and the expectation is taken with respect to $\boldsymbol{z}^*$ whose distribution is $N_k(\boldsymbol{0}, I_k)$. Note that $\boldsymbol{u} = \Gamma^{1/2}\boldsymbol{z}^*$ is distributed as $N_k(\boldsymbol{0}, \Gamma)$. Therefore, we can write

$$C_0(t) = \varphi^{h/2} E_{\boldsymbol{u}}\left[1 + \frac{1}{\sqrt{n}} Q_1(\boldsymbol{u}) + \frac{1}{n} Q_2(\boldsymbol{u}) \right] + O(n^{-3/2}). \tag{3.3}$$

Let the ab element of Γ and Ω denote by γ_{ab} and ω_{ab}, respectively. Using $\gamma_{ab} = \delta_{ab} + (\varphi - 1)\omega_{ab}$, we can see that

$$E_{\boldsymbol{u}}[H_{abc}(\boldsymbol{u})] = 0,$$

$$E_{\boldsymbol{u}}[H_{abcd}(\boldsymbol{u})] = (\varphi - 1)^2 \sum_{[3]} \omega_{ab}\omega_{cd},$$

$$E_{\boldsymbol{u}}[H_{abcdef}(\boldsymbol{u})] = (\varphi - 1)^3 \sum_{[15]} \omega_{ab}\omega_{cd}\omega_{ef}.$$

Note that

$$\sum_{a,b,c,d}^{k} \bar{\chi}_{abcd} \sum_{[3]} \omega_{ab}\omega_{cd} = 3 \sum_{a,b,c,d}^{k} \bar{\chi}_{abcd}\omega_{ab}\omega_{cd}$$
$$= n \sum_{j=1}^{n} (\boldsymbol{x}_j' \Upsilon \boldsymbol{x}_j)^2,$$

$$\sum_{a,b,c,d,ef}^{k} \bar{\chi}_{abc}\bar{\chi}_{edf} \sum_{[15]} \omega_{ab}\omega_{cd}\omega_{ef}$$
$$= \sum_{a,b,c,d,ef}^{k} \bar{\chi}_{abc}\bar{\chi}_{edf}(6\omega_{ad}\omega_{be}\omega_{cf} + 9\omega_{ab}\omega_{cd}\omega_{ef})$$
$$= 3n \sum_{i,j}^{n} \{2(\boldsymbol{x}_i'\Upsilon\boldsymbol{x}_j)^3 + 3(\boldsymbol{x}_i'\Upsilon\boldsymbol{x}_i)(\boldsymbol{x}_i'\Upsilon\boldsymbol{x}_j)(\boldsymbol{x}_j'\Upsilon\boldsymbol{x}_j)\},$$

where

$$\Upsilon = (X'X)^{-1/2}\Omega(X'X)^{-1/2}.$$

Let ψ_{ab} be the ab element of Ψ which is defined by

$$\Psi = X(X'X)^{-1}H'\{H(X'X)^{-1}H'\}^{-1}H(X'X)^{-1}X'.$$

Substituting these equations into (3.3) yields

$$C_0(t) = \varphi^{h/2}\left[1 + \frac{1}{8n}\kappa_4(\varphi-1)^2\{n\,\mathrm{tr}(D_{(\Psi)}^2)\}\right.$$
$$\left. + \frac{1}{24n}\kappa_3^2(\varphi-1)^3\mathbf{1}_n'(2n\Psi_{(3)} + 3nD_{(\Psi)}\Psi D_{(\Psi)})\mathbf{1}_n\right] + O(n^{-3/2}), \tag{3.4}$$

where $D_{(\Psi)} = \mathrm{diag}(\psi_{11}, \psi_{22}, \ldots, \psi_{nn})$ and $\Psi_{(3)}$ is an $n \times n$ matrix whose ab element is denoted by ψ_{ab}^3.

3.3. Computation of $C_1(t)$

Using Lemma 2.3(i), we can write

$$C_1(t) = \int_{\mathscr{R}^{k+1}} \exp\{it\boldsymbol{z}'\Omega\boldsymbol{z}\}(-it)v\boldsymbol{z}'\Omega\boldsymbol{z}$$
$$\times \phi_k(\boldsymbol{z})\phi(v;\kappa_3\bar{\boldsymbol{\chi}}_1'\boldsymbol{z},\zeta^2)\left\{1 - \frac{1}{6\sqrt{n}}g_{(1)}(\boldsymbol{z},v)\right\}d\boldsymbol{z}\,dv + O(n^{-1})$$
$$= E_{\boldsymbol{z}}E_{v|\boldsymbol{z}}[(-it)\exp\{it\boldsymbol{z}'\Omega\boldsymbol{z}\}$$
$$\times (\zeta w + \kappa_3\bar{\boldsymbol{\chi}}_1'\boldsymbol{z})\boldsymbol{z}'\Omega\boldsymbol{z}\left\{1 - \frac{1}{6\sqrt{n}}g_{(1)}(\boldsymbol{z},v)\right\}\Bigg] + O(n^{-1}).$$

Here the expectation in the last expression is taken with respect to $\boldsymbol{z}$ and v whose distribution is

$$v|\boldsymbol{z} \sim N(\kappa_3\bar{\boldsymbol{\chi}}_1'\boldsymbol{z},\zeta^2), \quad \boldsymbol{z} \sim N_k(\mathbf{0}, I_k). \tag{3.5}$$

Note that the conditional distribution of w given $\boldsymbol{z}$ is $N(0,1)$. Therefore, we have

$$E_{v|\boldsymbol{z}}[g^{(1)}_{abc}(\boldsymbol{z},v)] = \sum_{[3]} \delta_{ab}z_c - z_a z_b z_c,$$

$$E_{v|\boldsymbol{z}}[wg^{(1)}_{abc}(\boldsymbol{z},v)] = \zeta^{-1}\kappa_3 \sum_{[3]} (z_a z_b - \delta_{ab})\bar{\chi}_c,$$

$$E_{v|\boldsymbol{z}}[g^{(2)}_{a}(\boldsymbol{z},v)] = 0,$$

$$E_{v|\boldsymbol{z}}[wg^{(2)}_{a}(\boldsymbol{z},v)] = \zeta^{-1}(1 - z_a^2),$$

$$E_{v|\boldsymbol{z}}[g^{(3)}_{a}(\boldsymbol{z},v)] = 0, \quad E_{v|\boldsymbol{z}}[wg^{(3)}_{a}(\boldsymbol{z},v)] = 0,$$

$$E_{v|\boldsymbol{z}}[g^{(4)}(\boldsymbol{z},v)] = 0, \quad E_{v|\boldsymbol{z}}[wg^{(4)}(\boldsymbol{z},v)] = 0 \tag{3.6}$$

and

$$E_{\boldsymbol{z}}E_{v|\boldsymbol{z}}[(-it)\exp\{it\boldsymbol{z}'\Omega\boldsymbol{z})(\zeta w + \kappa_3\bar{\boldsymbol{\chi}}_1'\boldsymbol{z})\boldsymbol{z}'\Omega\boldsymbol{z}] = 0. \tag{3.7}$$

These results (3.6) and (3.7) imply

$$\begin{aligned}
C_1(t) =& \frac{2it}{12\sqrt{n}} E_{\boldsymbol{z}}\Bigg[\exp\{it\boldsymbol{z}'\Omega\boldsymbol{z}\}\boldsymbol{z}'\Omega\boldsymbol{z} \\
& \times \Bigg\{\kappa_3^2 \sum_{a,b,c}^{k} \bar{\chi}_{abc} \sum_{[3]} (z_a z_b - \delta_{ab})\bar{\chi}_c + 3\kappa_4 \sum_{a=1}^{k} (1 - z_a^2) \\
& + \kappa_3^2 \bar{\boldsymbol{\chi}}_1'\boldsymbol{z} \sum_{a,b,c}^{k} \bar{\chi}_{abc}\left(\sum_{[3]} \delta_{ab}z_c - z_a z_b z_c\right)\Bigg\}\Bigg] + O(n^{-1}) \\
=& \frac{(1-\varphi^{-1})}{12\sqrt{n}} i^{-1}\left\{\frac{d}{dt}R_1(t)\right\} + O(n^{-1}),
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
R_1(t) =& E_{\boldsymbol{z}}\Bigg[\exp\{it\boldsymbol{z}'\Omega\boldsymbol{z}\} \\
& \times \Bigg\{\kappa_3^2 \sum_{a,b,c}^{k} \bar{\chi}_{abc} \sum_{[3]} (z_a z_b - \delta_{ab})\bar{\chi}_c + 3\kappa_4 \sum_{a=1}^{k} (1 - z_a^2) \\
& + \kappa_3^2 \bar{\boldsymbol{\chi}}_1'\boldsymbol{z} \sum_{a,b,c}^{k} \bar{\chi}_{abc}\left(\sum_{[3]} \delta_{ab}z_c - z_a z_b z_c\right)\Bigg\}\Bigg].
\end{aligned}$$

In order to calculate $R_1(t)$, using the same idea as in the reduction of $C_0(t)$, we obtain

$$R_1(t) = \varphi^{h/2} E_{\boldsymbol{u}}\left[\kappa_3^2 \sum_{a,b,c}^{k} \bar{\chi}_{abc} \sum_{[3]} (u_a u_b - \delta_{ab})\bar{\chi}_c + 3\kappa_4 \sum_{a=1}^{k} (1 - u_a^2) \right.$$
$$\left. + \kappa_3^2 \bar{\boldsymbol{\chi}}_1' \boldsymbol{u} \sum_{a,b,c}^{k} \bar{\chi}_{abc} \left(\sum_{[3]} \delta_{ab} u_c - u_a u_b u_c \right)\right].$$

Here $\boldsymbol{u}$ is distributed as $N_k(\boldsymbol{0}, \Gamma)$. Therefore, we have

$$E_{\boldsymbol{u}}\left[\sum_{a,b,c}^{k} \bar{\chi}_{abc} \sum_{[3]} (u_a u_b - \delta_{ab})\bar{\chi}_c\right] = (\varphi - 1) \sum_{a,b,c}^{k} \bar{\chi}_{abc} \sum_{[3]} \omega_{ab}\bar{\chi}_c,$$

$$E_{\boldsymbol{u}}\left[\sum_{a=1}^{k} (1 - u_a^2)\right] = -(\varphi - 1) \sum_{a=1}^{k} \omega_{aa} = -(\varphi - 1)h,$$

$$E_{\boldsymbol{u}}\left[\bar{\boldsymbol{\chi}}_1' \boldsymbol{u} \sum_{a,b,c}^{k} \bar{\chi}_{abc} \left(\sum_{[3]} \delta_{ab} u_c - u_a u_b u_c \right)\right]$$
$$= -(\varphi - 1) \sum_{a,b,c}^{k} \bar{\chi}_{abc} \sum_{[3]} \omega_{ab}\bar{\chi}_c - (\varphi - 1)^2 \sum_{a,b,c,d}^{k} \bar{\chi}_{abc}\bar{\chi}_d \sum_{[3]} \omega_{ab}\omega_{cd}.$$

Note that

$$\sum_{a,b,c,d}^{k} \bar{\chi}_{abc}\bar{\chi}_d \sum_{[3]} \omega_{ab}\omega_{cd} = 3 \sum_{a,b,c,d}^{k} \bar{\chi}_{abc}\bar{\chi}_d \omega_{ab}\omega_{cd}$$
$$= 3 \sum_{i,j}^{n} (\boldsymbol{x}_i \Upsilon \boldsymbol{x}_i)(\boldsymbol{x}_i \Upsilon \boldsymbol{x}_j) = 3(\boldsymbol{1}_n' D_{(\Psi)} \Psi \boldsymbol{1}_n).$$

Using these results, we obtain

$$R_1(t) = -\varphi^{h/2}\{3\kappa_4(\varphi - 1)h + 3\kappa_3^2(\varphi - 1)^2(\boldsymbol{1}_n' D_{(\Psi)} \Psi \boldsymbol{1}_n)\}. \tag{3.9}$$

Because $(d/dt)\varphi^{h/2}(\varphi - 1) = i\varphi^{1+h/2}\{(h+2)\varphi - h\}$ and $(d/dt)\varphi^{h/2}(\varphi - 1)^2 = i\varphi^{1+h/2}(\varphi - 1)\{(h+4)\varphi - h\}$, substituting (3.9) into (3.8) yields

$$C_1(t) = -\frac{1}{4\sqrt{n}}\varphi^{h/2}(\varphi - 1)[\kappa_4\{(h+2)\varphi - h\}h$$
$$+ \kappa_3^2(\varphi - 1)\{(h+4)\varphi - h\}(\boldsymbol{1}_n' D_{(\Psi)} \Psi \boldsymbol{1}_n)] + O(n^{-1}). \tag{3.10}$$

3.4. Computation of $C_2(t)$

Using the random variables $\boldsymbol{z}$ and v as in (3.5), we can write

$$C_2(t) = E_{\boldsymbol{z}} E_{v|\boldsymbol{z}}[\exp\{it\boldsymbol{z}'\Omega\boldsymbol{z}\}\{itv^2 T_0 + \tfrac{1}{2}(it)^2(vT_0)^2\}] + O(n^{-1/2}).$$

Noting that $v = \zeta w + \kappa_3 \bar{\boldsymbol{\chi}}_1' \boldsymbol{z}$, we obtain

$$E_{v|\boldsymbol{z}}[v^2] = \zeta^2 + \kappa_3^2 (\bar{\boldsymbol{\chi}}_1' \boldsymbol{z})^2.$$

This implies that

$$\begin{aligned} C_2(t) &= E_{\boldsymbol{z}}[\exp\{it\boldsymbol{z}'\Omega\boldsymbol{z}\}\{\zeta^2 + \kappa_3^2(\bar{\boldsymbol{\chi}}_1'\boldsymbol{z})^2\}\boldsymbol{z}'\Omega\boldsymbol{z}\{it + \tfrac{1}{2}(it)^2\boldsymbol{z}'\Omega\boldsymbol{z}\}] + O(n^{-1/2}), \\ &= \frac{1}{2}(1-\varphi^{-1})i^{-1}\left\{\frac{d}{dt}R_2(t)\right\} + \frac{1}{8}(1-\varphi^{-1})^2 i^{-2}\left\{\frac{d^2}{dt^2}R_2(t)\right\} + O(n^{-1/2}), \end{aligned} \tag{3.11}$$

where

$$R_2(t) = E_{\boldsymbol{z}}[\exp\{it\boldsymbol{z}'\Omega\boldsymbol{z}\}\{\zeta^2 + \kappa_3^2(\bar{\boldsymbol{\chi}}_1'\boldsymbol{z})^2\}].$$

Using an argument similar to the one as in the reduction of $R_1(t)$ we can reduce $R_2(t)$ to

$$R_2(t) = \varphi^{h/2} E_{\boldsymbol{u}}[\zeta^2 + \kappa_3^2(\bar{\boldsymbol{\chi}}_1'\boldsymbol{u})^2],$$

where $\boldsymbol{u} \sim N_k(\boldsymbol{0}, \Gamma)$. Note that

$$\sum_{a,b}^{k} \bar{\chi}_a \bar{\chi}_b \omega_{ab} = \frac{1}{n}\sum_{i,j}^{n}(\boldsymbol{x}_i \Upsilon \boldsymbol{x}_j) = \frac{1}{n}\boldsymbol{1}_n'\Psi\boldsymbol{1}_n.$$

Therefore we can calculate

$$R_2(t) = \varphi^{h/2}[\kappa_4 + 2 + \kappa_3^2(\varphi - 1)\{n^{-1}(\boldsymbol{1}_n'\Psi\boldsymbol{1}_n)\}].$$

It is easy to evaluate that

$$\begin{aligned} \frac{d}{dt}R_2(t) &= i\varphi^{1+h/2}[h(\kappa_4+2) + \kappa_3^2\{(h+2)\varphi - h\}\{n^{-1}(\boldsymbol{1}_n'\Psi\boldsymbol{1}_n)\}], \\ \frac{d^2}{dt^2}R_2(t) &= i^2\varphi^{2+h/2}[h(h+2)(\kappa_4+2) \\ &\qquad + \kappa_3^2\{(h+2)(h+4)\varphi - h(h+2)\}\{n^{-1}(\boldsymbol{1}_n'\Psi\boldsymbol{1}_n)\}]. \end{aligned} \tag{3.12}$$

Finally, substituting (3.12) into (3.11) yields

$$\begin{aligned} C_2(t) &= \tfrac{1}{8}\varphi^{h/2}(\varphi-1)[h\{(h+2)\varphi - (h-2)\}(\kappa_4+2) \\ &\qquad + \kappa_3^2\{(h+2)(h+4)\varphi^2 - 2h(h+2)\varphi + h(h-2)\}\{n^{-1}(\boldsymbol{1}_n'\Psi\boldsymbol{1}_n)\}] \\ &\qquad + O(n^{-1/2}). \end{aligned} \tag{3.13}$$

3.5. *Final result*

Using (3.4), (3.10) and (3.13), we can obtain an expansion of the characteristic function of T given by

$$C_T(t) = \varphi^{h/2}\left[1 + \frac{1}{n}\sum_{j=0}^{3} b_j\varphi^j + o(n^{-1})\right],$$

where

$$b_0 = -\kappa_3^2\{a_2 - ha_3 + h(h-2)a_4\} + \kappa_4 a_1 + \tfrac{1}{4}h(h-2),$$

$$b_1 = \kappa_3^2\{3a_2 - (3h+4)a_3 + h(3h+2)a_4\} - 2\kappa_4 a_1 - \tfrac{1}{2}h^2,$$

$$b_2 = -\kappa_3^2\{3a_2 - (3h+8)a_3 + (h+2)(3h+4)a_4\} + \kappa_4 a_1 + \tfrac{1}{4}h(h+2),$$

$$b_3 = \kappa_3^2\{a_2 - (h+4)a_3 + (h+2)(h+4)a_4\} \tag{3.14}$$

and

$$a_1 = \tfrac{1}{8}\{n\,\mathrm{tr}(D_\Psi^2) - h(h+2)\},$$

$$a_2 = \frac{n}{24}\{\mathbf{1}_n'(2\Psi_{(3)} + 3D_\Psi \Psi D_\Psi)\mathbf{1}_n\},$$

$$a_3 = \frac{1}{12}\mathbf{1}_n' D_\Psi \Psi \mathbf{1}_n, \quad a_4 = \frac{1}{8n}\mathbf{1}_n'\Psi\mathbf{1}_n.$$

By inverting $C_T(t)$, we have the following Theorem 3.1.

Theorem 3.1. *Under assumptions* C3, C6 *and* C7, *the null distribution of T can be expanded as*

$$\mathrm{P}(T \leqslant x) = G_h(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{h+2j}(x) + o(n^{-1}), \tag{3.15}$$

where G_f is the distribution function of a central chi-squared distribution with f degrees of freedom and the coefficients b_j are given by (3.14).

It may be noted that the final result depends on the cumulants up to the fourth order. Therefore, it is conjectured that assumption C7 may be weakened to $E(\varepsilon_i^4) < \infty$. This fact has been proved for the t-statistic by Hall [6].

Before concluding this section, we state the next corollary which is different form of Theorem 3.1.

Corollary 3.2. *Under the same assumptions as in Theorem* 3.1, *the asymptotic expansion* (3.15) *can be written as*

$$\mathrm{P}(T \leqslant x) = G_h(x) - \frac{2x}{nh} g_h(x)\left\{b_1 + b_2 + b_3 + \frac{(b_2+b_3)x}{h+2} + \frac{b_3 x^2}{(h+2)(h+4)}\right\} + o(n^{-1}), \tag{3.16}$$

where $g_h(x)$ is the density function of a central chi-squared distribution with h degrees of freedom and the coefficients b_j are given by (3.14).

4. Some applications

4.1. One-way ANOVA test statistic

In this section, we obtain asymptotic expansions of the null distribution of ANOVA test statistics by applying Theorem 3.1.

Firstly, we consider one-way ANOVA test statistic. Let y_{ij} be the jth sample observation ($j = 1, \ldots, n_i$) from the ith population ($i = 1, \ldots, q$) with mean μ_i and common variance σ^2, where μ_i's and σ^2 are unknown. Consider testing for the null hypothesis

$$\mathrm{H}_0 : \mu_1 = \cdots = \mu_q.$$

Let $n = n_1 + n_2 + \cdots + n_q$, $\bar{y}_i = n_i^{-1} \sum_{j=1}^{n_i} y_{ij}$ and $\bar{y} = n^{-1} \sum_{i=1}^{q} \sum_{j=1}^{n_i} y_{ij}$. A commonly used test statistics is

$$T = (n-q) S_h^2 / S_e^2, \tag{4.1}$$

where $S_h^2 = \sum_{i=1}^{q} n_i(\bar{y}_i - \bar{y})^2$, $S_i^2 = (n_i - 1)^{-1} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2$, and $S_e^2 = (n_1 - 1)S_1^2 + \cdots + (n_q - 1)S_q^2$. In order to apply Theorem 3.1, we need to rewrite (4.1) like (1.2). Let

$$X = \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{1}_{n_q} \end{pmatrix} \quad (n \times q \text{ matrix}), \tag{4.2}$$

$\boldsymbol{y} = (y_{11}, \ldots, y_{1n_1}, y_{21}, \ldots, y_{2n_2}, \ldots, y_{q1}, \ldots, y_{qn_q})'$, $\boldsymbol{\beta} = (\mu_1, \mu_2, \ldots, \mu_q)'$ and

$$\Omega = (I_q - \boldsymbol{\rho}\boldsymbol{\rho}'), \tag{4.3}$$

where

$$\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_q)' = \left(\sqrt{\frac{n_1}{n}}, \sqrt{\frac{n_2}{n}}, \ldots, \sqrt{\frac{n_q}{n}} \right)'.$$

It is easily seen that $\mathrm{rank}(\Omega) = q - 1$. Then we can write the test statistic and the null hypothesis as

$$T = \frac{\boldsymbol{z}'\Omega\boldsymbol{z}}{\hat{\sigma}^2}, \quad \mathrm{H}_0 : PD\boldsymbol{\beta} = \mathbf{0},$$

where P is $(q-1) \times q$ matrix with $\mathrm{rank}(P) = q - 1$, $PP' = I_{q-1}$ and $P'P = I_q - \boldsymbol{\rho}\boldsymbol{\rho}'$. and D is $\mathrm{diag}(\sqrt{n_1}, \sqrt{n_2}, \ldots, \sqrt{n_q})$. Using (4.1) and (4.3), we can evaluate the coefficients in an asymptotic expansion of (4.1). Noting that

$$(X'X)^{-1/2}X' = \begin{pmatrix} n_1^{-1/2}\mathbf{1}'_{n_1} & \mathbf{0}' & \dots & \mathbf{0}' \\ \mathbf{0}' & n_2^{-1/2}\mathbf{1}'_{n_2} & \dots & \mathbf{0}' \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}' & \mathbf{0}' & \dots & n_q^{-1/2}\mathbf{1}'_{n_q} \end{pmatrix},$$

we can derive easily that

$$\begin{aligned} \Psi &= X(X'X)^{-1/2}\Omega(X'X)^{-1/2}X' \\ &= \begin{pmatrix} \omega_{11}n_1^{-1}\mathbf{1}_{n_1}\mathbf{1}'_{n_1} & \cdots & \omega_{1q}n_1^{-1/2}n_q^{-1/2}\mathbf{1}_{n_1}\mathbf{1}'_{n_q} \\ \vdots & \ddots & \vdots \\ \omega_{q1}n_q^{-1/2}n_1^{-1/2}\mathbf{1}_{n_q}\mathbf{1}'_{n_1} & \cdots & \omega_{qq}n_q^{-1}\mathbf{1}_{n_q}\mathbf{1}'_{n_q} \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} a_1 &= \frac{1}{8}\left\{ \sum_{a=1}^{q} \rho_a^{-2}\omega_{aa}^2 - (q-1)(q+1) \right\} = \frac{1}{8}\left(\sum_{a=1}^{q} \rho_a^{-2} - q^2 - 2q + 2 \right), \\ a_2 &= \frac{1}{24}\sum_{a,b}^{q} \rho_a^{-1}\rho_a^{-1}\left(2\omega_{ab}^3 + 3\omega_{aa}\omega_{ab}\omega_{bb}\right) = \frac{1}{24}\left(5\sum_{a=1}^{q} \rho_a^{-2} - 3q^2 - 6q + 4 \right), \\ a_3 &= \frac{1}{12}\sum_{a,b}^{q} \rho_a^{-1}\rho_b\omega_{aa}\omega_{ab} = 0, \quad a_4 = \frac{1}{8}\sum_{a,b}^{q} \rho_a\rho_b\omega_{ab} = 0. \end{aligned} \tag{4.4}$$

Next, we consider assumptions C3 and C7. It is easily shown that $n^{-1}\sum_{i=1}^{n} ||\boldsymbol{x}_i||^r = 1$ and $n/n_i \leqslant n/\lambda_n$, C3 and C7 are replaced by

C8. y_{ij} have the 10th absolute moment and $n/n_i = O(1)$ $(i = 1, 2, \dots, q)$.

Using these results yield the following Theorem 4.1.

Theorem 4.1. *Let the αth cumulant of $(y_{ij} - \mu_i)/\sigma$ be denoted by κ_α. Under assumptions* C6 *and* C8, *the null distribution of one-way ANOVA test statistic can be expanded as*

$$\mathrm{P}(T \leqslant x) = G_{q-1}(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{q-1+2j}(x) + o(n^{-1}),$$

where

$$\begin{aligned} b_0 &= \tfrac{1}{4}(q-1)(q-3) - a_2\kappa_3^2 + a_1\kappa_4, \\ b_1 &= -\tfrac{1}{2}(q-1)^2 + 3a_2\kappa_3^2 - 2a_1\kappa_4, \\ b_2 &= \tfrac{1}{4}(q^2-1) - 3a_2\kappa_3^2 + a_1\kappa_4, \quad b_3 = a_2\kappa_3^2. \end{aligned}$$

Here the coefficients a_1 and a_2 are given by (4.4).

The coefficients b_j in Theorem 4.1 are coincide with the ones in [5].

4.2. Two-way ANOVA test statistic with balanced replications

Secondly, we consider two-way ANOVA test statistics. Our model is

$$y_{ijk} = \mu + \eta_i + \theta_j + v_{ij} + \varepsilon_{ijk} \quad (1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q, 1 \leqslant k \leqslant m),$$

where $\varepsilon_{ijk} \sim$ i.i.d. with mean 0 and finite positive variance σ^2. Here we make the usual constraints on η_i, θ_j and v_{ij}, defined by $\sum_{i=1}^p \eta_i = 0$, $\sum_{j=1}^q \theta_j = 0$ and $\sum_{i=1}^p v_{ij} = \sum_{j=1}^q v_{ij} = 0$. We want to test for three different hypotheses; (1) All the $v_{ij} = 0$, (2) All the $\eta_i = 0$, (3) All the $\theta_j = 0$. Let $n = pqm$, $\bar{y}_{ij} = m^{-1} \sum_{k=1}^m y_{ijk}$, $\bar{y}_{i.} = (mq)^{-1} \sum_{j=1}^q \sum_{k=1}^m y_{ijk}$, $\bar{y}_{.j} = (mp)^{-1} \sum_{i=1}^p \sum_{k=1}^m y_{ijk}$ and $\bar{y}_{..} = n^{-1} \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^m y_{ijk}$. The usual test statistic for hypothesis (1) is

$$T_{(1)} = \frac{(n-pq)m \sum_{i=1}^p \sum_{j=1}^q (\bar{y}_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2}{\sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^m (y_{ijk} - \bar{y}_{ij})^2}. \tag{4.5}$$

The usual test statistic for hypothesis (2) is

$$T_{(2)} = \frac{(n-pq)mq \sum_{i=1}^p (\bar{y}_{i.} - \bar{y}_{..})^2}{\sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^m (y_{ijk} - \bar{y}_{ij})^2}. \tag{4.6}$$

Besides the test statistic for hypothesis (3) is similar to (4.6).

As in Section 4.1, we rewrite (4.5) and (4.6) like (1.2). Let

$$X = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_m & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_m \end{pmatrix} \quad (n \times pq \text{ matrix}),$$

$\boldsymbol{y} = (y_{111}, \dots, y_{11m}, y_{121}, \dots, y_{12m}, \dots, y_{pq1}, \dots, y_{pqm})'$, and

$$\Omega_{(1)} = I_{pq} - \frac{1}{q} I_p \otimes \mathbf{1}_q \mathbf{1}_q' - \frac{1}{p} \mathbf{1}_p \mathbf{1}_p' \otimes I_q + \frac{1}{pq} \mathbf{1}_{pq} \mathbf{1}_{pq}',$$

$$\Omega_{(2)} = \frac{1}{q} I_p \otimes \mathbf{1}_q \mathbf{1}_q' - \frac{1}{pq} \mathbf{1}_{pq} \mathbf{1}_{pq}'.$$

It is easily seen that $\mathrm{rank}(\Omega_{(1)}) = (p-1)(q-1)$ and $\mathrm{rank}(\Omega_{(2)}) = (p-1)$. Then, the test statistics (4.5) and (4.6) are rewritten as

$$T_{(1)} = \frac{\boldsymbol{z}' \Omega_{(1)} \boldsymbol{z}}{\hat{\sigma}^2}, \quad T_{(2)} = \frac{\boldsymbol{z}' \Omega_{(2)} \boldsymbol{z}}{\hat{\sigma}^2}.$$

Noting that

$$\psi_{\alpha\beta}^{(1)} = \frac{1}{m} \boldsymbol{x}_\alpha' \Omega_{(1)} \boldsymbol{x}_\beta, \quad \psi_{\alpha\beta}^{(2)} = \frac{1}{m} \boldsymbol{x}_\alpha' \Omega_{(2)} \boldsymbol{x}_\beta,$$

Table 1
Percentage points of $T_{(1)}$

				Upper 5% points				Upper 1% points			
Model	p	q	m	$t(u)$	u	$t_{\mathrm{E}}(u)$	$\hat{t}_{\mathrm{E}}(u)$	$t(u)$	u	$t_{\mathrm{E}}(u)$	$\hat{t}_{\mathrm{E}}(u)$
(i)	2	3	5	6.746	5.991	6.589	6.595	10.85	9.210	10.62	10.65
$\kappa_3 = 0.0$	2	3	10	6.525	5.991	6.290	6.295	10.72	9.210	9.917	9.936
$\kappa_4 = 0.0$	2	3	15	6.030	5.991	6.190	6.194	9.353	9.210	9.681	9.695
	3	3	5	10.68	9.488	10.28	10.29	16.02	13.28	14.94	15.00
	3	3	10	9.788	9.488	9.883	9.888	14.44	13.28	14.11	14.13
	3	3	15	9.780	9.488	9.751	9.762	13.90	13.28	13.83	13.87
	5	3	5	16.58	15.51	16.49	16.48	21.67	20.09	21.98	21.95
	5	3	10	16.02	15.51	16.00	16.01	20.73	20.09	21.03	21.05
	5	3	15	15.87	15.51	15.84	15.84	20.43	20.09	20.72	20.72
(ii)	2	3	5	6.648	5.991	6.539	6.438	10.79	9.210	10.42	10.01
$\kappa_3 = 0.0$	2	3	10	6.352	5.991	6.265	6.254	9.899	9.210	9.817	9.772
$\kappa_4 = 1.5$	2	3	15	6.220	5.991	6.174	6.192	9.677	9.210	9.615	9.689
	3	3	5	10.43	9.488	10.22	10.31	15.14	13.28	14.77	15.04
	3	3	10	9.837	9.4880	9.852	9.886	14.10	13.28	14.02	14.12
	3	3	15	9.786	9.488	9.731	9.731	13.69	13.28	13.78	13.78
	5	3	5	16.70	15.51	16.44	16.43	22.31	20.09	21.84	21.85
	5	3	10	16.17	15.51	15.97	15.97	21.38	20.09	20.97	20.96
	5	3	15	15.90	15.51	15.82	15.83	20.53	20.09	20.67	20.70
(iii)	2	3	5	6.675	5.991	6.291	6.593	10.92	9.210	9.424	10.64
$\kappa_3 = 0.0$	2	3	10	6.115	5.991	6.141	6.226	9.314	9.210	9.317	9.661
$\kappa_4 = 6.0$	2	3	15	6.271	5.991	6.091	6.172	9.851	9.210	9.281	9.610
	3	3	5	10.34	9.488	9.910	10.28	15.33	13.28	13.87	14.96
	3	3	10	9.912	9.488	9.699	9.850	14.19	13.28	13.58	14.02
	3	3	15	9.661	9.488	9.629	9.735	14.11	13.28	13.48	13.79
	5	3	5	16.64	15.51	16.15	16.49	22.10	20.09	21.17	21.98
	5	3	10	16.16	15.51	15.83	15.92	21.50	20.09	20.63	20.85
	5	3	15	15.84	15.51	15.72	15.72	20.85	20.09	20.45	20.45
(iv)	2	3	5	6.918	5.991	6.639	6.648	11.43	9.210	10.82	10.86
$\kappa_3 = 0.0$	2	3	10	6.326	5.991	6.315	6.321	10.36	9.210	10.02	10.04
$\kappa_4 = -1.2$	2	3	15	6.209	5.991	6.207	6.207	9.445	9.210	9.748	9.748
	3	3	5	10.61	9.488	10.34	10.33	16.35	13.28	15.12	15.11
	3	3	10	9.968	9.488	9.913	9.913	14.65	13.28	14.20	14.20
	3	3	15	9.831	9.488	9.772	9.773	13.92	13.28	13.89	13.90
	5	3	5	16.87	15.51	16.55	16.56	23.37	20.09	22.11	22.13
	5	3	10	16.10	15.51	16.03	16.03	21.07	20.09	21.10	21.11
	5	3	15	15.79	15.51	15.86	15.86	21.08	20.09	20.76	20.77
(v)	2	3	5	6.472	5.991	6.390	6.554	10.70	9.210	9.824	10.48
$\kappa_3 = \sqrt{8/3}$	2	3	10	6.462	5.991	6.191	6.252	10.16	9.210	9.517	9.762
$\kappa_4 = 4.0$	2	3	15	6.127	5.991	6.124	6.164	9.433	9.210	9.415	9.573
	3	3	5	10.38	9.488	10.02	9.893	15.05	13.28	14.24	14.03
	3	3	10	9.965	9.488	9.755	9.825	14.28	13.28	13.76	13.98
	3	3	15	9.936	9.488	9.666	9.734	13.86	13.28	13.60	13.80
	5	3	5	16.87	15.51	16.26	16.35	23.21	20.09	21.48	21.81

Table 1 (*continued*)

Model	p	q	m	Upper 5% points $t(u)$	u	$t_{\mathrm{E}}(u)$	$\hat{t}_{\mathrm{E}}(u)$	Upper 1% points $t(u)$	u	$t_{\mathrm{E}}(u)$	$\hat{t}_{\mathrm{E}}(u)$
	5	3	10	15.93	15.51	15.88	15.93	21.15	20.09	20.79	20.93
	5	3	15	15.97	15.51	15.76	15.80	21.07	20.09	20.55	20.66
(vi)	2	3	5	6.578	5.991	6.540	6.625	11.18	9.210	10.42	10.77
$\kappa_3 = 1.0$	2	3	10	6.164	5.991	6.265	6.297	9.967	9.210	9.817	9.942
$\kappa_4 = 1.5$	2	3	15	6.102	5.991	6.174	6.184	9.448	9.210	9.615	9.655
	3	3	5	10.50	9.488	10.21	9.905	15.59	13.28	14.78	14.02
	3	3	10	9.820	9.488	9.847	9.867	14.12	13.28	14.03	14.08
	3	3	15	9.810	9.488	9.727	9.743	13.94	13.28	13.78	13.82
	5	3	5	16.65	15.51	16.43	16.47	21.99	20.09	21.89	21.96
	5	3	10	15.96	15.51	15.97	16.00	21.38	20.09	20.99	21.04
	5	3	15	15.84	15.51	15.82	15.81	20.84	20.09	20.69	20.67

we can obtain easily that

$$\psi_{\alpha\beta}^{(1)} = \frac{(p-1)(q-1)}{mpq}, \quad \psi_{\alpha\beta}^{(2)} = \frac{(p-1)}{mpq},$$

$$\sum_{\alpha,\beta}^{n} \psi_{\alpha\beta}^{(1)} = \sum_{a,b}^{pq} \omega_{ab}^{(1)} = 0, \quad \sum_{\alpha,\beta}^{n} \psi_{\alpha\beta}^{(2)} = \sum_{a,b}^{pq} \omega_{ab}^{(2)} = 0.$$

From these equations, the coefficients a_j's can be calculated easily.

Next we consider assumptions C3 and C7. It is easily shown that $n^{-1} \sum_{i=1}^{n} ||\boldsymbol{x}_i||^r = 1$ and $n/\lambda_n = n/m$, C3 and C7 are replaced by

C9. ε_{ijk} have the 10th absolute moment and $n/m = O(1)$.

Using these results yield the following Theorems 4.2 and 4.3.

Theorem 4.2. *Let the αth cumulant of ε_{ijk}/σ be denoted by κ_α. Under assumptions* C6 *and* C9, *the null distribution of $T_{(1)}$ can be expanded as*

$$\mathrm{P}(T_{(1)} \leqslant x) = G_{(p-1)(q-1)}(x) + \frac{1}{n} \sum_{j=0}^{3} b_j G_{(p-1)(q-1)+2j}(x) + o(n^{-1}),$$

where

$$b_0 = -\frac{\kappa_3^2}{12}(p-1)(p-2)(q-1)(q-2) - \frac{\kappa_4}{4}(p-1)(q-1) + \frac{1}{4}(p-1)(q-1)(pq-p-q-1),$$

$$b_1 = \frac{\kappa_3^2}{4}(p-1)(p-2)(q-1)(q-2) + \frac{\kappa_4}{2}(p-1)(q-1) - \frac{1}{2}(p-1)^2(q-1)^2,$$

Table 2
Percentage points of $T_{(2)}$

Model	p	q	m	Upper 5% points $t(u)$	u	$t_E(u)$	$\hat{t}_E(u)$	Upper 1% points $t(u)$	u	$t_E(u)$	$\hat{t}_E(u)$
(i)	2	3	5	4.336	3.841	4.151	4.148	7.943	6.635	7.479	7.457
$\kappa_3 = 0.0$	2	3	10	3.976	3.841	3.996	3.994	6.859	6.635	7.057	7.038
$\kappa_4 = 0.0$	2	3	15	3.984	3.841	3.944	3.943	7.159	6.635	6.916	6.904
	3	3	5	6.391	5.991	6.390	6.381	10.17	9.210	10.15	10.12
	3	3	10	6.119	5.991	6.190	6.187	9.340	9.210	9.681	9.668
	3	3	15	6.129	5.991	6.124	6.118	10.07	9.210	9.524	9.500
	5	3	5	9.738	9.488	9.962	9.967	14.05	13.28	14.28	14.30
	5	3	10	9.656	9.488	9.725	9.721	13.74	13.28	13.78	13.77
	5	3	15	9.761	9.488	9.646	9.647	13.53	13.28	13.61	13.62
(ii)	2	3	5	4.138	3.841	4.178	4.246	7.727	6.635	7.680	8.110
$\kappa_3 = 0.0$	2	3	10	4.119	3.841	4.009	4.018	7.183	6.635	7.158	7.207
$\kappa_4 = 1.5$	2	3	15	3.919	3.841	3.953	3.943	6.690	6.635	6.983	6.909
	3	3	5	6.486	5.991	6.423	6.371	10.41	9.210	10.29	10.08
	3	3	10	6.259	5.991	6.207	6.188	9.493	9.210	9.748	9.673
	3	3	15	6.165	5.991	6.134	6.135	9.692	9.210	9.569	9.567
	5	3	5	10.03	9.488	9.989	9.989	14.15	13.28	14.36	14.37
	5	3	10	9.843	9.488	9.739	9.739	14.04	13.28	13.82	13.82
	5	3	15	9.760	9.488	9.655	9.649	14.22	13.28	13.64	13.62
(iii)	2	3	5	4.213	3.841	4.312	4.151	7.048	6.635	8.685	7.467
$\kappa_3 = 0.0$	2	3	10	4.088	3.841	4.077	4.031	7.211	6.635	7.660	7.315
$\kappa_4 = 6.0$	2	3	15	3.980	3.841	3.998	3.954	6.766	6.635	7.318	6.987
	3	3	5	6.408	5.991	6.589	6.387	10.02	9.210	10.95	10.14
	3	3	10	6.208	5.991	6.290	6.208	9.647	9.210	10.08	9.752
	3	3	15	6.083	5.991	6.190	6.133	9.465	9.210	9.791	9.559
	5	3	5	10.03	9.488	10.13	9.962	14.64	13.28	14.76	14.28
	5	3	10	9.741	9.488	9.808	9.762	13.66	13.28	14.02	13.89
	5	3	15	9.626	9.488	9.701	9.703	13.97	13.28	13.77	13.78
(iv)	2	3	5	4.156	3.841	4.124	4.119	7.837	6.635	7.278	7.242
$\kappa_3 = 0.0$	2	3	10	4.034	3.841	3.982	3.979	6.984	6.635	6.957	6.932
$\kappa_4 = -1.2$	2	3	15	4.108	3.841	3.935	3.935	7.239	6.635	6.849	6.849
	3	3	5	6.5212	5.991	6.357	6.359	10.72	9.210	10.02	10.03
	3	3	10	6.264	5.991	6.174	6.174	9.806	9.210	9.615	9.614
	3	3	15	6.197	5.991	6.113	6.112	9.476	9.210	9.480	9.477
	5	3	5	10.03	9.488	9.934	9.931	14.80	13.28	14.20	14.19
	5	3	10	9.967	9.488	9.711	9.709	14.17	13.28	13.74	13.73
	5	3	15	9.972	9.488	9.637	9.635	13.77	13.28	13.59	13.58
(v)	2	3	5	4.156	3.841	4.267	4.182	7.324	6.635	8.295	7.639
$\kappa_3 = \sqrt{8/3}$	2	3	10	3.866	3.841	4.054	4.023	6.901	6.635	7.465	7.221
$\kappa_4 = 4.0$	2	3	15	3.874	3.841	3.983	3.963	6.803	6.635	7.188	7.031
	3	3	5	6.461	5.991	6.522	6.571	10.34	9.210	10.69	10.88
	3	3	10	6.229	5.991	6.257	6.217	9.364	9.210	9.948	9.789
	3	3	15	6.129	5.991	6.168	6.131	9.620	9.210	9.702	9.553
	5	3	5	9.898	9.488	10.07	10.01	14.62	13.28	14.61	14.46

Table 2 (*continued*)

Model	p	q	m	Upper 5% points				Upper 1% points			
				$t(u)$	u	$t_{\mathrm{E}}(u)$	$\hat{t}_{\mathrm{E}}(u)$	$t(u)$	u	$t_{\mathrm{E}}(u)$	$\hat{t}_{\mathrm{E}}(u)$
	5	3	10	9.761	9.488	9.777	9.751	13.91	13.28	13.94	13.88
	5	3	15	9.734	9.488	9.681	9.660	13.92	13.28	13.72	13.66
(vi)	2	3	5	4.220	3.841	4.186	4.132	7.573	6.635	7.692	7.334
$\kappa_3 = 1.0$	2	3	10	4.040	3.841	4.014	3.994	7.271	6.635	7.164	7.033
$\kappa_4 = 1.5$	2	3	15	3.932	3.841	3.956	3.950	6.767	6.635	6.987	6.946
	3	3	5	6.382	5.991	6.423	6.570	10.49	9.210	10.29	10.88
	3	3	10	6.246	5.991	6.207	6.197	9.996	9.210	9.748	9.707
	3	3	15	6.131	5.991	6.135	6.127	9.652	9.210	9.569	9.536
	5	3	5	10.01	9.488	9.984	9.967	13.87	13.28	14.37	14.30
	5	3	10	9.827	9.488	9.736	9.724	14.11	13.28	13.82	13.79
	5	3	15	9.556	9.488	9.653	9.657	13.61	13.28	13.64	13.65

$$\begin{aligned} b_2 = & -\frac{\kappa_3^2}{4}(p-1)(p-2)(q-1)(q-2) \\ & -\frac{\kappa_4}{4}(p-1)(q-1)+\frac{1}{4}(p-1)(q-1)(pq-p-q+3), \\ b_3 = & \frac{\kappa_3^2}{12}(p-1)(p-2)(q-1)(q-2). \end{aligned}$$

Theorem 4.3. *Let the* α*th cumulant of* ε_{ijk}/σ *be denoted by* κ_α. *Under assumptions* C6 *and* C9, *the null distribution of* $T_{(2)}$ *can be expanded as*

$$\mathrm{P}(T_{(2)} \leqslant x) = G_{p-1}(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{p-1+2j}(x) + o(n^{-1}),$$

where

$$\begin{aligned} b_0 = & -\frac{\kappa_3^2}{12pq}\{(p-1)^3-(pq-1)\}+\frac{\kappa_4}{8}(pq-2p+1)+\frac{1}{4}(p-1)(p-3), \\ b_1 = & \frac{\kappa_3^2}{4pq}\{(p-1)^3-(pq-1)\}-\frac{\kappa_4}{4}(pq-2p+1)-\frac{1}{2}(p-1)^2, \\ b_2 = & -\frac{\kappa_3^2}{4pq}\{(p-1)^3-(pq-1)\}+\frac{\kappa_4}{8}(pq-2p+1)+\frac{1}{4}(p-1)(p+1), \\ b_0 = & \frac{\kappa_3^2}{12pq}\{(p-1)^3-(pq-1)\}. \end{aligned}$$

As stated above, using Theorem 3.1 we can obtain an asymptotic expansion easily when X and Ω are given in an explicit form.

Table 3
Actual test sizes of $T_{(1)}$

Model	p	q	m	Nominal 5% test			Nominal 1% test		
				α_1	α_2	α_3	α_1	α_2	α_3
(i)	2	3	5	6.7	5.2	5.2	1.9	1.1	1.1
$\kappa_3 = 0.0$	2	3	10	6.1	5.5	5.4	1.7	1.3	1.3
$\kappa_4 = 0.0$	2	3	15	5.1	4.7	4.6	1.1	0.8	0.8
	3	3	5	7.3	5.6	5.6	2.3	1.4	1.4
	3	3	10	5.7	4.8	4.8	1.4	1.1	1.1
	3	3	15	5.5	5.1	5.1	1.3	1.0	1.0
	5	3	5	6.8	5.1	5.1	1.7	1.0	1.0
	5	3	10	5.9	5.0	5.0	1.2	0.9	0.9
	5	3	15	5.5	5.1	5.1	1.1	0.9	0.9
(ii)	2	3	5	6.7	5.3	5.5	2.0	1.2	1.4
$\kappa_3 = 0.0$	2	3	10	5.9	5.3	5.3	1.3	1.0	1.1
$\kappa_4 = 1.5$	2	3	15	5.6	5.1	5.1	1.3	1.0	1.0
	3	3	5	6.7	5.4	5.2	1.9	1.2	1.0
	3	3	10	5.9	5.0	4.9	1.4	1.0	1.0
	3	3	15	5.7	5.2	5.2	1.2	1.0	1.0
	5	3	5	6.9	5.3	5.3	2.0	1.1	1.1
	5	3	10	6.0	5.3	5.3	1.5	1.2	1.2
	5	3	15	5.5	5.1	5.1	1.2	1.0	1.0
(iii)	2	3	5	6.8	5.9	5.2	1.9	1.8	1.1
$\kappa_3 = 0.0$	2	3	10	5.3	4.9	4.7	1.1	1.0	0.8
$\kappa_4 = 6.0$	2	3	15	5.6	5.4	5.2	1.3	1.3	1.1
	3	3	5	6.6	5.6	5.1	1.8	1.6	1.1
	3	3	10	6.0	5.6	5.2	1.4	1.3	1.1
	3	3	15	5.4	5.0	4.8	1.3	1.2	1.1
	5	3	5	7.0	5.7	5.2	1.7	1.3	1.0
	5	3	10	6.0	5.5	5.3	1.4	1.2	1.1
	5	3	15	5.7	5.2	5.3	1.3	1.1	1.1
(iv)	2	3	5	7.3	5.7	5.6	2.2	1.2	1.2
$\kappa_3 = 0.0$	2	3	10	5.9	5.0	5.0	1.6	1.1	1.1
$\kappa_4 = -1.2$	2	3	15	5.6	5.0	5.0	1.1	0.8	0.8
	3	3	5	7.1	5.4	5.5	2.3	1.4	1.5
	3	3	10	5.8	5.1	5.1	1.6	1.1	1.1
	3	3	15	5.6	5.1	5.1	1.3	1.0	1.0
	5	3	5	7.3	5.5	5.5	2.3	1.4	1.4
	5	3	10	6.0	5.1	5.1	1.3	1.0	1.0
	5	3	15	5.4	4.9	4.9	1.4	1.2	1.1
(v)	2	3	5	6.2	5.2	4.9	1.7	1.4	1.0
$\kappa_3 = \sqrt{8/3}$	2	3	10	6.1	5.5	5.4	1.4	1.3	1.1
$\kappa_4 = 4.0$	2	3	15	5.4	5.0	4.9	1.1	1.0	0.9
	3	3	5	7.0	5.8	6.1	1.8	1.4	1.4
	3	3	10	6.1	5.4	5.3	1.6	1.3	1.2
	3	3	15	5.9	5.4	5.3	1.3	1.2	1.1
	5	3	5	7.2	6.0	5.8	2.1	1.4	1.3

Table 3 (*continued*)

				Nominal 5% test			Nominal 1% test		
Model	p	q	m	α_1	α_2	α_3	α_1	α_2	α_3
	5	3	10	5.6	5.1	5.0	1.3	1.1	1.1
	5	3	15	5.6	5.3	5.2	1.3	1.1	1.1
(vi)	2	3	5	6.6	5.1	4.9	1.9	1.3	1.1
$\kappa_3 = 1.0$	2	3	10	5.4	4.8	4.8	1.5	1.1	1.0
$\kappa_4 = 1.5$	2	3	15	5.3	4.9	4.9	1.2	0.9	0.9
	3	3	5	6.9	5.5	5.9	1.9	1.2	1.5
	3	3	10	5.8	4.9	4.9	1.4	1.0	1.0
	3	3	15	5.6	5.1	5.1	1.2	1.1	1.0
	5	3	5	6.7	5.4	5.3	1.7	1.0	1.0
	5	3	10	5.8	5.0	4.9	1.4	1.1	1.1
	5	3	15	5.4	5.0	5.0	1.3	1.0	1.1

5. Numerical accuracies

Numerical accuracies are studied for approximations of the percentage points and actual test sizes of two-way ANOVA test statistics $T_{(1)}$ and $T_{(2)}$ in Section 4.2. The approximations considered are based on the limiting distributions and asymptotic expansions. In order to examine the influence of κ_3 and κ_4 on accuracies, we considered the following five nonnormal models and the normal model with $p = 2, 3, 5$ and $q = 3$;

(i) normal distribution,
(ii) $X + YZ$, where X, Y, Z are independent normal distribution $N(0, 1)$,
(iii) Student's t-distribution with 5 degrees of freedom,
(iv) symmetric uniform distribution $U(-5, 5)$,
(v) χ^2 distribution with 3 degrees of freedom,
(vi) χ^2 distribution with 8 degrees of freedom.

The first four models are symmetric, and have κ_3 =0. For (iii) Student's t-distribution we choose the one with 5 degrees of freedom, since κ_4 is biggest of all. For (iv) uniform distribution we use $U(-5, 5)$ with width 10, but our results are not much effected by width. In (v) and (vi) we choose χ^2 distributions with difference of degrees of freedom which are nonsymmetric. We consider the normal model as the model with $\kappa_3 = 0$, $\kappa_4 = 0$.

The true percentage points $t(u)$ of T were obtained by simulation experiments which were iterated 10,000 times. There was not a large difference among values of $t(u)$ which was constructed by other samples. The approximate percentage points were computed by using u, $t_{\mathrm{E}}(u)$ and $\hat{t}_{\mathrm{E}}(u)$. The $t_{\mathrm{E}}(u)$ is defined as follow. Let

$$\mathrm{P}(T \leqslant t(u)) = \mathrm{P}(\chi_h^2 \leqslant u),$$

Table 4
Actual test sizes of $T_{(2)}$

Model	p	q	m	Nominal 5% test			Nominal 1% test		
				α_1	α_2	α_3	α_1	α_2	α_3
(i)	2	3	5	6.5	5.6	5.6	1.8	1.3	1.3
$\kappa_3 = 0.0$	2	3	10	5.4	4.9	4.9	1.1	0.9	0.9
$\kappa_4 = 0.0$	2	3	15	5.4	5.2	5.2	1.2	1.1	1.1
	3	3	5	6.0	5.0	5.0	1.5	1.0	1.0
	3	3	10	5.3	4.8	4.8	1.1	0.8	0.9
	3	3	15	5.4	5.0	5.0	1.5	1.3	1.3
	5	3	5	5.7	4.6	4.6	1.3	0.9	0.9
	5	3	10	5.3	4.9	4.9	1.2	1.0	1.0
	5	3	15	5.5	5.3	5.3	1.1	0.9	0.9
(ii)	2	3	5	5.9	4.9	4.7	1.6	1.0	0.8
$\kappa_3 = 0.0$	2	3	10	5.8	5.3	5.3	1.3	1.0	1.0
$\kappa_4 = 1.5$	2	3	15	5.2	4.9	4.9	1.0	0.8	0.9
	3	3	5	6.2	5.1	5.2	1.7	1.0	1.1
	3	3	10	5.8	5.2	5.2	1.1	0.9	0.9
	3	3	15	5.4	5.1	5.1	1.3	1.1	1.1
	5	3	5	6.3	5.1	5.1	1.5	0.9	0.9
	5	3	10	5.9	5.2	5.2	1.3	1.1	1.1
	5	3	15	5.6	5.3	5.3	1.4	1.2	1.2
(iii)	2	3	5	6.1	4.7	5.2	1.3	0.5	0.9
$\kappa_3 = 0.0$	2	3	10	5.7	5.0	5.2	1.4	0.8	0.9
$\kappa_4 = 6.0$	2	3	15	5.5	4.9	5.1	1.1	0.8	0.9
	3	3	5	6.2	4.6	5.1	1.4	0.6	0.9
	3	3	10	5.6	4.8	5.0	1.3	0.8	1.0
	3	3	15	5.2	4.7	4.8	1.2	0.9	0.9
	5	3	5	6.2	4.8	5.2	1.6	0.9	1.1
	5	3	10	5.5	4.9	5.0	1.2	0.9	0.9
	5	3	15	5.3	4.9	4.9	1.3	1.1	1.0
(iv)	2	3	5	6.0	5.1	5.1	1.6	1.2	1.2
$\kappa_3 = 0.0$	2	3	10	5.7	5.2	5.2	1.2	1.0	1.0
$\kappa_4 = -1.2$	2	3	15	5.8	5.5	5.5	1.3	1.2	1.2
	3	3	5	6.3	5.2	5.2	1.8	1.4	1.3
	3	3	10	5.7	5.2	5.3	1.3	1.1	1.1
	3	3	15	5.6	5.2	5.2	1.2	1.0	1.0
	5	3	5	6.0	5.2	5.2	1.5	1.2	1.2
	5	3	10	5.9	5.5	5.5	1.4	1.1	1.2
	5	3	15	6.0	5.7	5.7	1.2	1.1	1.1
(v)	2	3	5	5.9	4.5	4.9	1.3	0.6	0.8
$\kappa_3 = \sqrt{8/3}$	2	3	10	5.1	4.5	4.6	1.1	0.8	0.8
$\kappa_4 = 4.0$	2	3	15	5.1	4.8	4.8	1.1	0.8	0.9
	3	3	5	6.1	4.9	4.8	1.5	0.9	0.8
	3	3	10	5.7	4.9	5.0	1.1	0.8	0.9
	3	3	15	5.4	4.9	5.0	1.2	1.0	1.0
	5	3	5	5.8	4.7	4.9	1.5	1.0	1.1

Table 4 (*continued*)

Model	p	q	m	Nominal 5% test			Nominal 1% test		
				α_1	α_2	α_3	α_1	α_2	α_3
	5	3	10	5.7	5.0	5.0	1.2	1.0	1.0
	5	3	15	5.6	5.1	5.2	1.3	1.1	1.1
(vi)	2	3	5	6.0	5.0	5.2	1.5	1.0	1.1
$\kappa_3 = 1.0$	2	3	10	5.6	5.1	5.1	1.4	1.1	1.2
$\kappa_4 = 1.5$	2	3	15	5.3	4.9	4.9	1.1	0.9	0.9
	3	3	5	5.8	4.9	4.6	1.6	1.1	0.9
	3	3	10	5.6	5.1	5.1	1.4	1.1	1.2
	3	3	15	5.4	5.0	5.0	1.2	1.1	1.1
	5	3	5	6.1	5.1	5.1	1.4	0.9	0.9
	5	3	10	5.8	5.3	5.3	1.4	1.1	1.1
	5	3	15	5.1	4.8	4.8	1.1	1.0	1.0

where χ_h^2 is a chi-squared variate with h degrees of freedom. Then, from (3.16), we can expand $t(u)$ as

$$t(u) = u + \frac{2u}{nh}\left\{b_1 + b_2 + b_3 + \frac{(b_2+b_3)u}{h+2} + \frac{b_3u^2}{(h+2)(h+4)}\right\} + o(n^{-1})$$
$$= t_{\mathrm{E}}(u) + o(n^{-1}).$$

Besides $\hat{t}_{\mathrm{E}}(u)$ is defined from $t_{\mathrm{E}}(u)$ by replacing unknown parameters κ_3 and κ_4 by $\hat{\gamma}$ and $\hat{\tau}$ which are estimators for κ_3 and κ_4, respectively. In a practical situation, we will need to use $\hat{t}_{\mathrm{E}}(u)$ instead of $t_{\mathrm{E}}(u)$, since we do not know population parameters. Tables 1 and 2 give the true percentage points $t(u)$ and approximate percentage points based on u, $t_{\mathrm{E}}(u)$ and $\hat{t}_{\mathrm{E}}(u)$ for $T_{(1)}$ and $T_{(2)}$, respectively.

Actual test sizes are denoted by

$$\alpha_1 = \mathrm{P}(T > u), \quad \alpha_2 = \mathrm{P}(T > t_{\mathrm{E}}(u)), \quad \alpha_3 = \mathrm{P}(T > \hat{t}_{\mathrm{E}}(u)). \tag{5.1}$$

The actual test sizes α_1, α_2 and α_3 for $T_{(1)}$ and $T_{(2)}$ are given in Tables 3 and 4, respectively, for nominal test sizes 5% and 1%.

From these tables we can see that the approximation $t_{\mathrm{E}}(u)$ or $\hat{t}_{\mathrm{E}}(u)$ gives a considerable improvement in a comparison with the limiting approximation. Besides, there seems to be little influence on accuracies by differences κ_3 and κ_4. We have tried to study for other several models, and have obtained similar results.

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