



Sibuya-type bivariate lack of memory property



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ABSTRACT

The main goal of this article is to generalize the bivariate lack-of-memory property introduced in Marshall & Olkin (1967). Several characterizations of bivariate continuous distributions possessing such a property are established and illustrated by examples.

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1. Introduction and preliminaries

Marshall and Olkin [7] introduced the multivariate exponential distribution. The stochastic representation of the model in the bivariate case is specified by

$$(X_1, X_2) = [\min(T_1, T_3), \min(T_2, T_3)], \quad (1)$$

where T_1 , T_2 and T_3 are independent exponentially distributed random variables.

Denote by $S_{X_1, X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$ the survival function of the random vector (X_1, X_2) . Relation (1) is equivalent to the functional equation

$$S_{X_1, X_2}(x_1 + t, x_2 + t) = S_{X_1, X_2}(x_1, x_2)S_{X_1, X_2}(t, t) \quad (2)$$

for $\forall x_1, x_2 \geq 0, t > 0$. Bivariate continuous distributions satisfying (2) possess the classical bivariate lack-of-memory property (BLMP). It can be seen that BLMP preserves the distribution of both (X_1, X_2) and its residual lifetime vector

$$\mathbf{X}_t = (X_{1t}, X_{2t}) = [(X_1 - t, X_2 - t) \mid X_1 > t, X_2 > t] \quad (3)$$

independent of $t \geq 0$.

The only solution of the functional equation (2) with exponential marginals is Marshall–Olkin's bivariate exponential distribution given by

$$S_{X_1, X_2}(x_1, x_2) = \exp\{-b_1x_1 - b_2x_2 - b_3 \max(x_1, x_2)\} \quad \text{for } b_1, b_2, b_3 > 0.$$

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However, there do exist distributions having BLMP with non-exponential marginals. Let $S_X(x) = P(X > x)$ be the survival function of a random variable X and denote by $r_X(x) = -\frac{d}{dx} \ln S_X(x)$ its failure rate. Kulkarni [4] presents various solutions of functional equation (2) where the marginals may have non-constant failure rates (increasing, decreasing, bathtub, etc.).

Denote by $r_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \ln S_{X_1, X_2}(x_1, x_2)$ the conditional failure rates (hazard gradient components), $i = 1, 2$. For continuous bivariate distributions with BLMP we have

$$r(x_1, x_2) = r_1(x_1, x_2) + r_2(x_1, x_2) = a_0 \quad \text{for } a_0 > 0 \quad (4)$$

for $\forall x_1, x_2 \geq 0$, see Theorem 2 in [4].

Pinto [8] introduced a class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ of nonnegative bivariate continuous distributions that satisfy the relation

$$r(x_1, x_2) = r_1(x_1, x_2) + r_2(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2 \quad \text{for } a_0, a_1, a_2 \geq 0 \quad (5)$$

and $\forall x_1, x_2 \geq 0$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{a} = (a_0, a_1, a_2)$ is the parameter vector.

The class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ is rich. It contains continuous bivariate distributions that are symmetric or asymmetric, positive or negative quadrant dependent, absolutely continuous or having a singular component. Survival copula representation of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ is given in [9].

The following statement characterizes the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$.

Theorem 1 (Pinto [8]). *If the first partial derivatives of $S_{X_1, X_2}(x_1, x_2)$ exist and are continuous then relation (5) is fulfilled if and only if the corresponding joint survival function can be represented by*

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} S_{X_1}(x_1 - x_2) \exp \left\{ -a_0 x_2 - a_1 x_1 x_2 - \frac{a_2 - a_1}{2} x_2^2 \right\}, & \text{if } x_1 \geq x_2 \geq 0, \\ S_{X_2}(x_2 - x_1) \exp \left\{ -a_0 x_1 - a_2 x_1 x_2 - \frac{a_1 - a_2}{2} x_1^2 \right\}, & \text{if } x_2 \geq x_1 \geq 0, \end{cases}$$

where $S_{X_i}(x_i)$ are the marginal survival functions.

Let $H_{X_i}(x_i)$ be the cumulative failure rates of X_i defined by $S_{X_i}(x_i) = \exp\{-H_{X_i}(x_i)\}$, $i = 1, 2$. The survival function of the lifetime vector (X_1, X_2) admits the exponential representation

$$S_{X_1, X_2}(x_1, x_2) = \exp\{-H_{X_1}(x_1) - H_{X_2}(x_2) + D_{X_1, X_2}(x_1, x_2)\}, \quad (6)$$

where $D_{X_1, X_2}(x_1, x_2)$ is the so-called *dependence function* satisfying the boundary conditions $D_{X_1, X_2}(0, x_2) = D_{X_1, X_2}(x_1, 0) = 0$, see Sections 2.1 and 2.2 in [2]. Thus,

$$\exp\{D_{X_1, X_2}(x_1, x_2)\} = \frac{S_{X_1, X_2}(x_1, x_2)}{S_{X_1}(x_1)S_{X_2}(x_2)}. \quad (7)$$

It happens that the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ defined by (5) can be characterized by the linear Sibuya BLMP introduced by Pinto [8] as follows.

Definition 1. The nonnegative continuous bivariate distribution $S_{X_1, X_2}(x_1, x_2)$ possesses linear Sibuya BLMP (to be abbreviated LS-BLMP), if and only if

$$\frac{S_{\mathbf{X}_t}(x_1, x_2)}{S_{X_{1t}}(x_1)S_{X_{2t}}(x_2)} = \frac{S_{X_1, X_2}(x_1, x_2)}{S_{X_1}(x_1)S_{X_2}(x_2)} \quad (8)$$

for $\forall x_1, x_2, t \geq 0$ and

$$S_{X_{it}}(x_i) = S_{X_i}(x_i) \exp\{-a_i x_i t\} \quad \text{with } a_i > 0, i = 1, 2, \quad (9)$$

where $S_{X_{1t}}(x_1)$ and $S_{X_{2t}}(x_2)$ are the marginal survival functions of residual lifetime vector \mathbf{X}_t corresponding to (X_1, X_2) .

Remark 1 (Existence of a Singular Component). The class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ contains absolutely continuous distributions and those having a singular component. It will be assumed hereafter that $S_{X_1, X_2}(x_1, x_2)$ is decomposable into an absolutely continuous component in the support of $\mathbf{R}_+^2 = \{(x_1, x_2) \mid x_1, x_2 \geq 0\}$ and a singular one with support on the set

$$\Omega = \{(x_1, x_2) \in \mathbf{R}_+^2 \mid x_1 = x_2\}. \quad (10)$$

The LS-BLMP transforms into BLMP (4) when $a_1 = a_2 = 0$ in (5). Hence, all BLMP distributions are members of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$.

As we noted, ignoring the assumption of exponential marginals, many bivariate distributions possess BLMP given in (2). Keeping a similar idea in mind, it would be fruitful to obtain a class of bivariate distributions relaxing condition (9) in Definition 1. Such a larger class would be characterized by a weaker version of LS-BLMP, named S-BLMP in Definition 2.

In other words, we wish to find the functional form of the sum $r_1(x_1, x_2) + r_2(x_1, x_2)$ if (8) is satisfied, i.e. in the case of memory-less dependence function $D_{X_1, X_2}(x_1, x_2)$ specified by (7). The solution of this problem and related characterizations are the aims of our article.

In Section 2 we define the S-BLMP based on relation (8). A characterizing functional equation in terms of dependence function $D_{X_1, X_2}(x_1, x_2)$ is established. In Section 3 we introduce an extension of the $\mathcal{L}(\mathbf{x}; \mathbf{a})$ class of bivariate continuous distributions which equivalently represents the S-BLMP. Several characterizations are obtained and illustrated by examples. In Section 4 we present a stochastic representation of extended class. We finish with a short discussion.

2. Weaker version the linear Sibuya-BLMP

We define first a weaker form of LS-BLMP and characterize the distributions with this property via a functional equation involving the dependence function $D_{X_1, X_2}(x_1, x_2)$.

Definition 2. The nonnegative continuous bivariate survival function $S_{X_1, X_2}(x_1, x_2)$ possesses Sibuya BLMP (to be abbreviated S-BLMP) if and only if relation (8) is satisfied for $\forall x_1, x_2, t \geq 0$.

Definition 2 can be interpreted as follows: the baseline random vector (X_1, X_2) and its residual lifetime vector \mathbf{X}_t given by (3) should share the same dependence function $D_{X_1, X_2}(x_1, x_2)$ specified by (7), which does not depend on $t \geq 0$. This new concept includes as a special case the preservation of joint distributions of (X_1, X_2) and corresponding \mathbf{X}_t advocated by BLMP.

Note that the marginal survival functions of the residual lifetime vector \mathbf{X}_t defined by (3) are given by

$$S_{X_{1t}}(x_1) = \frac{S_{X_1, X_2}(x_1 + t, t)}{S_{X_1, X_2}(t, t)} \quad \text{and} \quad S_{X_{2t}}(x_2) = \frac{S_{X_1, X_2}(t, x_2 + t)}{S_{X_1, X_2}(t, t)}.$$

Thus, it is direct to verify that condition (8) is equivalent to the relation

$$\frac{S_{X_1, X_2}(x_1 + t, x_2 + t)}{S_{X_1, X_2}(x_1 + t, t)S_{X_1, X_2}(t, x_2 + t)} = \frac{S_{X_1, X_2}(x_1, x_2)}{S_{X_1}(x_1)S_{X_2}(x_2)S_{X_1, X_2}(t, t)}. \quad (11)$$

The following characterization is valid.

Lemma 1. The S-BLMP is characterized by the functional equation

$$D_{X_1, X_2}(x_1 + t, x_2 + t) = D_{X_1, X_2}(x_1 + t, t) + D_{X_1, X_2}(t, x_2 + t) + D_{X_1, X_2}(x_1, x_2) - D_{X_1, X_2}(t, t), \quad (12)$$

which has to be fulfilled for $\forall x_1, x_2, t \geq 0$. The solution has to satisfy the inequalities

$$\begin{aligned} \max[0, \exp\{H_{X_1}(x_1)\} + \exp\{H_{X_2}(x_2)\} - \exp\{H_{X_1}(x_1) + H_{X_2}(x_2)\}] \\ \leq \exp\{D_{X_1, X_2}(x_1, x_2)\} \leq \min[\exp\{H_{X_1}(x_1)\}, \exp\{H_{X_2}(x_2)\}], \end{aligned}$$

where $H_{X_i}(x_i)$ are the marginal cumulative failure rates of X_i , $i = 1, 2$.

Proof. Apply the exponential representation (6) in (11) to get the functional equation (12). Inversely, use (7) in (12) to recover (11).

One can obtain the inequalities in Lemma 1 using (7) and the usual Fréchet–Hoeffding bounds for $S_{X_1, X_2}(x_1, x_2)$. \square

The trivial solution of (12) is $D_{X_1, X_2}(x_1, x_2) = 0$, which characterizes the independence between nonnegative continuous random variables X_1 and X_2 . In fact, if X_1 and X_2 are independent, so are the components of the corresponding residual lifetime vector \mathbf{X}_t .

As a direct consequence of Lemma 1 we have the following

Corollary 1. If $D_{Y_1, Y_2}(x_1, x_2)$ and $D_{Z_1, Z_2}(x_1, x_2)$ are two different solutions of (12), then their linear combination is also a solution of (12).

The statement of Corollary 1 may serve as a tool for generating distributions possessing S-BLMP, as shown in next two examples.

Example 1 (Gumbel's Type Bivariate Distribution with a Singular Component). Let the nonnegative random vector (T_1, T_2) follow the absolutely continuous Gumbel's type I bivariate exponential distribution with survival function

$$S_{T_1, T_2}(x_1, x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \theta \lambda_1 \lambda_2 x_1 x_2\}, \quad \lambda_1, \lambda_2 > 0, \theta \in [0, 1], \quad (13)$$

see [3]. Its dependence function is $D_{T_1, T_2}(x_1, x_2) = -\theta \lambda_1 \lambda_2 x_1 x_2$ and solves (12), i.e. the distribution (13) possesses S-BLMP. Remind that Gumbel's law does not follow BLMP.

Following the stochastic representation (1), let T_3 be exponentially distributed with parameter $\lambda_3 > 0$ and independent of (T_1, T_2) . The resulting joint survival function of (X_1, X_2) is

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp\{-(\lambda_1 + \lambda_3)x_1 - \lambda_2 x_2 - \theta \lambda_1 \lambda_2 x_1 x_2\}, & \text{if } x_1 \geq x_2 \geq 0, \\ \exp\{-\lambda_1 x_1 - (\lambda_2 + \lambda_3)x_2 - \theta \lambda_1 \lambda_2 x_1 x_2\}, & \text{if } x_2 \geq x_1 \geq 0. \end{cases} \quad (14)$$

This survival function has absolutely continuous and singular parts, the last one with support on the set Ω given by (10). We will refer to the distribution defined by (14) as a *Gumbel's type bivariate distribution with a singular component*. One can find that the corresponding dependence function

$$D_{X_1, X_2}(x_1, x_2) = \lambda_3 \min(x_1, x_2) - \theta \lambda_1 \lambda_2 x_1 x_2,$$

is a linear combination of dependence functions satisfying (12). Thus, according to Corollary 1, the distribution given by (14) possesses S-BLMP.

Example 2 (A Stochastic Representation Generating S-BLMP Distributions). Let (Y_1, Y_2) and (Z_1, Z_2) be two independent nonnegative continuous random vectors. Suppose that their dependence functions $D_{Y_1, Y_2}(x_1, x_2)$ and $D_{Z_1, Z_2}(x_1, x_2)$ satisfy the functional equation (12). Define

$$(X_1, X_2) = (\min\{Y_1, Z_1\}, \min\{Y_2, Z_2\})$$

and notice that

$$S_{X_1, X_2}(x_1, x_2) = S_{Y_1, Y_2}(x_1, x_2) S_{Z_1, Z_2}(x_1, x_2).$$

Therefore, $D_{X_1, X_2}(x_1, x_2) = D_{Y_1, Y_2}(x_1, x_2) + D_{Z_1, Z_2}(x_1, x_2)$ and according to Corollary 1, $S_{X_1, X_2}(x_1, x_2)$ possesses S-BLMP.

3. Alternative characterizations of Sibuya-BLMP

In this section we introduce an extension of the $\mathcal{L}(\mathbf{x}; \mathbf{a})$ class of bivariate continuous distributions based on additive decomposition of the sum $r_1(x_1, x_2) + r_2(x_1, x_2)$. Several characterizations of the extended class and its equivalence with the class of distributions possessing S-BLMP are established.

3.1. Definition and characterizations of the extended class

Observe that relation (11) can be rewritten as

$$S_{X_1, X_2}(x_1 + t, x_2 + t) = S_{X_1, X_2}(x_1, x_2) S_{X_1, X_2}(t, t) M_1(x_1, t) M_2(x_2, t), \quad (15)$$

where

$$M_1(x_1, t) = \frac{S_{X_1, X_2}(x_1 + t, t)}{S_{X_1, X_2}(x_1, t) S_{X_1}(x_1)} \quad \text{and} \quad M_2(x_2, t) = \frac{S_{X_1, X_2}(t, x_2 + t)}{S_{X_1, X_2}(t, t) S_{X_2}(x_2)}.$$

The functions $M_i(x_i, t)$ are such that $M_i(0, t) = M_i(x_i, 0) = 1$, $i = 1, 2$.

If $S_{X_1, X_2}(x_1, x_2)$ is differentiable at (x_1, x_2) , the sum of the hazard gradient components $r(x_1, x_2) = r_1(x_1, x_2) + r_2(x_1, x_2)$ can be equivalently represented by the derivative along the direction of the vector $\vec{u} = (1, 1)$ as

$$r(x_1, x_2) = \lim_{t \rightarrow 0} \frac{-\ln[S_{X_1, X_2}(x_1 + t, x_2 + t)]}{t} = -\frac{\partial}{\partial t} \ln[S_{X_1, X_2}(x_1 + t, x_2 + t)],$$

evaluated at $t = 0$. Thus, from (15)

$$r(x_1, x_2) = -\frac{\partial}{\partial t} \ln[M_1(x_1, t) M_2(x_2, t) S_{X_1, X_2}(t, t) S_{X_1, X_2}(x_1, x_2)] \Big|_{t=0}.$$

If all the derivatives do exist, then

$$r(x_1, x_2) = \left[-\frac{d}{dt} \ln[S_{X_1, X_2}(t, t)] - \frac{\partial}{\partial t} \ln[M_1(x_1, t)] - \frac{\partial}{\partial t} \ln[M_2(x_2, t)] \right] \Big|_{t=0}. \quad (16)$$

The conclusion is that $r(x_1, x_2)$ can be decomposed into a sum of a constant and two continuous and integrable functions, one depending only on x_1 and the other depending on x_2 only.

Therefore, based on additive decomposition (16), we propose the following definition as an extension of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$.

Definition 3. The nonnegative continuous bivariate random vector (X_1, X_2) belongs to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{A} = (a_0, A_1(x_1), A_2(x_2))$, if and only if the sum of the components of its hazard gradient can be represented as

$$r(x_1, x_2) = r_1(x_1, x_2) + r_2(x_1, x_2) = a_0 + A_1(x_1) + A_2(x_2), \quad (17)$$

where $a_0 > 0$ and the continuous integrable functions $A_i(x_i)$ are such that $A_i(0) = 0$ and $A_i(x_i) > -a_0$ for $\forall x_i > 0$, $i = 1, 2$.

Substituting $A_i(x_i) = a_i x_i$ in (17) one can recover relation (5). Additionally assuming $a_i = 0$, $i = 1, 2$, BLMP given in (4) is obtained. Therefore, all members of the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ (containing BLMP distributions) belong to the extended class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ introduced by Definition 3.

Remark 2 (Sum of Hazard Gradient Components Rule). Our construction (17) relies on the performance of $-\ln[S_{X_1, X_2}(x_1, x_2)]$ along the direction of the vector $\vec{u} = (1, 1)$ and, according to Remark 1, permits the existence of a singular component with support on the set Ω specified by (10). Therefore, the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ contains absolutely continuous distributions as well as distributions composed by absolutely continuous and singular parts.

When there exists a singular component, the hazard gradient elements are no longer defined when $(x_1, x_2) \in \Omega$, so one should assume that

$$r(x, x) = \frac{d}{dx} [-\ln S_{X_1, X_2}(x, x)].$$

The following statement is a direct consequence of Definition 3 and is applied in Theorem 3, where we establish the equivalence between the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ and distributions possessing S-BLMP.

Lemma 2. Relation (17) is equivalent to the functional equation

$$r(x_1 + t, x_2 + t) = r(x_1 + t, t) + r(t, x_2 + t) - r(t, t) \quad (18)$$

for $\forall x_1, x_2, t \geq 0$.

Substitute $\psi(x_1, x_2) = r(x_1, x_2) - a_0$ in (17) and let $A_1(x_1) = \psi(x_1, 0)$ and $A_2(x_2) = \psi(0, x_2)$. Hence we get the functional equation

$$\psi(x_1, x_2) = \psi(x_1, 0) + \psi(0, x_2) \quad (19)$$

with solution

$$\psi(x_1, x_2) = A_1(x_1) + A_2(x_2),$$

$\forall x_1, x_2, t \geq 0$. Therefore, the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ can be equivalently defined by relation (17) and functional equations (18) or (19).

In Theorem 2 we provide the general expression for the joint survival function of the members of the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$. We use the fact that the hazard gradient vector $\mathbf{R}(x_1, x_2) = (r_1(x_1, x_2), r_2(x_1, x_2))$ uniquely determines the bivariate distribution by means of the line integral

$$S_{X_1, X_2}(x_1, x_2) = \exp \left\{ - \int_{\mathcal{C}} \mathbf{R}(\mathbf{z}) d\mathbf{z}, \right\} \quad (20)$$

where \mathcal{C} is a sufficiently smooth continuous path beginning at $(0, 0)$ and finishing at (x_1, x_2) , see [6].

Theorem 2. If the first partial derivatives of $S_{X_1, X_2}(x_1, x_2)$ exist and are continuous, with a possible exception on the set Ω defined by (10), then relation (17) is fulfilled if and only if the corresponding joint survival function can be represented by

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} S_{X_1}(x_1 - x_2) \exp \{-a_0 x_2 - I_1(x_1) + I_1(x_1 - x_2) - I_2(x_2)\}, & \text{if } x_1 \geq x_2 \geq 0, \\ S_{X_2}(x_2 - x_1) \exp \{-a_0 x_1 - I_1(x_1) - I_2(x_2) + I_2(x_2 - x_1)\}, & \text{if } x_2 \geq x_1 \geq 0, \end{cases} \quad (21)$$

where $\frac{d}{dx} I_i(x) = A_i(x)$, $x_i > 0$, $i = 1, 2$.

Proof. Let (17) be satisfied. Consider a sufficiently smooth continuous path (curve) \mathcal{C} beginning at point $(0, 0)$ and finishing at point (x_1, x_2) in \mathbf{R}_+^2 . Suppose that along this path $S_{X_1, X_2}(x_1, x_2)$ is absolutely continuous and hazard components $r_i(x_1, x_2)$, $i = 1, 2$ do exist almost everywhere in the set \mathcal{C} .

Initially let $x_1 \geq x_2 \geq 0$ and choose a particular path \mathcal{C} from $(0, 0)$ to (x_1, x_2) as the union of two line segments: the first linking $(0, 0)$ and $(x_1 - x_2, 0)$ and the second joining $(x_1 - x_2, 0)$ with (x_1, x_2) . Denote these line segments by \mathcal{C}_1 and \mathcal{C}_2 , respectively. The curves \mathcal{C}_1 and \mathcal{C}_2 , can be parameterized as follows

$$\mathcal{C}_1 = \{(y_1(t), y_2(t)) \in \mathbf{R}_+^2 \text{ such that } (y_1(t), y_2(t)) = (t, 0), \text{ with } t \in [0, x_1 - x_2]\}$$

and

$$\mathcal{C}_2 = \{(y_1(t), y_2(t)) \in \mathbf{R}_+^2 \text{ such that } (y_1(t), y_2(t)) = (x_1 - x_2 + t, t), \text{ with } t \in [0, x_2]\}.$$

Along the path \mathcal{C}_1 we have $S_{X_1, X_2}(x_1, x_2) = S_{X_1, X_2}(x_1, 0) = S_{X_1}(x_1)$ and therefore $r_1(x_1, 0) = r_{X_1}(x_1)$. In addition, $y_2'(t) = 0$ and $y_1'(t) = 1$. Hence,

$$\int_{\mathcal{C}_1} \mathbf{R}(\mathbf{z}) d\mathbf{z} = \int_0^{x_1 - x_2} r_{X_1}(t) dt = H_{X_1}(x_1 - x_2).$$

Notice that along the line segment \mathcal{C}_2 we have $y_2(t) = t$ and $y_1(t) = x_1 - x_2 + t$, i.e. $y'_1(t) = y'_2(t) = 1$. The use of (20) and relation (17) implies

$$S_{X_1, X_2}(x_1, x_2) = \exp \left\{ -H_{X_1}(x_1 - x_2) - \int_0^{x_2} [a_0 + A_1(x_1 - x_2 + t) + A_2(t)] dt \right\},$$

which can be rewritten as

$$S_{X_1, X_2}(x_1, x_2) = S_{X_1}(x_1 - x_2) \exp \left\{ - \int_0^{x_2} [a_0 + A_1(x_1 - x_2 + t) + A_2(t)] dt \right\}.$$

Since $\frac{d}{dx} I_i(x) = A_i(x)$, we have $\int_a^b A_i(u) du = I_i(b) - I_i(a)$ for $a < b$, $i = 1, 2$. To evaluate $\int_0^{x_2} A_1(x_1 - x_2 + t) dt$, consider the change of variable $u = x_1 - x_2 + t$, i.e. $du = dt$ to obtain

$$\int_0^{x_2} A_1(x_1 - x_2 + t) dt = \int_{x_1 - x_2}^{x_1} A_1(u) du = I_1(x_1) - I_1(x_1 - x_2)$$

and we get the first relation in (21).

We proceed similarly when $x_2 \geq x_1 \geq 0$ and the necessary part of the statement is proved.

Now suppose that the joint survival $S_{X_1, X_2}(x_1, x_2)$ function is given by (21). Direct calculation shows that hazard gradient components $r_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \ln S_{X_1, X_2}(x_1, x_2)$, $i = 1, 2$, should satisfy relations

$$r_1(x_1, x_2) = \begin{cases} r_{X_1}(x_1 - x_2) + A_1(x_1) - A_1(x_1 - x_2), & \text{if } x_1 > x_2 \geq 0, \\ -r_{X_2}(x_2 - x_1) + a_0 + A_1(x_1) + A_2(x_2 - x_1), & \text{if } x_2 > x_1 \geq 0 \end{cases} \quad (22)$$

and

$$r_2(x_1, x_2) = \begin{cases} -r_{X_1}(x_1 - x_2) + a_0 + A_1(x_1 - x_2) + A_2(x_2), & \text{if } x_1 > x_2 \geq 0, \\ r_{X_2}(x_2 - x_1) + A_2(x_2) - A_2(x_2 - x_1), & \text{if } x_2 > x_1 \geq 0. \end{cases} \quad (23)$$

When $x_1 = x_2 = x \geq 0$, from (21) and Remark 2 we conclude

$$r(x, x) = \frac{d}{dx} [-\ln S_{X_1, X_2}(x, x)] = \frac{d}{dx} [a_0 x + I_1(x) + I_2(x)] = a_0 + A_1(x) + A_2(x).$$

Hence, condition (17) is fulfilled in all possible cases. \square

Remark 3 (Bounds for Marginal Failure Rates). The joint survival function $S_{X_1, X_2}(x_1, x_2)$ from (21) is proper only for certain marginal distributions of X_1 and X_2 . This implies corresponding restrictions in terms of marginal densities or failure rates.

Since $r_i(x_1, x_2) \geq 0$, $i = 1, 2$, substituting $x_1 = 0$ in (22) we have $0 \leq r_{X_2}(x_2) \leq a_0 + A_2(x_2)$. Analogously, from (23) we obtain $0 \leq r_{X_1}(x_1) \leq a_0 + A_1(x_1)$. Therefore, the marginal failure rates $r_{X_i}(x_i)$ are limited from above by a constant a_0 added to the corresponding functions $A_i(x_i)$, $i = 1, 2$.

These restrictions are important and have to be taken into account when generating members of the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$.

3.2. Equivalence between the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ and S-BLMP distributions

Pinto [8] established that the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ can be characterized by LS-BLMP. Following similar arguments, in Theorem 3 we show that S-BLMP characterizes the distributions belonging to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$. Having this equivalence at hand, we prove the characterization Theorem 4 which is, in fact, an additional bridge between the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ and S-BLMP.

Theorem 3. The S-BLMP characterizes the distributions belonging to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$.

Proof. Let $S_{X_1, X_2}(x_1, x_2)$ belong to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ defined by (17). From Theorem 2, $S_{X_1, X_2}(x_1, x_2)$ can be represented by (21) whenever its first partial derivatives exist and are continuous. Thus, from (21), it is direct to check that (11) holds true. Since (11) is equivalent to (8), the survival function possesses the S-BLMP.

Now suppose $S_{X_1, X_2}(x_1, x_2)$ possesses S-BLMP, so that (11) is satisfied. Let $x_1 \geq x_2 \geq 0$ to obtain

$$S_{X_1, X_2}(x_1 + t, x_2 + t) = S_{X_1}(x_1 - x_2) \exp \left\{ \int_0^{x_2 + t} r(x_1 - x_2 + u, u) du \right\}.$$

In a similar way one may get corresponding expressions for

$$S_{X_1, X_2}(x_1 + t, t), \quad S_{X_1, X_2}(t, x_2 + t), \quad S_{X_1, X_2}(x_1, x_2) \text{ and } S_{X_1, X_2}(t, t).$$

In particular, when $x_1 = x_2 = t \geq 0$ we assume that $\frac{d}{dt} [-\ln S_{X_1, X_2}(t, t)] = r(t, t)$. Substituting these quantities in (11), after some algebra (taking logarithm in both sides and differentiating with respect to t) we obtain the functional equation (18) which is valid for $x_1 \geq x_2 \geq 0$ and all $t \geq 0$. From Lemma 2, (18) is equivalent to (17).

The same is the conclusion when $x_2 \geq x_1 \geq 0$. Therefore, the random vector (X_1, X_2) belongs to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$. \square

In the next theorem we obtain a functional equation involving the joint survival function of (X_1, X_2) and characterizing members of the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$.

Theorem 4. The nonnegative continuous random vector (X_1, X_2) belongs to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ if and only if its joint survival function satisfies the functional equation (15), with functions

$$M_i(x_i, t) = \exp\{-[I_i(x_i + t) - I_i(x_i) - I_i(t)]\}, \quad (24)$$

where $\frac{d}{dx}I_i(x) = A_i(x)$, $i = 1, 2$.

Proof. Let (X_1, X_2) belong to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$. Then, from Theorem 2, the survival function $S_{X_1, X_2}(x_1, x_2)$ can be equivalently represented by (21).

Suppose $x_1 \geq x_2 \geq 0$. From (21) we have

$$S_{X_1, X_2}(t, t) = \exp\{-a_0 t - I_1(t) - I_2(t)\}.$$

Applying (21) again we obtain

$$S_{X_1, X_2}(x_1 + t, x_2 + t) = S_{X_1, X_2}(x_1 - x_2) \exp\{-a_0(x_2 + t) - I_1(x_1 + t) + I_1(x_1 - x_2) - I_2(x_2 + t)\},$$

which is identical to (15). The result is the same if $x_2 \geq x_1 \geq 0$.

Conversely, let (15) hold. Define the continuous function $Q(x_1, x_2)$ by

$$Q(x_1, x_2) = S_{X_1, X_2}(x_1, x_2) \exp\{I_1(x_1) + I_2(x_2)\}. \quad (25)$$

Notice that $Q(x_1, x_2)$ does not need to be a bivariate survival function. Substituting (25) in (15) we obtain the functional equation

$$Q(x_1 + t, x_2 + t) = Q(x_1, x_2)Q(t, t),$$

which is valid for $\forall x_1, x_2, t \geq 0$. Consider $x_1 \geq x_2 \geq 0$ to get the only continuous solution

$$Q(x_1, x_2) = Q(x_1 - x_2, 0) \exp\{-a_0 x_2\}$$

for some constant $a_0 > 0$. From (25) we find

$$Q(x_1 - x_2, 0) = S_{X_1, X_2}(x_1 - x_2, 0) \exp\{I_1(x_1 - x_2)\} = S_{X_1}(x_1 - x_2) \exp\{I_1(x_1 - x_2)\}.$$

Therefore,

$$S_{X_1, X_2}(x_1, x_2) \exp\{I_1(x_1) + I_2(x_2)\} = S_{X_1, X_2}(x_1 - x_2, 0) \exp\{I_1(x_1 - x_2)\} \exp\{-a_0 x_2\},$$

i.e. $S_{X_1, X_2}(x_1, x_2)$ is represented by (21).

Identical steps and conclusion follow if $x_2 \geq x_1 \geq 0$. \square

Notice that the equivalence between the expression of the joint survival function (21) and the functional equation (15) does not require continuity of partial derivatives of the joint survival function of (X_1, X_2) as in Theorem 2.

Since the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ is equivalent to bivariate distributions possessing S-BLMP, Theorem 4 indicates that the functional equation (15) characterizes S-BLMP as well.

The equivalences so far obtained are summarized in Fig. 1.

3.3. Absolutely continuous S-BLMP distributions

The nonnegative parameter a_0 plays important role in the class $\mathcal{L}(\mathbf{x}; \mathbf{a})$ being equivalent to LS-BLMP. In fact, denote by $f_{X_i}(x_i)$ the density of random variable X_i , $i = 1, 2$. It is known from Pinto [8] that if $a_0 = f_{X_1}(0) + f_{X_2}(0)$ then the joint survival function $S_{X_1, X_2}(x_1, x_2)$ is absolutely continuous and if $a_0 < f_{X_1}(0) + f_{X_2}(0)$, the distribution exhibits a singular component on the set Ω defined by (10). Similar property can be established for distributions possessing S-BLMP, as shown in the next characterizing statement.

Theorem 5. Consider the survival function (21) and let X_i be a nonnegative random variable with density $f_{X_i}(x_i)$, $i = 1, 2$. Then $a_0 \leq f_{X_1}(0) + f_{X_2}(0)$. The survival function (21) is absolutely continuous if and only if $f_{X_1}(0) + f_{X_2}(0) = a_0$.

Proof. Let $\alpha = P(X_1 = X_2) = 1 - P(X_1 > X_2) - P(X_2 > X_1) \in [0, 1]$. Standard calculus shows that

$$P(X_i > X_{3-i}) = 1 - \int_0^\infty [f_{X_i}(0) + A_i(y)] \exp\{-a_0 y - I_1(y) - I_2(y)\} dy, \quad i = 1, 2.$$

Notice that

$$\int_0^\infty [a_0 + A_1(y) + A_2(y)] \exp\{-a_0 y - I_1(y) - I_2(y)\} dy = 1,$$

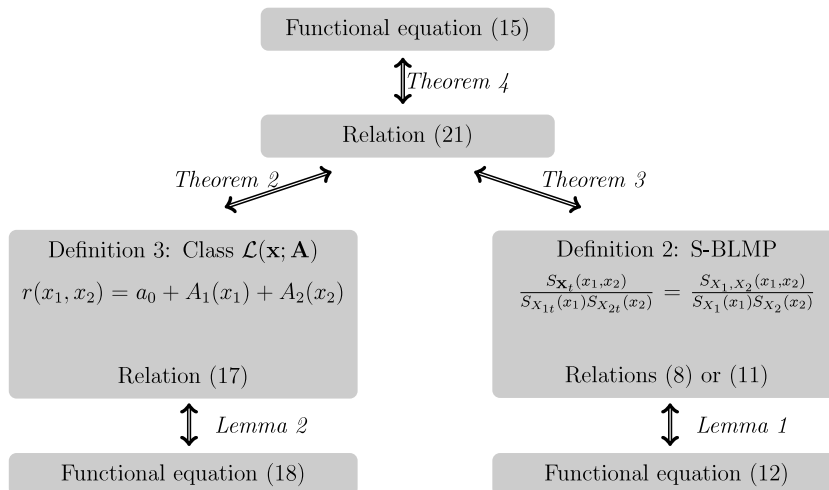


Fig. 1. Equivalent relations.

being the integral of the density function of the random variable $\min(X_1, X_2)$ over its support. Hence,

$$\alpha = \int_0^\infty [f_{X_1}(0) + f_{X_2}(0) - a_0] \exp\{-a_0 y - I_1(y) - I_2(y)\} dy.$$

Since $\exp\{-a_0 y - I_1(y) - I_2(y)\} \geq 0$ and $\alpha = P(X_1 = X_2) \in [0, 1]$, then $a_0 \leq f_{X_1}(0) + f_{X_2}(0)$.

To conclude the proof, observe that $S_{X_1, X_2}(x_1, x_2)$ in (21) is absolutely continuous if and only if $\alpha = 0$, which is equivalent to the required condition $f_{X_1}(0) + f_{X_2}(0) = a_0$. \square

Example 3 (Gumbel's Type Bivariate Distributions (13) and (14)). Consider Gumbel's type I bivariate exponential distribution given by (13). We have $S_{X_i}(x_i) = \exp\{-\lambda_i x_i\}$, i.e. $f_{X_i}(0) = \lambda_i$, $i = 1, 2$. From (17) we obtain $a_0 = \lambda_1 + \lambda_2$ and $A_i(x_i) = \theta \lambda_1 \lambda_2 x_i$, $i = 1, 2$. Thus, $f_{X_1}(0) + f_{X_2}(0) = a_0$ and we verify the absolute continuity of Gumbel's law.

Last conclusion fails for the survival function (14), where $S_{X_i}(x_i) = \exp\{-(\lambda_i + \lambda_3)x_i\}$ and $f_{X_i}(0) = \lambda_i + \lambda_3$, $i = 1, 2$. From (17) we have $a_0 = \lambda_1 + \lambda_2 + \lambda_3$. Thus, $f_{X_1}(0) + f_{X_2}(0) = a_0 + \lambda_3 > a_0$. Applying Theorem 5 we conclude that the corresponding joint survival function $S_{X_1, X_2}(x_1, x_2)$ has absolutely continuous and singular components.

4. Stochastic representation

In this section we obtain a stochastic representation for the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ assuming additional properties for the functions $A_i(x_i)$, $i = 1, 2$, given in Definition 3. In fact, we have the following result.

Theorem 6. Let the nonnegative continuous random vector (X_1, X_2) belong to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ and the continuous and integrable functions $A_i(x)$ in (17) be strictly increasing with $\lim_{x \rightarrow \infty} A_i(x) = \infty$, $i = 1, 2$. Then there exists a random vector (Z_1, Z_2) independent of (X_1, X_2) such that $\mathbf{X}_t = (\min\{X_1, Z_1\}, \min\{X_2, Z_2\})$.

Proof. We know from Theorem 2 that if (X_1, X_2) belongs to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ and the first partial derivatives of its survival function exist and are continuous, then $S_{X_1, X_2}(x_1, x_2)$ is represented by (21). According to Theorem 4, relations (21) and (15) are equivalent. Fix $t = t_0 > 0$ and introduce the functions

$$R_{Z_i}(x, t_0) = I_i(x + t_0) - I_i(x) - I_i(t_0), \quad i = 1, 2, \quad (26)$$

entering in the exponent of (24). Since $A_i(x) = \frac{d}{dx} I_i(x)$ is increasing in x , we have

$$\frac{d}{dx} [I_i(x + t_0) - I_i(x) - I_i(t_0)] = A_i(x + t_0) - A_i(x) > 0, \quad i = 1, 2,$$

i.e. the function $R_{Z_i}(x, t_0)$ in (26) is increasing in x as well. Observe that, when $x = 0$ we have $R_{Z_i}(0, t_0) = 0$.

Taking into account the conditions $\lim_{x \rightarrow \infty} A_i(x) = \infty$ we arrive to

$$\lim_{x \rightarrow \infty} R_{Z_i}(x, t_0) = \infty, \quad i = 1, 2.$$

Therefore the function $R_{Z_i}(x, t_0)$ in (26) shares the properties of a cumulative failure rate of some continuous nonnegative random variable, to be denoted by Z_i , $i = 1, 2$. The survival function of Z_i depends on t_0 and is given by $M_i(x_i, t_0)$ in (24),

$i = 1, 2$. Thus, the product $M_1(x_1, t_0)M_2(x_2, t_0)$ is the joint survival function of random vector (Z_1, Z_2) with independent marginals. Since $\frac{S_{X_1, X_2}(x_1+t, x_2+t)}{S_{X_1, X_2}(t, t)}$ is the joint survival function of residual lifetime vector \mathbf{X}_t , the required stochastic representation follows from (15). \square

Due to the form of the functions $A_1(x_1)$ and $A_2(x_2)$ involved, the stochastic relation between (X_1, X_2) and \mathbf{X}_t stated in Theorem 6 admits a simple interpretation in terms of shocks represented by random variables Z_1 and Z_2 governed by independent Poisson processes.

Example 4 (*Generalized Marshall–Olkin (GMO) Distributions*). Let us consider the GMO distributions introduced by Li and Pellerey [5]. The independent nonnegative absolutely continuous random variables T_i are not necessarily exponential and satisfy the stochastic representation (1), $i = 1, 2, 3$. Notice that GMO models exhibit a singular component along the set Ω defined by (10). The joint survival function of the GMO models is

$$S_{X_1, X_2}(x_1, x_2) = \exp\{-H_{T_1}(x_1) - H_{T_2}(x_2) - H_{T_3}(\max\{x_1, x_2\})\}. \quad (27)$$

The GMO distributions (27) satisfy (2) only in the case of exponentially distributed T_i 's, $i = 1, 2, 3$. Hence, GMO models do not possess BLMP in general.

Substitute $H_{T_3}(x) = a_0x$, $a_0 > 0$ and $H_{T_i}(x) = \lambda_i x^3$, $\lambda_i > 0$, $i = 1, 2$, in (27) to obtain $S_{X_1, X_2}(x_1, x_2) = \exp\{-\lambda_1 x_1^3 - \lambda_2 x_2^3 - a_0 \max(x_1, x_2)\}$. The sum of the components of the hazard gradient is $r(x_1, x_2) = a_0 + 3\lambda_1 x_1^2 + 3\lambda_2 x_2^2$, so that $A_i(x_i) = 3\lambda_i x_i^2$, $i = 1, 2$ in (17). From Theorem 2 this specific GMO distribution admits the representation

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp\{-\lambda_1(x_1 - x_2)^3 - a_0(x_1 - x_2)\} \exp\{-a_0 x_2 - \lambda_1 x_1^3 + \lambda_1(x_1 - x_2)^3 - \lambda_2 x_2^3\}, & \text{if } x_1 \geq x_2 \geq 0, \\ \exp\{-\lambda_2(x_2 - x_1)^3 - a_0(x_2 - x_1)\} \exp\{-a_0 x_1 - \lambda_1 x_1^3 - \lambda_2 x_2^3 + \lambda_2(x_2 - x_1)^3\}, & \text{if } x_2 \geq x_1 \geq 0. \end{cases}$$

Applying Theorem 4 we get

$$S_{X_1, X_2}(x_1 + t, x_2 + t) = S_{X_1, X_2}(x_1, x_2) S_{X_1, X_2}(t, t) \exp\{-\lambda_1[(x_1 + t)^3 - x_1^3 - t^3] - \lambda_2[(x_2 + t)^3 - x_2^3 - t^3]\}.$$

The functions $A_i(x_i) = 3\lambda_i x_i^2$ are strictly increasing in x_i , unbounded from above and $A_i(0) = 0$, $i = 1, 2$. From Theorem 6 the residual lifetime vector admits the stochastic representation $\mathbf{X}_t = (\min\{X_1, Z_1\}, \min\{X_2, Z_2\})$, where Z_1 and Z_2 are two absolutely continuous nonnegative independent random variables such that $S_{Z_i}(x, t) = \exp\{-3\lambda_i t(x_i^2 + tx_i)\}$, i.e., the distribution of Z_i depends on t and λ_i , $i = 1, 2$.

5. Summary and conclusions

We define in this paper the Sibuya BLMP (S-BLMP) by means of relation (8), covering a huge class of nonnegative bivariate continuous survival functions $S_{X_1, X_2}(x_1, x_2)$ such that their dependence function $D_{X_1, X_2}(x_1, x_2)$ defined by (7) is shared by the corresponding residual lifetime vector \mathbf{X}_t for all $t > 0$. The S-BLMP is a weaker version of both BLMP (which preserves the joint survival functions of (X_1, X_2) and \mathbf{X}_t) and LS-BLMP introduced by Pinto [8]. Therefore, the set of bivariate distributions possessing S-BLMP is very rich.

It happens that S-BLMP is equivalent to the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ defined by (17), where the functions $A_1(x_1)$ and $A_2(x_2)$ are continuous and integrable. This equivalence is established by characterization Theorems 2–4. Additional characterizations in terms of functional equations have been obtained as well, see Lemmas 1 and 2. The corresponding relations are displayed in Fig. 1. Another characterization is valid for absolutely continuous distributions possessing S-BLMP, see Theorem 5.

A stochastic representation of the class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ is specified in Theorem 6 under the additional assumption of strictly increasing functions $A_i(x_i)$ in (17), $i = 1, 2$.

We would like to note a contribution into functional equation theory: the expression (21) for the joint survival function $S_{X_1, X_2}(x_1, x_2)$ is, in fact, the solution of the functional equation (15).

We feel that the characterization results and established relationships between well-known and new classes of bivariate distributions obtained in this article will help to deep and complement the BLMP-knowledge by considering an appropriate “non-aging” dependence function as a base ($D_{X_1, X_2}(x_1, x_2)$ specified by (7) in our case). We do believe the multivariate case will be elaborated soon.

The class $\mathcal{L}(\mathbf{x}; \mathbf{A})$ has a set of reliability properties to be investigated. For example, notice that expression (26) besides increasing in x is also increasing in t_0 , meaning that

$$\frac{S_{X_1, X_2}(x_1 + t_1, x_2 + t_1)}{S_{X_1, X_2}(x_1, x_2)} \geq \frac{S_{X_1, X_2}(x_1 + t_2, x_2 + t_2)}{S_{X_1, X_2}(x_1, x_2)} \quad \text{for } t_1 \geq t_2 > 0.$$

Therefore, whenever $A_i(x_i)$ are strictly increasing in x_i , $i = 1, 2$, $S_{X_1, X_2}(x_1, x_2)$ is bivariate increasing failure rate, according to the definition suggested by Brindley and Thompson [1].

To give “life” to the models introduced, it will be necessary to develop corresponding inference procedures. These activities are related with our future research plans.

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