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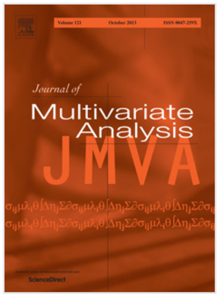

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Third-order local power properties of tests for a composite hypothesis, II

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Abstract

The Bartlett-type adjustment is a higher-order asymptotic method for improving the chi-squared approximation to the null distributions of various test statistics, which ensures that the resulting test has size $\alpha + o(N^{-1})$, where $0 < \alpha < 1$ is the significance level and N is the sample size. We continue our recent works on the third-order average local power properties of several Bartlett-type adjusted tests. Strengthening the results in the 90s, the third-order optimality of the adjusted Rao test in a sense has been established even if both the interest parameter and the nuisance parameter are multi-dimensional. We briefly discuss adjusted profile likelihood inference for handling the nuisance parameter.

AMS subject classifications: 62E20, 62F03, 62F05.

Keywords: Asymptotic expansion, Bartlett-type adjustment, local alternative, local power of test, nuisance parameter.

1. Introduction

We continue our recent works [10, 11, 12, 13] on higher-order asymptotic theory of several statistics for testing a composite hypothesis about a subvector of parameters. Here, the $N^{-i/2}$-term is referred to as being the $(i + 1)$th-order, where N is the sample size. A detailed historical review for comparing higher-order local powers, starting from the second-order local power analyses [22], is omitted here to save space; see Kakizawa [13] and the references cited therein.

In the absence of nuisance parameter, Mukerjee [17, 19] established that Rao's (score) test under the third-order conditions of size and local unbiasedness has the third-order optimality in terms of average local power criterion. Mukerjee [21] additionally showed that Rao's test even in the original form (not being adjusted for local unbiasedness and only the size condition is being retained) has the third-order optimality, where we observe that 'the test in the original form' is nothing but the size-adjusted test with substitution of Cornish-Fisher's type expansion for the percentile. On the other hand, not much work has yet been reported on the third-order local power properties in the presence of nuisance parameter, except that Mukerjee [16, 18] attempted to discuss the third-order optimality of Rao's (adjusted) test under the assumption of the global parameter orthogonality for the situation where both the interest parameter and the nuisance parameter are scalar. He mentioned that the same argument is applicable even when the nuisance parameter is multi-dimensional.

The present paper addresses the comments in the review paper [19] that

> *if both the interest parameter and the nuisance parameter be multi-dimensional, then, as noted in Cox and Reid, one may not in general be able to achieve an orthogonal parameterization. Anyway, it is strongly believed that the results discussed here should have their counterparts even in such a situation.*

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As a companion paper to Kakizawa [13], we are primarily concerned with the third-order local power properties of several Bartlett-type adjusted tests. The Bartlett-type adjustment dates back to different three methods proposed by Chandra and Mukerjee [2], Cordeiro and Ferrari [3], and Taniguchi [25] in alphabetical order. It is a higher-order asymptotic method for improving the chi-squared approximation to the null distributions of various test statistics, which ensures that the resulting test has size $\alpha+\mathrm{o}(N^{-1})$, as in the size-adjusted test based on Cornish-Fisher's type expansion for the percentile, where $0<\alpha<1$ is the significance level. Rao and Mukerjee [23, 24] have compared the third-order point-by-point local powers of three Bartlett-type adjustments [2, 3, 25] for the simple hypothesis on a scalar parameter. In recent years, there have been renewed interests [9, 10, 11] due to the existence of infinitely many Bartlett-type adjustments for the multi-parameter hypothesis testing.

By constructions (see Definitions 1 and 2 in Section 2), it will be convenient for us to define two types separately. One is the generalized Bartlett-type adjustment (for short GB). The other is the generalized Cordeiro-Ferrari Bartlett-type adjustment (for short GCF). We denote by $T^{\mathrm{GB}(N)}$ and $T^{\mathrm{GCF}(N)}$ the GB and GCF adjustments for a likelihood-based test statistic $T^{(N)} \in \mathcal{T}_{N,3}$ under consideration (see (3) below). Kakizawa [13] derived the third-order average local power of the GB-adjusted test $T^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}$, where $\chi^2_{p_1,\alpha}$ is the upper α-point of the central chi-squared distribution with p_1 degrees of freedom, and then established that even if both the interest parameter and the nuisance parameter are multi-dimensional, the GB-adjusted Rao test has the third-order optimality. So, Mukerjee's conjectual statement, as mentioned before, may be solved in a sense. However, we know that Rao's test statistic has many variants; e.g. $\mathrm{R}^{(N)}$ and $\mathrm{MR}^{(N)}$ (see (2) below), for which the adjusted tests $\mathrm{R}^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}$ and $\mathrm{MR}^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}$ have the identical average local power up to the third-order. That is, the GB adjustment smooths out the distinctive features between $\mathrm{R}^{(N)}$ and $\mathrm{MR}^{(N)}$, and hence it may be more interesting to compare them (announced at the end of Section 4 of [13]). This is the reason why we need to have further discussion, on the basis of the GCF adjustment.

The contribution of the present paper is three fold. First, our results allow both the interest parameter and the nuisance parameter to be multi-dimensional, for which there is no assumption regarding the global parameter orthogonality. Second, we elucidate that the adjusted Rao tests $\mathrm{R}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}$ and $\mathrm{MR}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}$ are, generally, discriminated in terms of the third-order average local power, and that the former test $\mathrm{R}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}$ has the third-order optimality in a large class of the GB and GCF-adjusted tests. Third, we briefly discuss adjusted profile likelihood inference (e.g. [4, 6]), which represents an important tool for handling the nuisance parameter.

Although we focus on the iid case for notational simplicity, we arrive at the same conclusions even in a non-iid case where some regularity conditions are met for the log-likelihood derivatives according to the situations under consideration. We retain throughout this paper the notation and conventions of Kakizawa [13] (see also [10, 11]). The rest of this paper is organized as follows. Section 2 contains the notation to be used throughout this paper. Section 3 derives an asymptotic expansion formula for the (average) local power of the GCF-adjusted test $T^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}$. Section 4 describes main results. Concluding remarks are given in Section 5.

2. Bartlett-type adjustments

2.1. Notation

We denote by $P_\theta^{(N)}$ the $\boldsymbol{\theta}$-distribution of $\mathbf{X}_1, \ldots, \mathbf{X}_N$, which are iid random vectors (taking values of $\mathbf{R}^{d_X}$) according to a density $f(\mathbf{x}, \boldsymbol{\theta})$, $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbf{R}^p$. For any sequence $\{Y^{(N)}\}_{N \geq 1}$ of random variables having the form $Y^{(N)} = g_N(\mathbf{X}_1, \ldots, \mathbf{X}_N)$, we use the pointwise notation $Y^{(N)} = \mathrm{o}_\theta^{(N)}(q, \beta)$ under $P_\theta^{(N)}$, if $P_\theta^{(N)}[|Y^{(N)}| > d(\log N)^\beta] = \mathrm{o}(N^{-q})$ as $N \to \infty$ for some $d > 0$, $q \geq 0$, and $\beta \geq 0$. In what follows, we assume the same regularity conditions as in Kakizawa [13]. Suppose that the parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$ is composed of two parts, a parameter of interest $\boldsymbol{\theta}_{(1)} = (\theta_1, \ldots, \theta_{p_1})'$ and a nuisance parameter $\boldsymbol{\theta}_{(2)} = (\theta_{p_1+1}, \ldots, \theta_{p_1+p_2})'$; $\boldsymbol{\theta} = (\boldsymbol{\theta}'_{(1)}, \boldsymbol{\theta}'_{(2)})' \in \boldsymbol{\Theta} = \boldsymbol{\Theta}_{(1)} \times \boldsymbol{\Theta}_{(2)}$ (say), where $p = p_1 + p_2$. We write $\mathcal{L}^{(N)}(\boldsymbol{\theta}) = \sum_{i=1}^N \log f(\mathbf{X}_i, \boldsymbol{\theta})$. We want to test a composite hypothesis $\boldsymbol{\theta}_{(1)} = \boldsymbol{\theta}_{(1)0}$ against $\boldsymbol{\theta}_{(1)} \neq \boldsymbol{\theta}_{(1)0}$, where $\boldsymbol{\theta}_{(1)0} \in \boldsymbol{\Theta}_{(1)}$ is specified while $\boldsymbol{\theta}_{(2)} \in \boldsymbol{\Theta}_{(2)}$ remains unspecified. Let $\widehat{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)} \in \boldsymbol{\Theta}$ be the (unrestricted) maximum likelihood estimator (MLE) of $\boldsymbol{\theta}$, and let $\widetilde{\boldsymbol{\theta}}_{(2)\mathrm{ML}}^{(N)} \in \boldsymbol{\Theta}_{(2)}$ be the restricted MLE of $\boldsymbol{\theta}_{(2)}$ under the constraint $\boldsymbol{\theta}_{(1)} = \boldsymbol{\theta}_{(1)0}$. We write

$$\widetilde{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)} = \begin{pmatrix} \boldsymbol{\theta}_{(1)0} \\ \widetilde{\boldsymbol{\theta}}_{(2)\mathrm{ML}}^{(N)} \end{pmatrix}, \quad \widehat{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)} = \begin{pmatrix} \widehat{\boldsymbol{\theta}}_{(1)\mathrm{ML}}^{(N)} \\ \widehat{\boldsymbol{\theta}}_{(2)\mathrm{ML}}^{(N)} \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\theta}^\dagger = \begin{pmatrix} \boldsymbol{\theta}_{(1)0} \\ \boldsymbol{\theta}_{(2)}^\dagger \end{pmatrix} \in \boldsymbol{\Theta},$$

with $\boldsymbol{\theta}_{(2)}^\dagger$ being the irrelevant true value of the nuisance parameter $\boldsymbol{\theta}_{(2)}$. For any (nonrandom/random) scalar or vector or matrix function $Q(\cdot)$, we use the notation $\widehat{Q}$, $\widetilde{Q}$, and Q instead of $Q(\widehat{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)})$, $Q(\widetilde{\boldsymbol{\theta}}_{\mathrm{ML}}^{(N)})$, and $Q(\boldsymbol{\theta}^\dagger)$, respectively.

The Rth partial derivative of the log density $\log f(\mathbf{x}, \boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$ is denoted by

$$\ell_{j_1 \cdots j_R}(\mathbf{x}, \boldsymbol{\theta}) = \frac{\partial}{\partial \theta_{j_1}} \cdots \frac{\partial}{\partial \theta_{j_R}} \log f(\mathbf{x}, \boldsymbol{\theta})$$

for $R \in \mathbf{N}$; $j_1, \ldots, j_R \in \{1, \ldots, p\}$. We introduce $I_R = j_1 \cdots j_R$ for notational simplicity and denote the cumulants of the $\ell_{I_R}(\mathbf{X}, \boldsymbol{\theta})$'s with respect to $\mathbf{X} \sim f(\cdot, \boldsymbol{\theta})$ by

$$\nu_{I_{R_1}, \ldots, I_{R_v}}(\boldsymbol{\theta}) = Cum_\theta[\ell_{I_{R_1}}(\mathbf{X}, \boldsymbol{\theta}), \ldots, \ell_{I_{R_v}}(\mathbf{X}, \boldsymbol{\theta})]$$

(descending order $R_1 \geq \cdots \geq R_v \geq 1$ on the size $R_i = |I_{R_i}|$ is assumed, since $\nu_{I_{R_1}, \ldots, I_{R_v}}(\boldsymbol{\theta})$ is symmetric under permutation of $\{I_{R_1}, \ldots, I_{R_v}\}$). We assume that

$$\nu_{j_1}(\boldsymbol{\theta}) = 0, \quad \nu_{j_1 j_2}(\boldsymbol{\theta}) + \nu_{j_1, j_2}(\boldsymbol{\theta}) = 0, \quad \nu_{j_1 j_2 j_3}(\boldsymbol{\theta}) + \langle 3 \rangle \nu_{j_1 j_2, j_3}(\boldsymbol{\theta}) + \nu_{j_1, j_2, j_3}(\boldsymbol{\theta}) = 0,$$
$$\nu_{j_1 j_2 j_3 j_4}(\boldsymbol{\theta}) + \langle 4 \rangle \nu_{j_1 j_2 j_3, j_4}(\boldsymbol{\theta}) + \langle 3 \rangle \nu_{j_1 j_2, j_3 j_4}(\boldsymbol{\theta}) + \langle 6 \rangle \nu_{j_1 j_2, j_3, j_4}(\boldsymbol{\theta}) + \nu_{j_1, j_2, j_3, j_4}(\boldsymbol{\theta}) = 0$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, where $\langle n \rangle$ before a term with indices is a sum of n similar terms obtained by index permutation. These Bartlett identities for the cumulants enable us to eliminate $\nu_{j_1 j_2}(\boldsymbol{\theta})$, $\nu_{j_1 j_2 j_3}(\boldsymbol{\theta})$, and $\nu_{j_1 j_2 j_3 j_4}(\boldsymbol{\theta})$ in subsequent calculations. We write

$$Z_{j_1 \cdots j_R}^{(N)}(\boldsymbol{\theta}) = \begin{cases} \dfrac{1}{N^{1/2}} \displaystyle\sum_{i=1}^N \ell_{j_1}(\mathbf{X}_i, \boldsymbol{\theta}), & R = 1 \\ \dfrac{1}{N^{1/2}} \displaystyle\sum_{i=1}^N \{\ell_{j_1 \cdots j_R}(\mathbf{X}_i, \boldsymbol{\theta}) - \nu_{j_1 \cdots j_R}(\boldsymbol{\theta})\}, & R = 2, 3. \end{cases}$$

According to the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}'_{(1)}, \boldsymbol{\theta}'_{(2)})'$, we stack the element $Z_j^{(N)}(\boldsymbol{\theta})$ and $\nu_{j,k}(\boldsymbol{\theta}) = -\nu_{jk}(\boldsymbol{\theta})$ as

$$[Z_j^{(N)}(\boldsymbol{\theta})]_{j=1,\ldots,p} = \begin{pmatrix} \mathbf{Z}_{(1)}^{(N)}(\boldsymbol{\theta}) \\ \mathbf{Z}_{(2)}^{(N)}(\boldsymbol{\theta}) \end{pmatrix} \quad \text{and} \quad [\nu_{j,k}(\boldsymbol{\theta})]_{j,k\in\{1,\ldots,p\}} = \begin{pmatrix} \boldsymbol{\nu}_{(11)}(\boldsymbol{\theta}) & \boldsymbol{\nu}_{(12)}(\boldsymbol{\theta}) \\ \boldsymbol{\nu}_{(21)}(\boldsymbol{\theta}) & \boldsymbol{\nu}_{(22)}(\boldsymbol{\theta}) \end{pmatrix}.$$

They are the $p \times 1$ score vector $\mathbf{Z}^{(N)}(\boldsymbol{\theta})$ and the $p \times p$ Fisher information matrix $\boldsymbol{\nu}(\boldsymbol{\theta}) = \mathrm{Var}_\theta[\mathbf{L}_1(\mathbf{X}, \boldsymbol{\theta})]$, respectively, where $\mathbf{L}_1(\mathbf{X}, \boldsymbol{\theta}) = [\ell_j(\mathbf{X}, \boldsymbol{\theta})]_{j=1,\ldots,p}$. Here, there is no assumption regarding the global parameter orthogonality $\boldsymbol{\nu}_{(12)}(\cdot) \equiv \mathbf{O}_{p_1,p_2}$, where $\mathbf{O}_{p_1,p_2}$ is the $p_1 \times p_2$ zero matrix.

We employ standard summation convention that if an index occurs twice in a product of two or more terms, then this means the summation over all values which this index may assume. Unless otherwise stated, we use the letters $\{j, k\}$ as indices of $\boldsymbol{\theta}$ that run from 1 to p, the letters $\{a, b\}$ as indices of $\boldsymbol{\theta}_{(1)}$ that run from 1 to p_1, and the letters $\{r, s\}$ as indices of $\boldsymbol{\theta}_{(2)}$ that run from $p_1 + 1$ to p. We denote by $\nu^{j,k}(\boldsymbol{\theta})$ the (j, k)th element of $\boldsymbol{\nu}^{-1}(\boldsymbol{\theta})$, where we assume that $\boldsymbol{\nu}(\boldsymbol{\theta})$ is nonsingular. Let $[\nu_{(22)}^{r,s}(\boldsymbol{\theta})]_{r,s\in\{p_1+1,\ldots,p\}}$ be the inverse of the matrix $\boldsymbol{\nu}_{(22)}(\boldsymbol{\theta}) = [\nu_{r,s}(\boldsymbol{\theta})]_{r,s\in\{p_1+1,\ldots,p\}}$. We denote by $\nu_{(11\cdot2)}^{a,b}(\boldsymbol{\theta})$ the (a, b)th element of $\boldsymbol{\nu}_{(11\cdot2)}^{-1}(\boldsymbol{\theta})$, where

$$\boldsymbol{\nu}_{(11\cdot2)}(\boldsymbol{\theta}) = [\nu_{(11\cdot2)a,a'}(\boldsymbol{\theta})]_{a,a'\in\{1,\ldots,p_1\}} = \boldsymbol{\nu}_{(11)}(\boldsymbol{\theta}) - \boldsymbol{\nu}_{(12)}(\boldsymbol{\theta})\boldsymbol{\nu}_{(22)}^{-1}(\boldsymbol{\theta})\boldsymbol{\nu}_{(21)}(\boldsymbol{\theta}).$$

Further, we denote by $\mathcal{G}_{j,a}(\boldsymbol{\theta})$ the (j, a)th element of

$$\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{I}_{p_1} \\ -\boldsymbol{\nu}_{(22)}^{-1}(\boldsymbol{\theta})\boldsymbol{\nu}_{(21)}(\boldsymbol{\theta}) \end{pmatrix},$$

where $\mathbf{I}_{p_1}$ is the $p_1 \times p_1$ identity matrix. We note $\boldsymbol{\nu}_{(11\cdot2)}(\boldsymbol{\theta}) = \boldsymbol{\mathcal{G}}'(\boldsymbol{\theta})\boldsymbol{\nu}(\boldsymbol{\theta})\boldsymbol{\mathcal{G}}(\boldsymbol{\theta})$ and

$$\boldsymbol{\mathcal{G}}(\boldsymbol{\theta})\boldsymbol{\nu}_{(11\cdot2)}^{-1}(\boldsymbol{\theta})\boldsymbol{\mathcal{G}}'(\boldsymbol{\theta}) = \boldsymbol{\nu}^{-1}(\boldsymbol{\theta}) - \begin{pmatrix} \mathbf{O}_{p_1,p_1} & \mathbf{O}_{p_1,p_2} \\ \mathbf{O}_{p_2,p_1} & \boldsymbol{\nu}_{(22)}^{-1}(\boldsymbol{\theta}) \end{pmatrix} = \mathbf{B}(\boldsymbol{\theta}) \quad \text{(say)}. \tag{1}$$

2.2. *A class of test statistics*

To cover the likelihood ratio (LR) test statistic, two variants of Rao's and Wald's test statistics as well as Terrell's gradient test statistic [26];

$$\begin{aligned}
\mathrm{LR}^{(N)} &= 2(\widehat{\mathcal{L}}^{(N)} - \widetilde{\mathcal{L}}^{(N)}), & \mathrm{grad}^{(N)} &= (\widetilde{\mathbf{Z}}_{(1)}^{(N)})' N^{1/2}(\widehat{\boldsymbol{\theta}}_{(1)\mathrm{ML}}^{(N)} - \boldsymbol{\theta}_{(1)0}), \\
\mathrm{R}^{(N)} &= (\widetilde{\mathbf{Z}}_{(1)}^{(N)})'\widetilde{\boldsymbol{\nu}}_{(11\cdot2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}, & \mathrm{W}^{(N)} &= N(\widehat{\boldsymbol{\theta}}_{(1)\mathrm{ML}}^{(N)} - \boldsymbol{\theta}_{(1)0})'\widehat{\boldsymbol{\nu}}_{(11\cdot2)}(\widehat{\boldsymbol{\theta}}_{(1)\mathrm{ML}}^{(N)} - \boldsymbol{\theta}_{(1)0}), \\
\mathrm{MR}^{(N)} &= (\widetilde{\mathbf{Z}}_{(1)}^{(N)})'\widehat{\boldsymbol{\nu}}_{(11\cdot2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}, & \mathrm{MW}^{(N)} &= N(\widehat{\boldsymbol{\theta}}_{(1)\mathrm{ML}}^{(N)} - \boldsymbol{\theta}_{(1)0})'\widetilde{\boldsymbol{\nu}}_{(11\cdot2)}(\widehat{\boldsymbol{\theta}}_{(1)\mathrm{ML}}^{(N)} - \boldsymbol{\theta}_{(1)0}),
\end{aligned} \tag{2}$$

Kakizawa [10, 11, 13] considered a class $\mathcal{T}_{N,3}$ of test statistics for testing the null hypothesis $\boldsymbol{\theta}_{(1)} = \boldsymbol{\theta}_{(1)0}$ against $\boldsymbol{\theta}_{(1)} \neq \boldsymbol{\theta}_{(1)0}$, as follows: Every test statistic $T^{(N)} = T_N(\mathbf{X}_1, \ldots, \mathbf{X}_N; \boldsymbol{\theta}_{(1)0}) \in \mathcal{T}_{N,3}$ admits a stochastic expansion of the form

$$T^{(N)} = T_{3\mathrm{rd}}^{(N)} + \frac{1}{N^{3/2}}\mathrm{o}_{\theta^\dagger}^{(N)}(1 + \xi, \beta) \quad \text{for some fixed } \beta, \xi > 0, \tag{3}$$

where

$$\begin{aligned}
T_{3\mathrm{rd}}^{(N)} &= (\widetilde{\mathbf{Z}}_{(1)}^{(N)})'\widetilde{\boldsymbol{\nu}}_{(11\cdot2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)} + \frac{2}{N^{1/2}}\Big(\widetilde{C}_{b_1b_2b_3}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\prod_{i=1}^{3}[\widetilde{\boldsymbol{\nu}}_{(11\cdot2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i} + \widetilde{C}_{b_1b_2,k_1k_2}^{\mathcal{G}\,\mathcal{G}}\widetilde{Z}_{k_1k_2}^{(N)}\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}_{(11\cdot2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}\Big) \\
&\quad + \frac{2}{N}\Big\{\widetilde{D}_{b_1b_2b_3b_4}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\prod_{i=1}^{4}[\widetilde{\boldsymbol{\nu}}_{(11\cdot2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i} + (\widetilde{D}_{b_1b_2b_3,k_1k_2}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\widetilde{Z}_{k_1k_2}^{(N)} + \widetilde{D}_{b_1b_2b_3,k_1k_2k_3}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\widetilde{Z}_{k_1k_2k_3}^{(N)})\prod_{i=1}^{3}[\widetilde{\boldsymbol{\nu}}_{(11\cdot2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i} \\
&\quad + \widetilde{D}_{b_1b_2,k_1k_2,k_3k_4}^{\mathcal{G}\,\mathcal{G}}\widetilde{Z}_{k_1k_2}^{(N)}\widetilde{Z}_{k_3k_4}^{(N)}\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}_{(11\cdot2)}^{-1}\widetilde{\mathbf{Z}}_{(1)}^{(N)}]_{b_i}\Big\}
\end{aligned} \tag{4}$$

(hereafter, $[\mathbf{v}]_i$ sometimes stands for the ith element v_i of any vector $\mathbf{v}$),

$$C^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot)=C_{j_1j_2j_3}(\cdot)\mathcal{G}_{j_1,a_1}(\cdot)\mathcal{G}_{j_2,a_2}(\cdot)\mathcal{G}_{j_3,a_3}(\cdot)\,,\quad C^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2}(\cdot)=C_{j_1j_2,k_1k_2}(\cdot)\mathcal{G}_{j_1,a_1}(\cdot)\mathcal{G}_{j_2,a_2}(\cdot)\,.$$

We adopt similar definitions for $D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3a_4}(\cdot)$, $D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3,k_1\cdots k_v}(\cdot)$, $D^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2,k_3k_4}(\cdot)$, and so on. Notice that β,ξ, C-functions; $C_{j_1j_2j_3}(\cdot)$'s and $C_{j_1j_2,k_1k_2}(\cdot)$'s, and D-functions; $D_{j_1j_2j_3j_4}(\cdot)$'s, $D_{j_1j_2j_3,k_1\cdots k_v}(\cdot)$'s, and $D_{j_1j_2,k_1k_2,k_3k_4}(\cdot)$'s may vary from one test statistic to another, where these C(or D)-functions are of class $\mathcal{C}^2(\mathbf{\Theta})$ (or $\mathcal{C}^1(\mathbf{\Theta})$). Without loss of generality, we assume that $C_{j_1j_2j_3}(\cdot)$, $C_{j_1j_2,k_1k_2}(\cdot)$, $D_{j_1j_2j_3j_4}(\cdot)$, $D_{j_1j_2j_3,k_1\cdots k_v}(\cdot)$, and $D_{j_1j_2,k_1k_2,k_3k_4}(\cdot)$ are symmetric under permutation of $\{j_1,j_2,j_3,j_4\}$ and that $D_{j_1j_2,k_1k_2,k_3k_4}(\cdot)=D_{j_1j_2,k_3k_4,k_1k_2}(\cdot)$.

Note that all test statistics in (2) are members in $\bigcup_{c\in\mathrm{R}}\mathcal{T}^c_{N,3}$, where $\mathcal{T}^c_{N,3}$ is a subclass of $\mathcal{T}_{N,3}$ consisting of those members in $\mathcal{T}_{N,3}$ for which $C^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2}(\cdot)$, $a_1,a_2\in\{1,\ldots,p_1\}$, $k_1,k_2\in\{1,\ldots,p\}$ have the form

$$C^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2}(\cdot)=\frac{c}{2}\left\{\mathcal{G}_{k_1,a_1}(\cdot)\mathcal{G}_{k_2,a_2}(\cdot)+\mathcal{G}_{k_1,a_2}(\cdot)\mathcal{G}_{k_2,a_1}(\cdot)\right\},\tag{5}$$

with c being a constant. Especially, we have

$$c_{\mathrm{Rao}}=c_{\text{modified Rao}}=0\,,\quad c_{\mathrm{LR}}=\frac{1}{2}\,,\quad c_{\mathrm{Wald}}=c_{\text{modified Wald}}=c_{\text{gradient}}=1\tag{6}$$

for the LR, Rao's, Wald's, and Terrell's gradient test statistics.

2.3. GB and GCF adjustments

From now on, unless otherwise stated, the term with superscript C (or CD) indicates that it depends on the C (or C,D)-functions of $T^{(N)}\in\mathcal{T}_{N,3}$. We define

$$C^{+\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\;a_1a_2a_3}(\cdot)=C^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot)+\frac{\langle 3\rangle}{3}\,C^{\mathcal{G}\,\mathcal{G}}_{a_1a_2,k_1k_2}(\cdot)\nu^{\quad\;\;\mathcal{G}}_{k_1k_2,a_3}(\cdot)\,,\quad a_1,a_2,a_3\in\{1,\ldots,p_1\}\,.$$

It is esay to see (e.g. [10, 12]) that the expressions for the C^+-functions for the LR, Rao's, Wald's, and Terrell's gradient test statistics are respectively given by

$$\begin{aligned}
{}_{\mathrm{LR}}C^{+\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\;a_1a_2a_3}(\cdot)&=-\frac{1}{6}\,\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1,a_2,a_3}(\cdot)\,,\quad {}_{\text{gradient}}C^{+\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\;a_1a_2a_3}(\cdot)=-\frac{1}{12}\left\{\langle 3\rangle\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2,a_3}(\cdot)+3\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1,a_2,a_3}(\cdot)\right\},\\
{}_{\mathrm{Rao}}C^{+\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\;a_1a_2a_3}(\cdot)&\equiv 0\,,\qquad {}_{\mathrm{Wald}}C^{+\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\;a_1a_2a_3}(\cdot)=-\frac{1}{6}\left\{\langle 3\rangle\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2,a_3}(\cdot)+3\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1,a_2,a_3}(\cdot)\right\},\qquad(7)\\
{}_{\text{modified Rao}}C^{+\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\;a_1a_2a_3}(\cdot)&=-\frac{1}{6}\{2\langle 3\rangle\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2,a_3}(\cdot)+3\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1,a_2,a_3}(\cdot)\}\,,\quad {}_{\text{modified Wald}}C^{+\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\;a_1a_2a_3}(\cdot)=\frac{1}{6}\,\langle 3\rangle\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2,a_3}(\cdot)\,.
\end{aligned}$$

The meaning of the notation of the C^+,D^+-functions (the definition of D^+-function is similar to that of C^+-function) will be apparent from (4) by rewritting

$$\widetilde{Z}^{(N)}_{j_1\cdots j_R}=(\widetilde{Z}^{(N)}_{j_1\cdots j_R}-\widetilde{\nu}_{j_1\cdots j_R,b_{R+1}}^{\qquad\quad\;\mathcal{G}}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_{R+1}})+\widetilde{\nu}_{j_1\cdots j_R,b_{R+1}}^{\qquad\quad\;\mathcal{G}}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_{R+1}}\,,\quad R=2,3\,.$$

We are ready to describe the Bartlett-type adjustments [10, 11] for these likelihood-based test statistics.

Definition 1 (*GB* [10]*, including the extensions of* [2, 25] *to the multi-parameter setting as special cases*) With

$$\Gamma^C_{a_1a_2a_3}(\cdot)=-\frac{1}{6}\left\{\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1,a_2,a_3}(\cdot)+6C^{+\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\;a_1a_2a_3}(\cdot)\right\},\quad a_1,a_2,a_3\in\{1,\ldots,p_1\}\,,$$

we say that

$$T^{\mathrm{GB}(N)} = T^{(N)} + \frac{2}{N^{1/2}}\,\widetilde{\Gamma}^{C}_{b_1b_2b_3}\prod_{i=1}^{3}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} + \frac{2}{N}\Big\{(\widetilde{\Gamma}^{\mathrm{GB}}_{b_1b_2b_3b_4} + \widetilde{\Delta}^{\mathrm{GB}}_{b_1b_2b_3b_4})\prod_{i=1}^{4}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i}$$
$$+\widetilde{\Delta}^{\mathrm{GB}}_{b_1b_2b_3,k_1k_2}\widetilde{Z}^{(N)}_{k_1k_2}\prod_{i=1}^{3}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} + \widetilde{\Gamma}^{\mathrm{GB}}_{b_1b_2}\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i}\Big\} \tag{8}$$

is the generalized Bartlett-type adjustment of $T^{(N)} \in \mathcal{T}_{N,3}$, if

$$P^{(N)}_{\theta^\dagger}[T^{\mathrm{GB}(N)} \le x] = \Pr[\chi^2_{p_1} \le x] + \mathrm{o}(N^{-1}) \tag{9}$$

($\Gamma^{\mathrm{GB}}_{a_1a_2}(\cdot)$'s, $\Gamma^{\mathrm{GB}}_{a_1a_2a_3a_4}(\cdot)$'s, $\Delta^{\mathrm{GB}}_{a_1a_2a_3a_4}(\cdot)$'s, and $\Delta^{\mathrm{GB}}_{a_1a_2a_3,k_1k_2}(\cdot)$'s are assumed to be of class $\mathcal{C}^1(\boldsymbol{\Theta})$), with $\Delta^{\mathrm{GB}}_{a_1a_2a_3a_4}(\cdot)$'s and $\Delta^{\mathrm{GB}}_{a_1a_2a_3,k_1k_2}(\cdot)$'s being specified in advance (see [10]).

Definition 2 (*GCF* [11], *including* [3] *as a special case*) We say that

$$T^{\mathrm{GCF}(N)} = T^{(N)} + \frac{2}{N}\sum_{R=2,4,6}\widetilde{\Gamma}_{b_1\cdots b_R}\prod_{i=1}^{R}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} \tag{10}$$

is the generalized Cordeiro-Ferrari Bartlett-type adjustment of $T^{(N)} \in \mathcal{T}_{N,3}$, if

$$P^{(N)}_{\theta^\dagger}[T^{\mathrm{GCF}(N)} \le x] = \Pr[\chi^2_{p_1} \le x] + \mathrm{o}(N^{-1})\,, \tag{11}$$

where functions $\Gamma_{a_1\cdots a_R}(\cdot)$'s are assumed to be of class $\mathcal{C}^1(\boldsymbol{\Theta})$.

The name 'GCF' and the idea behind the form of (10) stem from the original proposal in Cordeiro and Ferrari [3];

$$T^{\mathrm{CF}(N)} = \Big[1 - \frac{2}{N}\Big\{\frac{\widetilde{\beta}^C_3}{p_1(p_1+2)(p_1+4)}\,(T^{(N)})^2 + \frac{\widetilde{\beta}^{CD}_2}{p_1(p_1+2)}\,T^{(N)} + \frac{\widetilde{\beta}^{CD}_1}{p_1}\Big\}\Big]T^{(N)}\,, \quad T^{(N)} \in \mathcal{T}_{N,3}\,,$$

where

$$\beta^C_3(\cdot) = \frac{1}{24}\Big[3\nu^{b_1,b_2}_{(11\cdot2)}(\cdot)\{\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{b_1,b_2,b}(\cdot) + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1b_2b}(\cdot)\}\nu^{b,b'}_{(11\cdot2)}(\cdot)\{\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{b'_1,b'_2,b'}(\cdot) + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b'_1b'_2b'}(\cdot)\}\nu^{b'_1,b'_2}_{(11\cdot2)}(\cdot)$$
$$+ 2\{\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{b_1,b_3,b_5}(\cdot) + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1b_3b_5}(\cdot)\}\nu^{b_1,b_2}_{(11\cdot2)}(\cdot)\nu^{b_3,b_4}_{(11\cdot2)}(\cdot)\nu^{b_5,b_6}_{(11\cdot2)}(\cdot)\{\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{b_2,b_4,b_6}(\cdot) + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_2b_4b_6}(\cdot)\}\Big]\,,$$

and the closed-form expressions for $\beta^{CD}_1(\cdot)$ and $\beta^{CD}_2(\cdot)$ are found in Kakizawa [11] ([3] contains the formulas for the Rao test statistic $\mathrm{R}^{(N)}$).

Remark 1 By definition, we can rewrite (8) as

$$T^{\mathrm{GB}(N)} = T^{\star(N)} + \frac{2}{N}\sum_{R=2,4}\widetilde{\Gamma}_{b_1\cdots b_R}\prod_{i=1}^{R}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i}\,,$$

where

$$T^{\star(N)} = T^{(N)} + \frac{2}{N^{1/2}}\,\widetilde{\Gamma}^{C}_{b_1b_2b_3}\prod_{i=1}^{3}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i}$$
$$+\frac{2}{N}\Big\{\widetilde{\Delta}^{\mathrm{GB}}_{b_1b_2b_3b_4}\prod_{i=1}^{4}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} + \widetilde{\Delta}^{\mathrm{GB}}_{b_1b_2b_3,k_1k_2}\widetilde{Z}^{(N)}_{k_1k_2}\prod_{i=1}^{3}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i}\Big\}$$

is an adjustment (it is still not Bartlett correctable generally; the second-term of $T^{\star(N)}$ was introduced by [12] in order to assert the second-order point-by-point local power identity) for a given $T^{(N)} \in \mathcal{T}_{N,3}$. That is, $T^{\mathrm{GB}(N)}$ is regarded as a double adjustment $(T^{\star})^{\mathrm{GCF}(N)}$ (we set $\Gamma_{a_1\cdots a_6}(\cdot) \equiv 0$ in Definition 2), where, compared to the original form (3), the $C^{\mathcal{G}}, D^{\mathcal{G}}$-functions associated with $T^{\star(N)} \in \mathcal{T}_{N,3}$ are given by $C^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot) + \Gamma^{C}_{a_1a_2a_3}(\cdot)$, $D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3a_4}(\cdot) + \Delta^{\mathrm{GB}}_{a_1a_2a_3a_4}(\cdot)$'s, and $D^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3,k_1k_2}(\cdot) + \Delta^{\mathrm{GB}}_{a_1a_2a_3,k_1k_2}(\cdot)$'s, while the other $C^{\mathcal{G}}, D^{\mathcal{G}}$-functions are not changed. This interpretation $T^{\mathrm{GB}(N)} = (T^{\star})^{\mathrm{GCF}(N)}$ is important, since the GB adjustment [10] can be treated as the GCF adjustment [11].

In order to determine the symmetric arrays $[\Gamma^{\mathrm{GB}}_{a_1\cdots a_R}(\cdot)]_{a_1,\ldots,a_R\in\{1,\ldots,p_1\}}$, $R = 2, 4$, Kakizawa [10, 13] gave necessary and sufficient conditions for (9), and then discussed the third-order local power properties of the resulting GB-adjusted tests, as mentioned in Introduction. On the other hand, Kakizawa [11] gave necessary and sufficient conditions (see (A.6)–(A.8)) on the symmetric arrays $[\Gamma_{a_1\cdots a_R}(\cdot)]_{a_1,\ldots,a_R\in\{1,\ldots,p_1\}}$, $R = 2, 4, 6$, such that (11) holds. In what follows, we will investigate the third-order local power properties of the resulting GCF-adjusted tests.

3. Asymptotic expansion of $T^{\mathrm{GCF}(N)}$ under a sequence of local alternatives

We write

$$G_\nu(x;\omega^2) = \int_0^x g_\nu(t;\omega^2)\,dt\,,$$

where $g_\nu(\cdot;\omega^2)$ denotes the density function of the noncentral chi-squared distribution with ν degrees of freedom and noncentrality parameter ω^2.

By a tedious algebra similar to Kakizawa [10, 11, 12, 13], one obtains the following result. The proof, outlined in Appendix A, is straightforward but becomes much longer than the derivation of asymptotic expansion [13] for $P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T^{\mathrm{GB}(N)} > x]$, since, by definition (see (8) or Remark 1), $T^{\mathrm{GB}(N)}$ has the identical C^+-functions; $C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot) = -(1/6)\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1,a_2,a_3}(\cdot) = {}_{\mathrm{LR}}C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot)$, $a_1, a_2, a_3 \in \{1,\ldots,p_1\}$ (such a structure not only led to the second-order point-by-point local power identity but also made several coefficients for $\pi^{\mathrm{GB}(2)}_\alpha(\mathbf{h}_{(1)})$ disappear drastically, as shown in [12, 13]). It may be true that Proposition 1 is very heavy, depending on the C, D-functions associated with $T^{(N)} \in \mathcal{T}_{N,3}$, but averaging this power function (with respect to $\mathbf{h}_{(1)}$) along the sphere $\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)} = \lambda$, where $\lambda > 0$, it turns out that the expression (14) below is independent of the D-functions. The emphasis here is on the patterns in the coefficients for $\pi^{\mathrm{GCF}(2)}_\alpha(\mathbf{h}_{(1)})$ (rather than the derivation).

Proposition 1 *Let $T^{(N)} \in \mathcal{T}_{N,3}$. Suppose that* (A.6)–(A.8) *hold. Then*

$$P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}] = 1 - G_{p_1}(\chi^2_{p_1,\alpha};\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)}) + \sum_{\ell=1}^{2}\frac{2}{N^{\ell/2}}\,\pi^{\mathrm{GCF}(\ell)}_\alpha(\mathbf{h}_{(1)}) + \mathrm{o}(N^{-1})$$

for any $\mathbf{h}_{(1)} = (h_1,\ldots,h_{p_1})' \in \mathbf{R}^{p_1}$, where

$$\pi^{\mathrm{GCF}(\ell)}_\alpha(\mathbf{h}_{(1)}) = \sum_{v=1}^{3\ell}\mathcal{P}^{\mathrm{GCF}(\ell)}_v(\mathbf{h}_{(1)})g_{p_1+2v}(\chi^2_{p_1,\alpha};\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot2)}\mathbf{h}_{(1)})\,,\quad \ell = 1, 2\,.$$

Here,

$$
\begin{aligned}
\mathcal{P}_1^{\mathrm{GCF}(1)}(\mathbf{h}_{(1)}) &= -\frac{1}{2}\,(\nu_{rr',\diamond}{}^{\mathcal{G}} + \nu_{r,r',\diamond}{}^{\mathcal{G}})\nu_{(22)}^{r,r'} - \frac{1}{6}\,(3\nu_{\diamond\diamond,\diamond}^{\mathcal{GGG}} + 2\nu_{\diamond,\diamond,\diamond}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}) + \frac{1}{2}\,(2\nu_{\diamond\diamond,\diamond}{}^{\mathcal{G}\,\mathcal{G}} + \nu_{\diamond,\diamond,\diamond}{}^{\mathcal{G}\,\mathcal{G}})\,,\\
\mathcal{P}_2^{\mathrm{GCF}(1)}(\mathbf{h}_{(1)}) &= \frac{1}{2}\,(\nu_{\diamond,b,b'}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}} + 6C^{+}{}_{\diamond\, b\, b'}^{\mathcal{GGG}})\nu_{(11\cdot 2)}^{b,b'} + \frac{1}{6}\,\nu_{\diamond,\diamond,\diamond}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}\,, \quad \mathcal{P}_3^{\mathrm{GCF}(1)}(\mathbf{h}_{(1)}) = \frac{1}{6}\,(\nu_{\diamond,\diamond,\diamond}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}} + 6C^{+}{}_{\diamond\diamond\diamond}^{\mathcal{GGG}})
\end{aligned}
$$

(*we used the notation* $Q_{\ldots\diamond\ldots} = Q_{\ldots a\ldots}h_a$ *for simplicity*) *and*

$$
\begin{aligned}
\mathcal{P}_1^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)}) &= \mathcal{Q}_{1[2]}^{C(2)} + \mathcal{Q}_{1[4]}^{(2)} + \mathcal{Q}_{1[6]}^{(2)}\,,\\
\mathcal{P}_2^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)}) &= (\Gamma_{\diamond\diamond} + \mathcal{Q}_{2[2]}^{CD(2)}) + \mathcal{Q}_{2[4]}^{C(2)} + \mathcal{Q}_{2[6]}^{(2)}\,,\\
\mathcal{P}_3^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)}) &= (6\Gamma_{\diamond\diamond bb'}\nu_{(11\cdot 2)}^{b,b'} + \mathcal{Q}_{3[2]}^{CD(2)}) + \mathcal{Q}_{3[4]}^{C(2)} + \mathcal{Q}_{3[6]}^{C(2)}\,,\\
\mathcal{P}_4^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)}) &= (45\Gamma_{\diamond\diamond b_1 b_1' b_2 b_2'}\nu_{(11\cdot 2)}^{b_1,b_1'}\nu_{(11\cdot 2)}^{b_2,b_2'} + \mathcal{Q}_{4[2]}^{C(2)}) + (\Gamma_{\diamond\diamond\diamond\diamond} + \mathcal{Q}_{4[4]}^{CD(2)}) + \mathcal{Q}_{4[6]}^{C(2)}\,,\\
\mathcal{P}_5^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)}) &= (15\Gamma_{\diamond\diamond\diamond\diamond bb'}\nu_{(11\cdot 2)}^{b,b'} + \mathcal{Q}_{5[4]}^{C(2)}) + \mathcal{Q}_{5[6]}^{C(2)}\,,\\
\mathcal{P}_6^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)}) &= (\Gamma_{\diamond\diamond\diamond\diamond\diamond\diamond} + \mathcal{Q}_{6[6]}^{C(2)})\,,
\end{aligned}
$$

where the $\mathcal{Q}_{v[2i]}^{(2)}$*'s (without superscript* C *or* CD*), being independent of the* C, D*-functions, and the* $\mathcal{Q}_{v[2i]}^{C(2)}$*'s and* $\mathcal{Q}_{v[2i]}^{CD(2)}$*'s, are homogeneous polynomials of degree* $2i$ *in* $\mathbf{h}_{(1)}$ *(the details are omitted here).*

It is worth noting that the third-order point-by-point local power of the Cordeiro-Ferrari adjustment [3] is the same as that of the size-adjusted test based on Cornish-Fisher's type expansion for the percentile (this statement is consistent with [15] on several tests for the admissibility of a subset of instrumental variables and [8] on the normal-based GMANOVA tests under a non-Gaussian error). That is, we obtain

$$
\begin{aligned}
&P^{(N)}_{\theta^\dagger + N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}\Big[T^{(N)} > \Big\{1 + \frac{2}{N}\Big(\frac{\widetilde{\beta}_3^{C}}{p_1(p_1+2)(p_1+4)}\,x^2 + \frac{\widetilde{\beta}_2^{CD}}{p_1(p_1+2)}\,x + \frac{\widetilde{\beta}_1^{CD}}{p_1}\Big)\Big\}x\Big]\\
&\quad = P^{(N)}_{\theta^\dagger + N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T^{\mathrm{CF}(N)} > x] + \mathrm{o}(N^{-1})
\end{aligned}
\tag{12}
$$

as a special case of Proposition 1 with

$$
\begin{aligned}
\Gamma_{\diamond\diamond}^{\mathrm{CF}} &= -\frac{\beta_1^{CD}}{p_1}\,(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)})\,,\\
6\Gamma_{\diamond\diamond bb'}^{\mathrm{CF}}\nu_{(11\cdot 2)}^{bb'} &= -\frac{2\beta_2^{CD}}{p_1}\,(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)})\,,\\
45\Gamma_{\diamond\diamond b_1 b_1' b_2 b_2'}^{\mathrm{CF}}\nu_{(11\cdot 2)}^{b_1 b_1'}\nu_{(11\cdot 2)}^{b_2 b_2'} &= -\frac{3\beta_3^{C}}{p_1}\,(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)})\,,\\
\Gamma_{\diamond\diamond\diamond\diamond}^{\mathrm{CF}} &= -\frac{\beta_2^{CD}}{p_1(p_1+2)}\,(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)})^2\,,\\
15\Gamma_{\diamond\diamond\diamond\diamond bb'}^{\mathrm{CF}}\nu_{(11\cdot 2)}^{bb'} &= -\frac{3\beta_3^{C}}{p_1(p_1+2)}\,(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)})^2\,,\\
\Gamma_{\diamond\diamond\diamond\diamond\diamond\diamond}^{\mathrm{CF}} &= -\frac{\beta_3^{C}}{p_1(p_1+2)(p_1+4)}\,(\mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)})^3\,.
\end{aligned}
$$

Then, it turns out that unless $p_1 = 1$, the third-order point-by-point local power of the adjusted test $T^{\mathrm{CF}(N)} > \chi^2_{p_1,\alpha}$ (equivalently the size-adjusted test based on Cornish-Fisher's type expansion for the percentile) generally depends on the C, D-functions associated with $T^{(N)} \in \mathcal{T}_{N,3}$, which contrasts with Rao and Mukerjee [24].

We now adopt the average criterion [17]. That is, let

$$\operatorname*{ave}_{S_\lambda}\{\pi(\mathbf{h}_{(1)})\} = \frac{\int_{S_\lambda} \pi(\mathbf{h}_{(1)})\, d\mathbf{h}_{(1)}}{\int_{S_\lambda} d\mathbf{h}_{(1)}}$$

be the average of $\pi(\mathbf{h}_{(1)})$ along the sphere $S_\lambda = \{\mathbf{h}_{(1)} \in \mathbf{R}^{p_1} : \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)} = \lambda\}$, $\lambda > 0$. Not surprisingly, we have

$$\operatorname*{ave}_{S_\lambda}\{\pi_\alpha^{\mathrm{GCF}(1)}(\mathbf{h}_{(1)})\} = \sum_{v=1}^{3} \operatorname*{ave}_{S_\lambda}\{\mathcal{P}_v^{\mathrm{GCF}(1)}(\mathbf{h}_{(1)})\} g_{p_1+2v}(\chi^2_{p_1,\alpha};\lambda) = 0 \tag{13}$$

(see Lemma B.1). Furthermore, regardless of the infinitely many choices for the symmetric arrays $[\Gamma_{a_1\cdots a_R}(\cdot)]_{a_1,\ldots,a_R\in\{1,\ldots,p_1\}}$, $R = 2, 4, 6$ (see [11]), it is shown that

$$\operatorname*{ave}_{S_\lambda}\{\pi_\alpha^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)})\} = \sum_{v=1}^{6} \operatorname*{ave}_{S_\lambda}\{\mathcal{P}_v^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)})\} g_{p_1+2v}(\chi^2_{p_1,\alpha};\lambda) \tag{14}$$

is independent of the D-functions associated with $T^{(N)} \in \mathcal{T}_{N,3}$ (see (C.1)–(C.6) for the details), such that

$$\operatorname*{ave}_{S_\lambda}\{\pi_\alpha^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)})\} = \frac{\lambda}{2p_1}\,\{U^C_{1,\alpha} g_{p_1+2}(\chi^2_{p_1,\alpha};0) + U^C_{2,\alpha} g_{p_1+4}(\chi^2_{p_1,\alpha};0)\} + \mathrm{O}(\lambda^2)\,, \tag{15}$$

$$U^C_{1,\alpha} = U^C_{11} + U^C_{12}\chi^2_{p_1,\alpha} \quad \text{and} \quad U^C_{2,\alpha} = (U^C_{21} - U_{21} + U_{210}) + (U^C_{22} - U_{22} + U_{220})\chi^2_{p_1,\alpha}$$

(we used $xg_\nu(x;\lambda) = \nu g_{\nu+2}(x;\lambda) + \lambda g_{\nu+4}(x;\lambda)$), where

$$\begin{aligned}
U^C_{11} &= \nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)} C^{\mathcal{G}\,\mathcal{G}}_{b_1b_2,k_1k_2}\mathcal{M}_{k_1k_2,}{}^{\mathcal{G}\,\mathcal{G}}_{b_3b_4} \quad (\text{we write } \mathcal{M}_{j_1j_2,j_3j_4} = \nu_{j_1j_2,j_3j_4} - \nu_{j_1j_2,k}\nu^{k,k'}\nu_{j_3j_4,k'}),\\
U^C_{12} &= -\frac{1}{p_1+2}\,\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\langle 3\rangle C^{\mathcal{G}\,\mathcal{G}}_{b_1b_2,k_1k_2} C^{\mathcal{G}\,\mathcal{G}}_{b_3b_4,k_3k_4}\mathcal{M}_{k_1k_2,k_3k_4}\,,\\
U^C_{21} &= [3\nu^{r,r'}_{(22)}(\nu_{rr',}{}^{\mathcal{G}}_{b_4} + \nu_{r,r',}{}^{\mathcal{G}}_{b_4}) C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ b_1b_2b_3} + 3C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ b_1b_2b}\nu^{b,b'}_{(11\cdot 2)}\nu^{\mathcal{G}\,\mathcal{G}\ \mathcal{G}}_{b_3b_4,b'}\\
&\quad +6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ b_1b_3b}\nu^{b,b'}_{(11\cdot 2)}\{2\nu^{\mathcal{G}\,\mathcal{G}\ \mathcal{G}}_{b_2b_4,b'} + \nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{b_2,b_4,b'} - (2\nu_{b_2}{}^{\mathcal{G}}_{b_4,}{}^{\mathcal{G}}_{b'} + \nu_{b_2,}{}^{\mathcal{G}}_{b_4,}{}^{\mathcal{G}}_{b'})\}\\
&\quad +6(C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ b_1b_2b_3/b_4} - C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}\ \ \mathcal{G}}_{\ b_1b_2b_3/b_4})]\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)} \quad \left(\text{we write } C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3/k} = \frac{\partial}{\partial\theta_k}\,C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3}(\boldsymbol{\theta}^\dagger)\right),\\
U^C_{22} &= -\frac{1}{p_1+4}\,C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ b_1b_2b_3} C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ b_4b_5b_6}\langle 15\rangle\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\nu^{b_5,b_6}_{(11\cdot 2)}\,,
\end{aligned}$$

U_{2v}'s are defined as U^C_{2v}'s with $C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3}(\cdot)$'s replaced by $-(1/6)\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{a_1,a_2,a_3}(\cdot)$'s, and U_{2v0}'s are common for any $T^{(N)} \in \mathcal{T}_{N,3}$. In the rest of this paper, (15) (and its GB-counterpart (16) below) will be seen to serve as the basis for the third-order average local power comparison.

Remark 2 ((18)–(22) in [13] as a particular case) Kakizawa [13] obtained the third-order average local power of the GB-adjusted test $T^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}$, which was a special case of (13) and (14), especially (C.1)–(C.6) with $C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3}(\cdot)$'s replaced by $-(1/6)\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{a_1,a_2,a_3}(\cdot)$'s (see (8) or Remark 1). That is,

$$\begin{aligned}
&\operatorname*{ave}_{S_\lambda}\{\pi_\alpha^{\mathrm{GB}(1)}(\mathbf{h}_{(1)})\} = 0\,,\\
&\operatorname*{ave}_{S_\lambda}\{\pi_\alpha^{\mathrm{GB}(2)}(\mathbf{h}_{(1)})\} = \frac{\lambda}{2p_1}\,\{U^C_{1,\alpha} g_{p_1+2}(\chi^2_{p_1,\alpha};0) + (U_{210} + U_{220}\chi^2_{p_1,\alpha}) g_{p_1+4}(\chi^2_{p_1,\alpha};0)\} + \mathrm{O}(\lambda^2)\,.
\end{aligned} \tag{16}$$

4. Main results

4.1. Comparison between GCF and GB

First of all, we are interested in comparing two methods (GCF and GB). We notice that whenever $\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{a_1,a_2,a_3}(\cdot)+6C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3}(\cdot)\equiv 0$, $a_1,a_2,a_3\in\{1,\ldots,p_1\}$, $T^{\mathrm{GB}(N)}$ (with $\Delta^{\mathrm{GB}}_{a_1a_2a_3a_4}(\cdot)=\Delta^{\mathrm{GB}}_{a_1a_2a_3,k_1k_2}(\cdot)\equiv 0$) is nothing but $T^{\mathrm{GCF}(N)}$ (even for the nonzero Δ^{GB}-functions, they are equivalent in terms of the third-order average local power; see (C.1)–(C.6)). To avoid trivialities, we therefore consider the case

$$\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{a_1',a_2',a_3'}(\cdot)+6C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1'a_2'a_3'}(\cdot)\neq 0 \quad \text{at least one } (a_1',a_2',a_3');\ a_1',a_2',a_3'\in\{1,\ldots,p_1\}.$$

Then, (15) and (16) immediately yield:

Theorem 2 *Let $T^{(N)}\in\mathcal{T}_{N,3}$. Then*

$$\begin{aligned}&\lim_{\lambda\to 0}\frac{1}{\lambda}\operatorname*{ave}_{S_\lambda}\Big\{\lim_{N\to\infty}N\Big(P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}_{(1)}',0_{p_2}')'}[T^{\mathrm{GCF}(N)}>\chi^2_{p_1,\alpha}]-P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}_{(1)}',0_{p_2}')'}[T^{\mathrm{GB}(N)}>\chi^2_{p_1,\alpha}]\Big)\Big\}\\&\quad=\frac{1}{p_1}\{(U^C_{21}-U_{21})+(U^C_{22}-U_{22})\chi^2_{p_1,\alpha}\}g_{p_1+4}(\chi^2_{p_1,\alpha};0)\,,\end{aligned}$$

so that this limit is positive (negative) for small α, if $U^C_{22}>U_{22}$ ($U^C_{22}<U_{22}$).

Letting $C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3}(\cdot)\equiv 0$, $a_1,a_2,a_3\in\{1,\ldots,p_1\}$ (i.e., $U^C_{22}=0$), we have:

Corollary 3 *If $\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{a_1,a_2,a_3}(\cdot)\equiv 0$, $a_1,a_2,a_3\in\{1,\ldots,p_1\}$, then*

$$\operatorname*{ave}_{S_\lambda}\Big\{\lim_{N\to\infty}N\Big(P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}_{(1)}',0_{p_2}')'}[\mathrm{R}^{\mathrm{GCF}(N)}>\chi^2_{p_1,\alpha}]-P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}_{(1)}',0_{p_2}')'}[\mathrm{R}^{\mathrm{GB}(N)}>\chi^2_{p_1,\alpha}]\Big)\Big\}=0\,.$$

On the other hand, if $\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{a_1',a_2',a_3'}(\cdot)\neq 0$ at least one (a_1',a_2',a_3'); $a_1',a_2',a_3'\in\{1,\ldots,p_1\}$, then $-U_{22}>0$;

$$\begin{aligned}&\lim_{\lambda\to 0}\frac{1}{\lambda}\operatorname*{ave}_{S_\lambda}\Big\{\lim_{N\to\infty}N\Big(P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}_{(1)}',0_{p_2}')'}[\mathrm{R}^{\mathrm{GCF}(N)}>\chi^2_{p_1,\alpha}]-P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}_{(1)}',0_{p_2}')'}[\mathrm{R}^{\mathrm{GB}(N)}>\chi^2_{p_1,\alpha}]\Big)\Big\}\\&\quad=\frac{1}{p_1}\{(-U_{21})+(-U_{22})\chi^2_{p_1,\alpha}\}g_{p_1+4}(\chi^2_{p_1,\alpha};0)>0\end{aligned}$$

for small α. Hence, in general, the $\mathrm{R}^{\mathrm{GCF}(N)}$-test is locally superior to the $\mathrm{R}^{\mathrm{GB}(N)}$-test (here, 'locally' means that both $\lambda>0$ and $0<\alpha<1$ are small).

This finding is a substantial extension of Rao and Mukerjee [23, 24] to the framework of a composite hypothesis about a subvector of the parameters. Note that the GCF-adjusted test does not always outperform the GB-adjusted test; for example, assuming that $\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{a_1,a_2,a_3}(\cdot)\equiv 0$, $a_1,a_2,a_3\in\{1,\ldots,p_1\}$ (i.e., $U_{22}=0$), then, the $T^{\mathrm{GB}(N)}$-test is locally superior to the $T^{\mathrm{GCF}(N)}$-test, whenever $T^{(N)}\in\mathcal{T}_{N,3}$ has the C-functions such that $C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1'a_2'a_3'}(\cdot)\neq 0$ at least one (a_1',a_2',a_3'); $a_1',a_2',a_3'\in\{1,\ldots,p_1\}$ (i.e., $U^C_{22}<0$).

4.2. Optimality of the GCF-adjusted Rao test in a class of the GCF or GB-adjusted tests

From (15) (or (16)), the optimality of the adjusted Rao test about a subvector of the parameters can be established. The point is that for the subclass $\bigcup_{c\in\mathrm{R}}\mathcal{T}^c_{N,3}(\subset\mathcal{T}_{N,3})$, the scalar c in (5) and the C^+-functions $C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3}(\cdot)$'s, given by the formulas (7), play a crucial role to discuss the third-order optimality in the present set-up, which is contrast to Kakizawa [13] with the identical C^+-functions; $C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3}(\cdot)=-(1/6)\nu^{\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{a_1,a_2,a_3}(\cdot)={}_{\mathrm{LR}}C^{+\mathcal{G}\ \mathcal{G}\ \mathcal{G}}_{\ a_1a_2a_3}(\cdot)$, $a_1,a_2,a_3\in\{1,\ldots,p_1\}$.

Theorem 4 *Let $T^{(N)} \in \mathcal{T}^c_{N,3}$. Then*

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\operatorname*{ave}_{S_\lambda}\Big\{\lim_{N\to\infty} N\Big(P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[\mathrm{R}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}] - P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}]\Big)\Big\}$$
$$= \frac{1}{p_1}\,[U_{1,\alpha}(c)g_{p_1+2}(\chi^2_{p_1,\alpha};0) + \{(-U^C_{21}) + (-U^C_{22})\chi^2_{p_1,\alpha}\}g_{p_1+4}(\chi^2_{p_1,\alpha};0)]\,,$$

where

$$U_{1,\alpha}(c) = -c\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\mathcal{M}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1b_2,b_3b_4} + \frac{\chi^2_{p_1,\alpha}c^2}{p_1+2}\,\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\langle 3\rangle\mathcal{M}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1b_2,b_3b_4}\,.$$

Remark 3 (Theorem 4 (iii) in [13] as a particular case) Let $T^{(N)} \in \mathcal{T}^c_{N,3}$. Then, (16) implies that

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\operatorname*{ave}_{S_\lambda}\Big\{\lim_{N\to\infty} N\Big(P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[\mathrm{R}^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}] - P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}]\Big)\Big\}$$
$$= \frac{1}{p_1}\,U_{1,\alpha}(c)g_{p_1+2}(\chi^2_{p_1,\alpha};0)\,.$$

We recall that the coefficients of $\chi^2_{p_1,\alpha}g_{p_1+2v}(\chi^2_{p_1,\alpha};0)$, $v=1,2$ in Theorem 4 are nonnegative, i.e.

$$-U^C_{22} \geq 0 \quad \text{and} \quad \nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\langle 3\rangle\mathcal{M}^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1b_2,b_3b_4} = E_{\theta^\dagger}[\{\mathrm{tr}(\mathbf{L}^\perp_2\mathbf{B})\}^2] + 2E_{\theta^\dagger}[\mathrm{tr}\{(\mathbf{L}^\perp_2\mathbf{B})^2\}] \geq 0\,,$$

where $\mathbf{B}(\boldsymbol{\theta})$ is defined in (1) and $\mathbf{L}^\perp_2(\mathbf{X},\boldsymbol{\theta}) = [\ell^\perp_{j_1j_2}(\mathbf{X},\boldsymbol{\theta}) - \nu_{j_1j_2}(\boldsymbol{\theta})]_{j_1,j_2\in\{1,\ldots,p\}}$, with

$$\ell^\perp_{j_1j_2}(\mathbf{X},\boldsymbol{\theta}) = \ell_{j_1j_2}(\mathbf{X},\boldsymbol{\theta}) - \nu_{j_1j_2,k}(\boldsymbol{\theta})\nu^{k,k'}(\boldsymbol{\theta})\ell_{k'}(\mathbf{X},\boldsymbol{\theta})\,.$$

Therefore, Theorem 4 demonstrates that the GCF-adjusted Rao test $\mathrm{R}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}$, with $c=0$ and $C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2a_3}(\cdot) \equiv 0$, $a_1,a_2,a_3 \in \{1,\ldots,p_1\}$, is, in general, locally optimal in a class of the GCF-adjusted tests; $T^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}$ for $T^{(N)} \in \bigcup_{c\in\mathrm{R}}\mathcal{T}^c_{N,3}$. From (12), this optimality holds even for the size-adjusted tests based on Cornish-Fisher's type expansion, which is an extension of Mukerjee [21] to the framework of a composite hypothesis about a subvector of the parameters.

This kind of result is consistent with Kakizawa [13], i.e., Remark 3 tells us that the GB-adjusted Rao test $\mathrm{R}^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}$ (or $\mathrm{MR}^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}$), with $c=0$, is, in general, locally optimal in a class of the GB-adjusted tests; $T^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}$ for $T^{(N)} \in \bigcup_{c\in\mathrm{R}}\mathcal{T}^c_{N,3}$. However, using the GB-adjustment, it was impossible to discriminate any test statistic belonging to the subclass $\mathcal{T}^c_{N,3}$, i.e.

$$\operatorname*{ave}_{S_\lambda}\Big\{\lim_{N\to\infty} N\Big(P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T_1^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}] - P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[T_2^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}]\Big)\Big\} = 0$$

for any $T_1^{(N)}, T_2^{(N)} \in \mathcal{T}^c_{N,3}$ (e.g. $\mathrm{R}^{(N)}, \mathrm{MR}^{(N)} \in \mathcal{T}^0_{N,3}$ or $\mathrm{W}^{(N)}, \mathrm{MW}^{(N)}, \mathrm{grad}^{(N)} \in \mathcal{T}^1_{N,3}$; see (6)). Also, any GB-adjusted test $T^{\mathrm{GB}(N)} > \chi^2_{p_1,\alpha}$ for $T^{(N)} \in \bigcup_{c\in\mathrm{R}}\mathcal{T}^c_{N,3}$ has the identical third-order average power if $\mathcal{M}_{j_1j_2,j_3j_4}(\cdot) \equiv 0$, $j_1,j_2,j_3,j_4 \in \{1,\ldots,p\}$ (see Theorem 4 (ii) of [13]).

On the other hand, using the GCF-adjustment, even if $\mathcal{M}_{j_1j_2,j_3j_4}(\cdot) \equiv 0$, $j_1,j_2,j_3,j_4 \in \{1,\ldots,p\}$, we have

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\operatorname*{ave}_{S_\lambda}\Big\{\lim_{N\to\infty} N\Big(P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[\mathrm{R}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}] - P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[\mathrm{MR}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}]\Big)\Big\} > 0$$

for small α, if $(\langle 3\rangle/3)\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a'_1a'_2,a'_3}(\cdot) + (1/2)\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a'_1,a'_2,a'_3}(\cdot) \neq 0$ at least one (a'_1,a'_2,a'_3); $a'_1,a'_2,a'_3 \in \{1,\ldots,p_1\}$ (note that if $(\langle 3\rangle/3)\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1a_2,a_3}(\cdot) + (1/2)\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{a_1,a_2,a_3}(\cdot) \equiv 0$, $a_1,a_2,a_3 \in \{1,\ldots,p_1\}$, then

$$\operatorname*{ave}_{S_\lambda}\Big\{\lim_{N\to\infty} N\Big(P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[\mathrm{R}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}] - P^{(N)}_{\theta^\dagger+N^{-1/2}(\mathrm{h}'_{(1)},0'_{p_2})'}[\mathrm{MR}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}]\Big)\Big\} = 0\,,$$

since $\mathrm{MR}^{(N)} \in \mathcal{T}^{0}_{N,3}$ has the C^{+}-functions; $-\{(\langle 3\rangle/3)\nu^{\mathcal{G}}_{a_1}{}^{\mathcal{G}}_{a_2,a_3}{}^{\mathcal{G}}(\cdot)+(1/2)\nu^{\mathcal{G}}_{a_1,a_2,a_3}{}^{\mathcal{G}}{}^{\mathcal{G}}(\cdot)\}$, $a_1, a_2, a_3 \in \{1, \dots, p_1\}$).

Combining Theorem 4, Remark 3, and Corollary 3, together with the identity (12), we establish the following result which considerably strengthens the finding in Kakizawa [13].

Theorem 5 *The GCF-adjusted Rao test* $\mathrm{R}^{\mathrm{GCF}(N)} > \chi^2_{p_1,\alpha}$ *(or the size-adjusted Rao test based on Cornish-Fisher's type expansion) is, in general, locally optimal in a class of the* $T^{\mathrm{GCF}(N)}$*-tests (or the size-adjusted* $T^{(N)}$*-test based on Cornish-Fisher's type expansion) and the* $T^{\mathrm{GB}(N)}$*-tests for any* $T^{(N)} \in \bigcup_{c\in\mathrm{R}} \mathcal{T}^{c}_{N,3}$.

5. Concluding remarks

We notice that an adjusted LR test statistic [20] considered in the $N^{-1/2}$-level does not belong to the class $\mathcal{T}_{N,2}$, which is the class $\mathcal{T}_{N,3}$ (see (3)) with the omission of the $\mathrm{O}_p(N^{-1})$-term. Thus, letting

$$M_j(\cdot) = \frac{1}{2}\{\nu_{rr',j}(\cdot) + \nu_{r,r',j}(\cdot)\}\nu^{r,r'}_{(22)}(\cdot)\,, \quad j = 1, \dots, p\,, \tag{17}$$

the class $\mathcal{T}_{N,3}$ may be enlarged to the class $\mathcal{T}_{N,3,M}$, as follows:

$$T^{(N)}_{M} = T^{(N)}_{\mathrm{3rd},M} + \frac{1}{N^{3/2}}\,\mathrm{o}^{(N)}_{\theta^\dagger}(1+\xi,\beta) \quad \text{for some fixed } \beta > 0 \text{ and } \xi \geq 0, \tag{18}$$

where

$$\begin{aligned} T^{(N)}_{\mathrm{3rd},M} = T^{(N)}_{\mathrm{3rd}} &+ \frac{2}{N^{1/2}}\,\widetilde{M}^{\mathcal{G}}_{b_1}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_1} \\ &+ \frac{2}{N}\Big(\frac{1}{2}\,\widetilde{M}^{\mathcal{G}}_{b_1}\widetilde{\nu}^{b_1,b_2}_{(11\cdot 2)}\widetilde{M}^{\mathcal{G}}_{b_2} + {}_{M}\widetilde{D}^{\mathcal{G}}_{b_1}{}^{\mathcal{G}}_{b_2}\prod_{i=1}^{2}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_i} + {}_{M}\widetilde{D}^{\mathcal{G}}_{b_1,k_1k_2}\widetilde{Z}^{(N)}_{k_1k_2}[\widetilde{\boldsymbol{\nu}}^{-1}_{(11\cdot 2)}\widetilde{\mathbf{Z}}^{(N)}_{(1)}]_{b_1}\Big). \end{aligned} \tag{19}$$

Here, the additional D-functions ${}_{M}D_{j_1j_2}(\cdot) = {}_{M}D_{j_2j_1}(\cdot)$ and ${}_{M}D_{j_1,k_1k_2}(\cdot)$, which may vary from one test statistic to another, are assumed to be of class $\mathcal{C}^1(\boldsymbol{\Theta})$, such that ${}_{M}D_{j_1j_2}(\cdot) = {}_{M}D_{j_1,k_1k_2}(\cdot) \equiv 0$ if $M_j(\cdot) \equiv 0$, $j = 1, \dots, p$. The choice (17) is related with the so-called adjusted profile likelihood inference (e.g. [4, 6]), as pointed out by Mukerjee [20] (see also [12]). Using the conditional likelihood approach, Mukerjee [16, 18] and Ghosh and Mukerjee [7] essentially considered (18) and (19) under the global parameter orthogonality (with $p_1 = 1$). Thus, strengthening the results [7, 16, 18], it is hoped that the finding in the present paper would be extended to the class $\mathcal{T}_{N,3,M}$ of test statistics from the adjusted profile inference. The details on this topic will be reported elsewhere.

Finally, it would be interesting in future to make a small sample comparison. After seminal papers by Chandra and Mukerjee [2], Cordeiro and Ferrari [3], and Taniguchi [25], many researchers in this area (the reference lists from past decades are found in [13]) often have reported the Cordeiro-Ferrari Bartlett-type adjustment, which reveals the finite sample improvements for the Rao and LR tests, where the adjustment for the LR case is the traditional Bartlett adjustment. Since these researchers' numerical power analyses have been done without the third-order asymptotic theory under the contiguous alternative, it is hoped that our present higher-order average local power analyses would fill up this gap to some extent.

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Appendix A: Outline of the proof of Proposition 1

Recall that (10) admits the stochastic expansion

$$T^{\mathrm{GCF}(N)} = U_a^{\Gamma_{2,4,6}(N)} \nu_{(11\cdot 2)}^{a,b} U_b^{\Gamma_{2,4,6}(N)} + \frac{1}{N^{3/2}}\, \mathrm{o}_{\theta^\dagger}^{(N)}(1+\min(\delta/2,\xi),\max(9/2,\beta)) \tag{A.1}$$

(see [10, 11]) with

$$U_a^{\Gamma_{2,4,6}(N)} = [\mathbf{Z}_{(1)}^{0(N)}]_a + \frac{1}{N^{1/2}}\, U_a^{C(N)} + \frac{1}{N}\Big(U_a^{CD(N)} + \sum_{R=2,4,6} \Gamma_{b_1\cdots b_{R-1}a} \prod_{i=1}^{R-1} [\boldsymbol{\nu}_{(11\cdot 2)}^{-1}\mathbf{Z}_{(1)}^{0(N)}]_{b_i}\Big)\,, \tag{A.2}$$

$a = 1,\ldots,p_1$. Note that (A.2) is a certain polynomial in $\mathbf{Z}_{(1)}^{0(N)} = \boldsymbol{\mathcal{G}}'\mathbf{Z}^{(N)}$, $\boldsymbol{\rho}^{0(N)} = \boldsymbol{\nu}_{(22)}^{-1}\mathbf{Z}_{(2)}^{(N)}$, and $[Z_{j_1j_2}^{\perp(N)}, Z_{j_1j_2j_3}^{\perp(N)}]_{j_1,j_2,j_3\in\{1,\ldots,p\}}$, where $Z_{j_1\cdots j_R}^{\perp(N)} = Z_{j_1\cdots j_R}^{(N)} - \nu_{j_1\cdots j_R,k}\nu^{k,k'} Z_{k'}^{(N)}$, $R=2,3$. As in Kakizawa [13], (A.1) implies

$$T^{\mathrm{GCF}(N)} = U_a^{\Gamma_{2,4,6}(N)} \nu_{(11\cdot 2)}^{a,b} U_b^{\Gamma_{2,4,6}(N)} + \frac{1}{N^{3/2}}\, \mathrm{o}_{\theta^{\dagger(N)}}^{(N)}(1+\min(\delta/2,\xi),\max(9/2,\beta))\,, \tag{A.3}$$

where

$$\boldsymbol{\theta}^{\dagger(N)} = \boldsymbol{\theta}^\dagger + N^{-1/2}\mathbf{h}\,.$$

Then, by making use of Chibisov's lemma applied to (A.3) (see [14]), the non-null distribution of $T^{\mathrm{GCF}(N)}$ admits the third-order asymptotic expansion which is the same as that of $U_a^{\Gamma_{2,4,6}(N)} \nu_{(11\cdot 2)}^{a,b} U_b^{\Gamma_{2,4,6}(N)}$, obtained via the Bhattacharya and Ghosh argument [1] (see e.g. [11, 13]) by computing the non-null cumulants of (A.2) up to $\mathrm{o}(N^{-1})$, as follows:

$$\begin{aligned}
E_{\theta^{\dagger(N)}}^{(N)}[U_{a_1}^{\Gamma_{2,4,6}(N)}] &= \nu_{(11\cdot 2)a_1,\diamond} + \frac{\dot\kappa_{a_1}^{C}}{N^{1/2}} + \frac{\dot\kappa_{a_1}^{CD}}{N}\\
&\quad + \frac{1}{N}\{\Gamma_{\diamond a_1} + (\langle 3\rangle \Gamma_{\diamond bb'a_1}\nu_{(11\cdot 2)}^{b,b'} + \Gamma_{\diamond\diamond\diamond a_1})\\
&\quad + (\langle 15\rangle \Gamma_{\diamond b_1b_1'b_2b_2'a_1}\nu_{(11\cdot 2)}^{b_1,b_1'}\nu_{(11\cdot 2)}^{b_2,b_2'} + \langle 10\rangle \Gamma_{\diamond\diamond\diamond bb'a_1}\nu_{(11\cdot 2)}^{b,b'} + \Gamma_{\diamond\diamond\diamond\diamond\diamond a_1})\}\\
&\quad + \mathrm{o}(N^{-1})\,,\\
Cov_{\theta^{\dagger(N)}}^{(N)}(U_{a_1}^{\Gamma_{2,4,6}(N)}, U_{a_2}^{\Gamma_{2,4,6}(N)}) &= \nu_{(11\cdot 2)a_1,a_2} + \frac{\dot\kappa_{a_1,a_2}^{C}}{N^{1/2}} + \frac{\dot\kappa_{a_1,a_2}^{CD}}{N}\\
&\quad + \frac{\langle 2\rangle}{N}\{\Gamma_{a_1a_2} + (\langle 3\rangle \Gamma_{bb'a_1a_2}\nu_{(11\cdot 2)}^{b,b'} + \langle 3\rangle \Gamma_{\diamond\diamond a_1a_2})\\
&\quad + (\langle 15\rangle \Gamma_{b_1b_1'b_2b_2'a_1a_2}\nu_{(11\cdot 2)}^{b_1,b_1'}\nu_{(11\cdot 2)}^{b_2,b_2'} + \langle 30\rangle \Gamma_{\diamond\diamond bb'a_1a_2}\nu_{(11\cdot 2)}^{b,b'}\\
&\quad + \langle 5\rangle \Gamma_{\diamond\diamond\diamond\diamond a_1a_2})\} + \mathrm{o}(N^{-1})\,,\\
Cum_{\theta^{\dagger(N)}}^{(N)}(U_{a_1}^{\Gamma_{2,4,6}(N)},\ldots,U_{a_3}^{\Gamma_{2,4,6}(N)}) &= \frac{\kappa_{a_1,a_2,a_3}^{C}}{N^{1/2}} + \frac{\dot\kappa_{a_1,a_2,a_3}^{CD}}{N}\\
&\quad + \frac{\langle 3!\rangle}{N}\{\langle 3\rangle \Gamma_{\diamond a_1a_2a_3} + (\langle 30\rangle \Gamma_{\diamond bb'a_1a_2a_3}\nu_{(11\cdot 2)}^{b,b'} + \langle 10\rangle \Gamma_{\diamond\diamond\diamond a_1a_2a_3})\}\\
&\quad + \mathrm{o}(N^{-1})\,,\\
Cum_{\theta^{\dagger(N)}}^{(N)}(U_{a_1}^{\Gamma_{2,4,6}(N)},\ldots,U_{a_4}^{\Gamma_{2,4,6}(N)}) &= \frac{\kappa_{a_1,a_2,a_3,a_4}^{CD}}{N} + \frac{\langle 4!\rangle}{N}\{\Gamma_{a_1a_2a_3a_4}\\
&\quad + (\langle 10\rangle \Gamma_{bb'a_1a_2a_3a_4}\nu_{(11\cdot 2)}^{b,b'} + \langle 10\rangle \Gamma_{\diamond\diamond a_1a_2a_3a_4})\} + \mathrm{o}(N^{-1})\,,\\
Cum_{\theta^{\dagger(N)}}^{(N)}(U_{a_1}^{\Gamma_{2,4,6}(N)},\ldots,U_{a_5}^{\Gamma_{2,4,6}(N)}) &= \frac{\langle 5!\rangle}{N}\,\langle 5\rangle \Gamma_{\diamond a_1a_2a_3a_4a_5} + \mathrm{o}(N^{-1})\,,\\
Cum_{\theta^{\dagger(N)}}^{(N)}(U_{a_1}^{\Gamma_{2,4,6}(N)},\ldots,U_{a_6}^{\Gamma_{2,4,6}(N)}) &= \frac{\langle 6!\rangle}{N}\,\Gamma_{a_1a_2a_3a_4a_5a_6} + \mathrm{o}(N^{-1})
\end{aligned}$$

with the forms

$$\begin{aligned}
&\dot{\kappa}^{C}_{a_1} = \kappa^{C}_{a_1} + \kappa^{C\langle 2\rangle}_{a_1}\,, \qquad \dot{\kappa}^{C}_{a_1,a_2} = \kappa^{C\langle 1\rangle}_{a_1,a_2}\,, \\
&\dot{\kappa}^{CD}_{a_1} = \kappa^{CD\langle 1\rangle}_{a_1} + \kappa^{CD\langle 3\rangle}_{a_1}\,, \quad \dot{\kappa}^{CD}_{a_1,a_2} = \kappa^{CD}_{a_1,a_2} + \kappa^{CD\langle 2\rangle}_{a_1,a_2}\,, \quad \dot{\kappa}^{CD}_{a_1,a_2,a_3} = \kappa^{CD\langle 1\rangle}_{a_1,a_2,a_3}\,.
\end{aligned} \tag{A.4}$$

The null cumulant coefficients $\kappa^{C}_{a_1}$'s, $\kappa^{CD}_{a_1,a_2}$'s, $\kappa^{C}_{a_1,a_2,a_3}$'s, and $\kappa^{CD}_{a_1,a_2,a_3,a_4}$'s, together with the expressions in (A.4) with superscript $\langle i\rangle$, being homogeneous polynomials of degree i in $\mathbf{h} = (h_1, \ldots, h_p)'$, are available from Kakizawa [13].

In this way (the details are omitted here to save space), we have

$$\begin{aligned}
P^{(N)}_{\theta^{\dagger(N)}}[T^{\mathrm{GCF}(N)} > x] = {}& 1 - G_{p_1}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) + \sum_{\ell=1}^{2} \frac{2}{N^{\ell/2}} \sum_{v=1}^{3\ell} \dot{\mathcal{Q}}^{(\ell)}_{v}(\mathbf{h}) g_{p_1+2v}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) \\
& + \frac{2}{N}\,\{\Gamma_{b_1 b_2}\nu^{b_1,b_2}_{(11\cdot 2)} g_{p_1+2}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) \\
& \quad + (3\Gamma_{b_1 b_2 b_3 b_4}\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)} + \Gamma_{\diamond\diamond}) g_{p_1+4}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) \\
& \quad + (15\Gamma_{b_1 b_2 b_3 b_4 b_5 b_6}\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\nu^{b_5,b_6}_{(11\cdot 2)} + 6\Gamma_{\diamond\diamond b b'}\nu^{b,b'}_{(11\cdot 2)}) g_{p_1+6}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) \\
& \quad + (45\Gamma_{\diamond\diamond b_1 b'_1 b_2 b'_2}\nu^{b_1,b'_1}_{(11\cdot 2)}\nu^{b_2,b'_2}_{(11\cdot 2)} + \Gamma_{\diamond\diamond\diamond\diamond}) g_{p_1+8}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) \\
& \quad + 15\Gamma_{\diamond\diamond\diamond\diamond b b'}\nu^{b,b'}_{(11\cdot 2)} g_{p_1+10}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)}) \\
& \quad + \Gamma_{\diamond\diamond\diamond\diamond\diamond\diamond} g_{p_1+12}(x; \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)})\} \\
& + \mathrm{o}(N^{-1})\,,
\end{aligned} \tag{A.5}$$

where

$$\begin{aligned}
&\dot{\mathcal{Q}}^{(1)}_{1}(\mathbf{h}) = -\frac{1}{2}\,(\nu_{rr',\diamond}{}^{\mathcal{G}} + \nu_{r,r',\diamond}{}^{\mathcal{G}})\nu^{r,r'}_{(22)} - \frac{1}{6}\,(3\nu^{\mathcal{G}\mathcal{G}\mathcal{G}}_{\diamond\diamond,\diamond} + 2\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\diamond,\diamond,\diamond}) + \frac{1}{2}\,(2\nu_{\bullet\diamond,\diamond}{}^{\mathcal{G}\,\mathcal{G}} + \nu_{\bullet,\diamond,\diamond}{}^{\mathcal{G}\,\mathcal{G}})\,, \\
&\dot{\mathcal{Q}}^{(1)}_{2}(\mathbf{h}) = \frac{1}{2}\,(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\diamond,b,b'} + 6C^{+}{}^{\mathcal{G}\mathcal{G}\mathcal{G}}_{\diamond\, b\, b'})\nu^{b,b'}_{(11\cdot 2)} + \frac{1}{6}\,\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\diamond,\diamond,\diamond} \quad \dot{\mathcal{Q}}^{(1)}_{3}(\mathbf{h}) = \frac{1}{6}\,(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\diamond,\diamond,\diamond} + 6C^{+}{}^{\mathcal{G}\mathcal{G}\mathcal{G}}_{\diamond\diamond\diamond})
\end{aligned}$$

(we used the notation $Q_{\ldots\bullet\ldots} = Q_{\ldots j\ldots}h_j$). The remaining coefficients $\dot{\mathcal{Q}}^{(2)}_{v}(\mathbf{h})$'s have the forms

$$\dot{\mathcal{Q}}^{(2)}_{v}(\mathbf{h}) = \begin{cases} \beta^{CD}_{v} + \displaystyle\sum_{i=1}^{3} \dot{\mathcal{Q}}^{(2)}_{v\langle 2i\rangle}\,, & v = 1, 2\,, \\ \beta^{C}_{3} + \displaystyle\sum_{i=1}^{3} \dot{\mathcal{Q}}^{(2)}_{3\langle 2i\rangle}\,, & v = 3\,, \\ \displaystyle\sum_{i=v-3}^{3} \dot{\mathcal{Q}}^{(2)}_{v\langle 2i\rangle}\,, & v = 4, 5, 6\,, \end{cases}$$

where the closed-form expressions of $\dot{\mathcal{Q}}^{(2)}_{v\langle 2i\rangle}$'s, being homogeneous polynomials of degree $2i$ in $\mathbf{h}$, are available from the author.

On the other hand, Kakizawa [11] showed that (11) holds iff

$$0 \equiv \beta^{CD}_{1}(\cdot) + \Gamma_{b_1 b_2}(\cdot)\nu^{b_1,b_2}_{(11\cdot 2)}(\cdot)\,, \tag{A.6}$$

$$0 \equiv \beta^{CD}_{2}(\cdot) + 3\Gamma_{b_1 b_2 b_3 b_4}(\cdot)\nu^{b_1,b_2}_{(11\cdot 2)}(\cdot)\nu^{b_3,b_4}_{(11\cdot 2)}(\cdot)\,, \tag{A.7}$$

$$0 \equiv \beta^{C}_{3}(\cdot) + 15\Gamma_{b_1 b_2 b_3 b_4 b_5 b_6}(\cdot)\nu^{b_1,b_2}_{(11\cdot 2)}(\cdot)\nu^{b_3,b_4}_{(11\cdot 2)}(\cdot)\nu^{b_5,b_6}_{(11\cdot 2)}(\cdot)\,. \tag{A.8}$$

Thus, Proposition 1 is shown by letting $\mathbf{h}_{(2)} = \mathbf{0}_{p_2}$ in (A.5). $\square$

Appendix B: Auxiliary lemma

In deriving the third-order average local power along $S_\lambda = \{\mathbf{h}_{(1)} \in \mathbf{R}^{p_1} : \mathbf{h}'_{(1)}\boldsymbol{\nu}_{(11\cdot 2)}\mathbf{h}_{(1)} = \lambda\}$, $\lambda > 0$, we need to evaluate $\mathrm{ave}_{S_\lambda}\{\mathcal{P}(\mathbf{h}_{(1)})\}$, where $\mathcal{P}(\cdot)$ is a polynomial.

Lemma B.1 *Suppose that $Q_{a_1\cdots a_v}$'s are independent of $\mathbf{h}_{(1)} = (h_1, \dots, h_{p_1})'$. Then*

$$\begin{aligned}
\underset{S_\lambda}{\mathrm{ave}}\Big(Q_{a_1\cdots a_v}\prod_{i=1}^{v} h_{a_i}\Big) &= 0\,, \quad v = 1, 3\,,\\
\underset{S_\lambda}{\mathrm{ave}}\Big(Q_{a_1a_2}\prod_{i=1}^{2} h_{a_i}\Big) &= \frac{\lambda}{p_1}\,Q_{a_1a_2}\nu^{a_1,a_2}_{(11\cdot 2)}\,,\\
\underset{S_\lambda}{\mathrm{ave}}\Big(Q_{a_1a_2a_3a_4}\prod_{i=1}^{4} h_{a_i}\Big) &= \frac{\lambda^2}{p_1(p_1+2)}\,Q_{a_1a_2a_3a_4}\langle 3\rangle\nu^{a_1,a_2}_{(11\cdot 2)}\nu^{a_3,a_4}_{(11\cdot 2)}\,,\\
\underset{S_\lambda}{\mathrm{ave}}\Big(Q_{a_1a_2a_3a_4a_5a_6}\prod_{i=1}^{6} h_{a_i}\Big) &= \frac{\lambda^3}{p_1(p_1+2)(p_1+4)}\,Q_{a_1a_2a_3a_4a_5a_6}\langle 15\rangle\nu^{a_1,a_2}_{(11\cdot 2)}\nu^{a_3,a_4}_{(11\cdot 2)}\nu^{a_5,a_6}_{(11\cdot 2)}\,.
\end{aligned}$$

Proof. We define

$$\boldsymbol{\Delta}_{(1)} = (\Delta_1, \dots, \Delta_{p_1})' = \lambda^{-1/2}\boldsymbol{\nu}^{1/2}_{(11\cdot 2)}\mathbf{h}_{(1)} \quad \text{and} \quad Q^{a_1\cdots a_v} = Q_{a'_1\cdots a'_v}[\boldsymbol{\nu}^{-1/2}_{(11\cdot 2)}]_{a'_1a_1}\cdots[\boldsymbol{\nu}^{-1/2}_{(11\cdot 2)}]_{a'_va_v}\,.$$

Then,

$$\frac{\int_{S_\lambda} Q_{a_1\cdots a_v}h_{a_1}\cdots h_{a_v}\,d\mathbf{h}_{(1)}}{\int_{S_\lambda} d\mathbf{h}_{(1)}} = \lambda^{v/2}\frac{\int_S Q^{a_1\cdots a_v}\Delta_{a_1}\cdots\Delta_{a_v}\,d\boldsymbol{\Delta}_{(1)}}{\int_S d\boldsymbol{\Delta}_{(1)}} = \lambda^{v/2}Q^{a_1\cdots a_v}E[U_{a_1}\cdots U_{a_v}]$$

with $S = \{\boldsymbol{\Delta}_{(1)} \in \mathbf{R}^{p_1} : \boldsymbol{\Delta}'_{(1)}\boldsymbol{\Delta}_{(1)} = 1\}$, where $\mathbf{U}^{(p_1)} = (U_1, \dots, U_{p_1})'$ is distributed uniformly on the unit sphere surface S in $\mathbf{R}^{p_1}$ (e.g. [5, chapter 2]). Let $(Z_1, \dots, Z_{p_1})'$ be distributed as $\mathrm{N}(\mathbf{0}_{p_1}, \mathbf{I}_{p_1})$. Using the fact that the distribution of $\mathrm{N}(\mathbf{0}_{p_1}, \mathbf{I}_{p_1})$ is the same as that of $R\mathbf{U}^{(p_1)}$, where R, being distributed as χ_{p_1}, is independent of $\mathbf{U}^{(p_1)}$, we have $E[Z_{a_1}\cdots Z_{a_v}] = E[R^v]E[U_{a_1}\cdots U_{a_v}]$, which completes the proof. □

Appendix C: $\mathrm{ave}_{S_\lambda}\{\mathcal{P}^{\mathrm{GCF}(2)}_v(\mathbf{h}_{(1)})\}$'s

We finally present the closed-form expressions for (14), as follows:

$$\underset{S_\lambda}{\mathrm{ave}}\{\mathcal{P}^{\mathrm{GCF}(2)}_1(\mathbf{h}_{(1)})\} = \frac{\lambda}{2p_1}\,\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}C_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_2},k_1k_2}\mathcal{M}_{k_1k_2,\overset{\mathcal{G}}{b_3}\overset{\mathcal{G}}{b_4}} + \sum_{i=1}^{3}\lambda^i\mathcal{A}_{1,2i}\,, \tag{C.1}$$

$$\begin{aligned}
\underset{S_\lambda}{\mathrm{ave}}\{\mathcal{P}^{\mathrm{GCF}(2)}_2(\mathbf{h}_{(1)})\} = \frac{\lambda}{2p_1}\,&[-(C_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_2},k_1k_2}C_{\overset{\mathcal{G}}{b_3}\overset{\mathcal{G}}{b_4},k_3k_4}\mathcal{M}_{k_1k_2,k_3k_4} + 2C_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_3},k_1k_2}C_{\overset{\mathcal{G}}{b_2}\overset{\mathcal{G}}{b_4},k_3k_4}\mathcal{M}_{k_1k_2,k_3k_4})\\
&+3\nu^{r,r'}_{(22)}(\nu_{rr',\overset{\mathcal{G}}{b_4}} + \nu_{r,r',\overset{\mathcal{G}}{b_4}})C^{+}_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_2}\overset{\mathcal{G}}{b_3}} + 3C^{+}_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_2}\overset{\mathcal{G}}{b}}\nu^{b,b'}_{(11\cdot 2)}\nu_{\overset{\mathcal{G}}{b_3}\overset{\mathcal{G}}{b_4},\overset{\mathcal{G}}{b'}}\\
&+6C^{+}_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_3}\overset{\mathcal{G}}{b}}\nu^{b,b'}_{(11\cdot 2)}\{2\nu_{\overset{\mathcal{G}}{b_2}\overset{\mathcal{G}}{b_4},\overset{\mathcal{G}}{b'}} + \nu_{\overset{\mathcal{G}}{b_2},\overset{\mathcal{G}}{b_4},\overset{\mathcal{G}}{b'}} - (2\nu_{b_2\overset{\mathcal{G}}{b_4},\overset{\mathcal{G}}{b'}} + \nu_{b_2,\overset{\mathcal{G}}{b_4},\overset{\mathcal{G}}{b'}})\}\\
&+6(C^{+}_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_2}\overset{\mathcal{G}}{b_3}/b_4} - C^{+}_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_2}\overset{\mathcal{G}}{b_3}/\overset{\mathcal{G}}{b_4}})]\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\\
&+\frac{\lambda^2}{2p_1(p_1+2)}\\
&[C_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_2},k_1k_2}\mathcal{M}_{k_1k_2,\overset{\mathcal{G}}{b_3}\overset{\mathcal{G}}{b_4}}\\
&+\nu^{b,b'}_{(11\cdot 2)}C^{+}_{\overset{\mathcal{G}}{b}\overset{\mathcal{G}}{b'}\overset{\mathcal{G}}{b_4}}\{3\nu_{\overset{\mathcal{G}}{b_1}\overset{\mathcal{G}}{b_2},\overset{\mathcal{G}}{b_3}} + 2\nu_{\overset{\mathcal{G}}{b_1},\overset{\mathcal{G}}{b_2},\overset{\mathcal{G}}{b_3}} - 3(2\nu_{b_1\overset{\mathcal{G}}{b_2},\overset{\mathcal{G}}{b_3}} + \nu_{b_1,\overset{\mathcal{G}}{b_2},\overset{\mathcal{G}}{b_3}})\}]\langle 3\rangle\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\\
&+\sum_{i=1}^{3}\lambda^i\mathcal{A}_{2,2i}\,,
\end{aligned} \tag{C.2}$$

$$
\begin{aligned}
\operatorname*{ave}_{S_\lambda}\{\mathcal{P}_3^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)})\} = {} & \frac{\lambda}{2p_1}\Big\{\frac{1}{4}\,\nu_{(11\cdot 2)}^{b_1,b_2}(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1,b_2,b} + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_2b})\nu^{b,b'}_{(11\cdot 2)}(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b'_1,b'_2,b'} - 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b'_1b'_2b'})\nu^{b'_1,b'_2}_{(11\cdot 2)} \\
& \quad + \frac{1}{6}\,(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1,b_3,b_5} + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_3b_5})\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\nu^{b_5,b_6}_{(11\cdot 2)}(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_2,b_4,b_6} - 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_2b_4b_6})\Big\} \\
& + \frac{\lambda^2}{2p_1(p_1+2)} \\
& [-C^{\mathcal{G}\,\mathcal{G}}_{b_1b_2,k_1k_2}C^{\mathcal{G}\,\mathcal{G}}_{b_3b_4,k_3k_4}\mathcal{M}_{k_1k_2,k_3k_4} \\
& + \nu^{r,r'}_{(22)}(\nu_{rr',b_4}^{\quad\ \ \mathcal{G}} + \nu_{r,r',b_4}^{\quad\ \ \mathcal{G}})C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_2b_3} + 3\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1b_2,b}\nu^{b,b'}_{(11\cdot 2)}C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_3b_4b'} \\
& + 3\nu^{b,b'}_{(11\cdot 2)}C^{+\mathcal{G}\mathcal{G}\mathcal{G}}_{\ \ b\,b'b_4}\{-(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1b_2,b_3} + \nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1,b_2,b_3}) + 2\nu_{b_1b_2,b_3}^{\quad\mathcal{G}\,\mathcal{G}} + \nu_{b_1,b_2,b_3}^{\quad\mathcal{G}\,\mathcal{G}}\} \\
& + 2(C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_2b_3/b_4} - C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}\,\ \mathcal{G}}_{\ \ b_1b_2b_3/b_4})]\langle 3\rangle\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)} \\
& + \frac{\lambda^3}{36p_1(p_1+2)(p_1+4)}\{3\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_4b_5,b_6} + 2\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_4,b_5,b_6} - 3(2\nu_{b_4b_5,b_6}^{\quad\mathcal{G}\,\mathcal{G}} + \nu_{b_4,b_5,b_6}^{\quad\mathcal{G}\,\mathcal{G}})\} \\
& \qquad (\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1,b_2,b_3} + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_2b_3})\langle 15\rangle\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\nu^{b_5,b_6}_{(11\cdot 2)} \\
& + \sum_{i=2}^{3}\lambda^i\mathcal{A}_{3,2i}\,, \qquad \text{(C.3)}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname*{ave}_{S_\lambda}\{\mathcal{P}_4^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)})\} = {} & \frac{\lambda^2}{p_1(p_1+2)}\Big\{\frac{1}{4}\,\nu_{(11\cdot 2)}^{b_1,b_2}(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1,b_2,b} + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_2b})\nu^{b,b'}_{(11\cdot 2)}(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b'_1,b'_2,b'} - 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b'_1b'_2b'})\nu^{b'_1,b'_2}_{(11\cdot 2)} \\
& \quad + \frac{1}{6}\,(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1,b_3,b_5} + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_3b_5})\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\nu^{b_5,b_6}_{(11\cdot 2)}(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_2,b_4,b_6} - 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_2b_4b_6})\Big\} \\
& + \frac{\lambda^3}{12p_1(p_1+2)(p_1+4)}\{-(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_4b_5,b_6} + \nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_4,b_5,b_6}) + 2\nu_{b_4b_5,b_6}^{\quad\mathcal{G}\,\mathcal{G}} + \nu_{b_4,b_5,b_6}^{\quad\mathcal{G}\,\mathcal{G}}\} \\
& \qquad (\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1,b_2,b_3} + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_2b_3})\langle 15\rangle\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\nu^{b_5,b_6}_{(11\cdot 2)} \\
& + \lambda^3\mathcal{A}_{4,6}\,, \qquad \text{(C.4)}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname*{ave}_{S_\lambda}\{\mathcal{P}_5^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)})\} = {} & \frac{\lambda^3}{72p_1(p_1+2)(p_1+4)} \\
& (\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_1,b_2,b_3} + 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_1b_2b_3})(\nu^{\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{b_4,b_5,b_6} - 6C^{+\mathcal{G}\,\mathcal{G}\,\mathcal{G}}_{\ \ b_4b_5b_6})\langle 15\rangle\nu^{b_1,b_2}_{(11\cdot 2)}\nu^{b_3,b_4}_{(11\cdot 2)}\nu^{b_5,b_6}_{(11\cdot 2)}\,, \qquad \text{(C.5)}
\end{aligned}
$$

$$
\operatorname*{ave}_{S_\lambda}\{\mathcal{P}_6^{\mathrm{GCF}(2)}(\mathbf{h}_{(1)})\} = 0\,, \qquad \text{(C.6)}
$$

with $\mathcal{A}_{v,2i}$'s being independent of $\lambda > 0$, $T^{(N)} \in \mathcal{T}_{N,3}$, and $[\Gamma_{a_1\cdots a_R}(\cdot)]_{a_1,\ldots,a_R\in\{1,\ldots,p_1\}}$, $R = 2, 4, 6$.

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