

A Weak Law for Normed Weighted Sums of Random Elements in Rademacher Type p Banach Spaces

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For weighted sums $\sum_{j=1}^n a_j V_j$ of independent random elements $\{V_n, n \geq 1\}$ in real separable, Rademacher type p ($1 \leq p \leq 2$) Banach spaces, a general weak law of large numbers of the form $(\sum_{j=1}^n a_j V_j - v_n)/b_n \xrightarrow{P} 0$ is established, where $\{v_n, n \geq 1\}$ and $b_n \rightarrow \infty$ are suitable sequences. It is assumed that $\{V_n, n \geq 1\}$ is stochastically dominated by a random element V , and the hypotheses involve both the behavior of the tail of the distribution of $\|V\|$ and the growth behaviors of the constants $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$. No assumption is made concerning the existence of expected values or absolute moments of the $\{V_n, n \geq 1\}$. © 1991 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, $\{V_n, n \geq 1\}$ are independent random elements defined on a probability space (Ω, \mathcal{F}, P) and taking values in a real separable, Rademacher type p ($1 \leq p \leq 2$) Banach space \mathcal{X} with norm $\|\cdot\|$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants with $a_n \neq 0$, $n \geq 1$,

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$0 < b_n \rightarrow \infty$, and let $\{v_n, n \geq 1\} \subseteq \mathcal{X}$. Then $\{a_n V_n, n \geq 1\}$ is said to obey the general *weak law of large numbers* (WLLN) with centering elements $\{v_n, n \geq 1\}$ and norming constants $\{b_n, n \geq 1\}$ if the normed and centered weighted sum $(\sum_{j=1}^n a_j V_j - v_n)/b_n \xrightarrow{P} 0$. It is supposed that the sequence $\{V_n, n \geq 1\}$ is *stochastically dominated* by a random element V in the sense that for some constant $D < \infty$ the condition (2.1) prevails. This assumption is, of course, automatically satisfied if the $\{V_n, n \geq 1\}$ are identically distributed. The theorem proved herein furnishes conditions on the growth behaviors of $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ and on the distribution of V which ensure that $\{a_n V_n, n \geq 1\}$ obeys the WLLN (2.7). The condition (2.6) is of the spirit of the condition $nP\{|X_1| > n\} = o(1)$ of the classical WLLN for independent, identically distributed (i.i.d.) (real-valued) random variables $\{X_n, n \geq 1\}$ attributed to Feller by Chow and Teicher [8, p. 128].

Conditions for $\{a_n V_n, n \geq 1\}$ to obey the general *strong law of large numbers* (SLLN) $\sum_{j=1}^n a_j (V_j - EV_j)/b_n \rightarrow 0$ almost certainly (a.c.) have been obtained by Mikosch and Norvaiša [17, 18] and by Adler, Rosalsky, and Taylor [5].

The theorem in the current work generalizes an earlier result of Adler and Rosalsky [4] which dealt with i.i.d. random variables. That result of Adler and Rosalsky extended and indeed was motivated by a result of Klass and Teicher [12] which is a WLLN analogue of Feller's [10] famous generalization of the Marcinkiewicz-Zygmund SLLN (see, e.g., Chow and Teicher [8, p. 125]).

WLLNs have recently been found by Klass and Teicher [12] and by Adler [2] to be worthwhile tools in connection with their proving generalized one-sided laws of the iterated logarithm (LIL) for sums of independent asymmetric random variables barely with or without finite mean. The weights $\{a_n, n \geq 1\}$ considered in this paper for the WLLN fit the general framework considered by Adler [2]. Under strong moment conditions, Ledoux and Talagrand [15] related the WLLN to a LIL in Banach spaces.

The referee so kindly suggested to the authors a number of articles wherein the relationship between the WLLN and the SLLN for random elements in a Banach space is investigated. For example, see Kuelbs and Zinn [14], de Acosta [1], Etemadi [9], Mikosch and Norvaiša [17, 18], Alt [6], Heinkel [11], and Ledoux and Talagrand [15]. A common feature of these results is that they supply the additional hypotheses required for the WLLN and the SLLN to be equivalent. However, conditions for the WLLN (2.7) are given in the theorem below and the corresponding SLLN need not necessarily hold as will be discussed later.

Throughout the paper, the symbol C denotes a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

2. MAINSTREAM

With these preliminaries accounted for, the theorem may now be established. It should be noted that it is not being assumed that the $\{V_n, n \geq 1\}$ have expected values. Moreover, as will be shown in the proof of the theorem, its hypotheses entail (2.11) and so necessarily $b_n \rightarrow \infty$. However, it is not assumed that $\{b_n, n \geq 1\}$ is monotone.

THEOREM. *Let $\{V_n, n \geq 1\}$ be independent random elements in a real separable, Rademacher type p ($1 \leq p \leq 2$) Banach space \mathcal{X} . Suppose that $\{V_n, n \geq 1\}$ is stochastically dominated by a random element V in the sense that for some constant $D < \infty$,*

$$P\{\|V_n\| > t\} \leq DP\{\|DV\| > t\}, \quad t \geq 0, n \geq 1. \quad (2.1)$$

Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants with $a_n \neq 0$, $b_n > 0$, $n \geq 1$, and suppose that either

$$\frac{b_n}{|a_n|} \uparrow, \frac{b_n}{n|a_n|} \downarrow, \sum_{j=1}^n |a_j|^p = o(b_n^p), \quad \text{and} \quad \sum_{j=1}^n \frac{b_j^p}{j^2 |a_j|^p} = O\left(\frac{b_n^p}{\sum_{j=1}^n |a_j|^p}\right) \quad (2.2)$$

or

$$\frac{b_n}{|a_n|} \uparrow, \frac{b_n}{n|a_n|} \rightarrow \infty, \quad \sum_{j=1}^n |a_j|^p = O(n|a_n|^p), \quad \text{and} \quad \sum_{j=1}^n \frac{b_j^p}{j^2 |a_j|^p} = O\left(\frac{b_n^p}{\sum_{j=1}^n |a_j|^p}\right) \quad (2.3)$$

or

$$p > 1, \frac{b_n}{n|a_n|} \uparrow, \quad \text{and} \quad \sum_{j=1}^n |a_j|^p = O(n|a_n|^p) \quad (2.4)$$

or

$$\frac{b_n}{n|a_n|} \uparrow \quad \text{and} \quad \sum_{j=1}^n |a_j| = O\left(\frac{n|a_n|}{\log n}\right) \quad (2.5)$$

hold. If

$$nP \left\{ \|DV\| > \frac{b_n}{|a_n|} \right\} = o(1), \quad (2.6)$$

then the WLLN

$$\frac{\sum_{j=1}^n a_j (V_j - EV_j I(\|V_j\| \leq b_n/|a_n|))}{b_n} \xrightarrow{P} 0 \quad (2.7)$$

obtains.

Proof. Let

$$c_n = \frac{b_n}{|a_n|}, \quad V_{nj} = V_j I(\|V_j\| \leq c_n), \quad 1 \leq j \leq n, n \geq 1.$$

Note at the outset that $E\|V_{nj}\| < \infty$, $1 \leq j \leq n$, $n \geq 1$, and so (see, e.g., Taylor [19, p. 40]) the $\{V_{nj}, 1 \leq j \leq n, n \geq 1\}$ all have expected values. For arbitrary $\varepsilon > 0$,

$$\begin{aligned} P\left\{\frac{\|\sum_{j=1}^n a_j(V_j - V_{nj})\|}{b_n} > \varepsilon\right\} &\leq P\left\{\bigcup_{j=1}^n [V_j \neq V_{nj}]\right\} \\ &\leq \sum_{j=1}^n P\{\|V_j\| > c_n\} \\ &\leq Dn P\{\|DV\| > c_n\} \quad (\text{by (2.1)}) \\ &= o(1) \quad (\text{by (2.6)}); \end{aligned}$$

whence

$$\frac{\sum_{j=1}^n a_j(V_j - V_{nj})}{b_n} \xrightarrow{P} 0.$$

Thus, it suffices to show that

$$\frac{\sum_{j=1}^n a_j(V_{nj} - EV_{nj})}{b_n} \xrightarrow{P} 0. \quad (2.8)$$

To this end, for arbitrary $\varepsilon > 0$,

$$\begin{aligned} P\left\{\frac{\|\sum_{j=1}^n a_j(V_{nj} - EV_{nj})\|}{b_n} > \varepsilon\right\} &\leq \frac{C}{b_n^p} E \left\| \sum_{j=1}^n a_j(V_{nj} - EV_{nj}) \right\|^p \\ &\leq \frac{C}{b_n^p} \sum_{j=1}^n |a_j|^p E \|V_{nj} - EV_{nj}\|^p \quad (\text{since } \mathcal{X} \text{ is of Rademacher type } p) \\ &\leq \frac{C}{b_n^p} \sum_{j=1}^n |a_j|^p E \|V_{nj}\|^p \\ &\leq \frac{C}{|a_n|^p} \sum_{j=1}^n |a_j|^p P\{\|DV\| > c_n\} + \frac{C}{b_n^p} \sum_{j=1}^n |a_j|^p E \|V\|^p I(\|DV\| \leq c_n), \end{aligned}$$

employing integration by parts (see Lemma 1 of Adler and Rosalsky [3]).

The proof will be completed once it is verified that

$$\sum_{j=1}^n |a_j|^p P\{\|DV\| > c_n\} = o(|a_n|^p) \quad (2.9)$$

and

$$\sum_{j=1}^n |a_j|^p E\|V\|^p I(\|DV\| \leq c_n) = o(b_n^p) \quad (2.10)$$

hold.

Note that under (2.5), p may and will be taken to be 1 without any loss of generality.

To prove (2.9), observe that under (2.2),

$$\begin{aligned} \frac{1}{|a_n|^p} \sum_{j=1}^n |a_j|^p P\{\|DV\| > c_n\} \\ &\leq \frac{Cb_n^p P\{\|DV\| > c_n\}}{|a_n|^p \sum_{j=1}^n (c_j^p/j^2)} \\ &\leq \frac{Cc_n^p P\{\|DV\| > c_n\}}{n(c_n^p/n^2)} = CnP\{\|DV\| > c_n\} = o(1) \end{aligned}$$

by (2.6). On the other hand, under (2.3), (2.4), or (2.5)

$$\frac{1}{|a_n|^p} \sum_{j=1}^n |a_j|^p P\{\|DV\| > c_n\} \leq CnP\{\|DV\| > c_n\} = o(1)$$

again by (2.6) and so (2.9) obtains.

To prove (2.10), note that $c_n \uparrow$ under (2.2), (2.3), (2.4), or (2.5) and that (2.3), (2.4), and (2.5) individually ensure

$$\sum_{j=1}^n |a_j|^p = o(b_n^p). \quad (2.11)$$

Thus, (2.11) holds under (2.2), (2.3), (2.4), or (2.5). Let $c_0 = 0$ and $d_n = c_n/n$, $n \geq 1$. Define an array $\{B_{nk}, 0 \leq k \leq n, n \geq 1\}$ by

$$B_{nk} = \begin{cases} \left(\frac{1}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\frac{c_{k+1}^p - c_k^p}{k} \right) & \text{for } 1 \leq k \leq n-1, n \geq 2 \\ 0 & \text{for } k=0, n, n \geq 1. \end{cases}$$

It will now be shown that $\{B_{nk}, 0 \leq k \leq n, n \geq 1\}$ is a Toeplitz array; that is,

$$\sum_{k=0}^n |B_{nk}| = O(1) \quad (2.12)$$

and

$$B_{nk} \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ for all fixed } k \geq 0. \quad (2.13)$$

Clearly (2.11) entails (2.13). To verify (2.12), note that $B_{nk} \geq 0$, $0 \leq k \leq n$, $n \geq 1$, since $c_n \uparrow$. Now $2^p - 1 \leq 3$, and by the mean value theorem for $k \geq 2$,

$$(k+1)^p - k^p \leq p(k+1)^{p-1} \leq \frac{3k(k+1)^p}{(k+1)^2} \leq 3k^{p-1}.$$

Thus,

$$(k+1)^p - k^p \leq 3k^{p-1}, \quad k \geq 1,$$

implying for $k \geq 1$ that

$$\frac{c_{k+1}^p - c_k^p}{k} = \frac{(k+1)^p d_{k+1}^p - k^p d_k^p}{k} \leq (k^{p-1} + 3k^{p-2}) d_{k+1}^p - k^{p-1} d_k^p. \quad (2.14)$$

Then under (2.2), since $d_n \downarrow$,

$$\frac{c_{k+1}^p - c_k^p}{k} \leq \frac{3d_k^p}{k^{2-p}} = \frac{3c_k^p}{k^2}, \quad k \geq 1;$$

whence for $n \geq 2$,

$$\sum_{k=0}^n B_{nk} \leq \left(\frac{3}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} \frac{c_k^p}{k^2} \right) = O(1)$$

and so (2.12) holds. Now under (2.3), (2.4), or (2.5), for $n \geq 2$,

$$\begin{aligned} \sum_{k=0}^n B_{nk} &\leq \left(\frac{1}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} ((k^{p-1} + 3k^{p-2}) d_{k+1}^p - k^{p-1} d_k^p) \right) \\ &\quad \text{(by (2.14))} \\ &\leq \left(\frac{1}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} ((k+1)^{p-1} d_{k+1}^p - k^{p-1} d_k^p) \right) \\ &\quad + \left(\frac{3}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} k^{p-2} d_{k+1}^p \right) \\ &\leq \frac{Cn}{c_n^p} n^{p-1} d_n^p + \left(\frac{3}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} k^{p-2} d_{k+1}^p \right) \\ &= C + \left(\frac{3}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} k^{p-2} d_{k+1}^p \right). \end{aligned} \quad (2.15)$$

Under (2.3), for $n \geq 2$,

$$\left(\frac{3}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} k^{p-2} d_{k+1}^p \right) \leq \left(\frac{C}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^n \frac{c_k^p}{k^2} \right) = O(1).$$

Under (2.4) or (2.5), for $n \geq 2$,

$$\begin{aligned} & \left(\frac{3}{b_n^p} \sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} k^{p-2} d_{k+1}^p \right) \\ & \leq \frac{3d_n^p}{b_n^p} \left(\sum_{j=1}^n |a_j|^p \right) \left(\sum_{k=1}^{n-1} k^{p-2} \right) \quad (\text{since } d_n \uparrow) \\ & = \begin{cases} \frac{C}{n^{p-1}} O(n^{p-1}) & \text{if (2.4) holds} \\ \frac{C}{\log n} O(\log n) & \text{if (2.5) holds} \end{cases} \\ & = O(1). \end{aligned}$$

Thus, under (2.3), (2.4), or (2.5), recalling (2.15),

$$\sum_{k=0}^n B_{nk} = O(1)$$

and again (2.12) holds, thereby proving that $\{B_{nk}, 0 \leq k \leq n, n \geq 1\}$ is a Toeplitz array.

By (2.6) and the Toeplitz lemma (see, e.g., Knopp [13, p. 74] or Loève [16, p. 250]),

$$\sum_{k=0}^n B_{nk} k P\{\|DV\| > c_k\} = o(1). \quad (2.16)$$

Next, note that

$$\begin{aligned} & \frac{1}{b_n^p} \sum_{j=1}^n |a_j|^p E \|V\|^p I(\|DV\| \leq c_n) \\ & = \frac{1}{b_n^p} \sum_{j=1}^n |a_j|^p \sum_{k=1}^n E \|V\|^p I(c_{k-1} < \|DV\| \leq c_k) \\ & \leq \frac{D^{-p}}{b_n^p} \sum_{j=1}^n |a_j|^p \sum_{k=1}^n c_k^p P\{c_{k-1} \leq \|DV\| \leq c_k\} \\ & = \frac{D^{-p}}{b_n^p} \sum_{j=1}^n |a_j|^p \sum_{k=1}^n c_k^p (P\{\|DV\| > c_{k-1}\} - P\{\|DV\| > c_k\}) \end{aligned}$$

$$= \frac{D^{-p}}{b_n^p} \sum_{j=1}^n |a_j|^p \left(c_1^p P\{\|DV\| > 0\} - c_n^p P\{\|DV\| > c_n\} \right. \\ \left. + \sum_{k=1}^{n-1} (c_{k+1}^p - c_k^p) P\{\|DV\| > c_k\} \right)$$

(by the Abel "summation of parts" lemma)

$$\leq \frac{D^{-p}}{b_n^p} \sum_{j=1}^n |a_j|^p \sum_{k=1}^{n-1} \frac{c_{k+1}^p - c_k^p}{k} k P\{\|DV\| > c_k\} + o(1) \quad (\text{by (2.11)}) \\ = D^{-p} \sum_{k=0}^n B_{nk} k P\{\|DV\| > c_k\} + o(1) \\ = o(1) \quad (\text{by (2.16)}),$$

thereby establishing (2.10) and the theorem. ■

3. FINAL REMARKS

Some remarks pertaining to the theorem are in order.

(i) If the hypotheses of the theorem obtain with $p = 1$, then

$$\left\| \sum_{j=1}^n a_j E V_j I(\|V_j\| \leq c_n) \right\| \leq \sum_{j=1}^n |a_j| E \|V_j\| I(\|V_j\| \leq c_n) = o(b_n)$$

was shown in the *proof* of (2.8) and, consequently,

$$\sum_{j=1}^n a_j V_j / b_n \xrightarrow{P} 0.$$

Moreover, a perusal of the argument reveals that the hypothesis of independence is not needed when $p = 1$.

(ii) The example of weighted i.i.d. random variables presented in Section 5 of Adler and Rosalsky [4] satisfies the hypotheses of the theorem and hence the WLLN (2.7) obtains. On the other hand, Adler and Rosalsky [4] showed that the corresponding SLLN fails. It should be noted that the random variables in that example are not integrable and the real line is of Rademacher type p for all $1 \leq p \leq 2$.

(iii) The necessity of (2.6) for (2.7) can be illustrated by considering $a_n \equiv 1$, $b_n = n$, $n \geq 1$, and $\{V_n, n \geq 1\}$ a sequence of symmetric, i.i.d. random elements in a real separable, Rademacher type p ($1 < p \leq 2$) Banach space. Then (2.4) is satisfied and by Theorem V.9.1 and Proposition V.5.1(ii) of Woyczyński [20], (2.7) holds iff (2.6) holds.

(iv) The following example (due to Beck [7]) shows that the theorem can fail even for well-behaved sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ when

$$\frac{b_n}{n|a_n|} \uparrow \quad \text{and} \quad \sum_{j=1}^n |a_j| = O(n|a_n|). \quad (3.1)$$

Consider the real separable (Rademacher type $p=1$) Banach space l_1 of absolutely summable real sequences $v = \{v_i, i \geq 1\}$ with norm $\|v\| = \sum_{i=1}^{\infty} |v_i|$. Let $v^{(n)}$ be the element having 1 in its n th position and 0 elsewhere. Define a sequence $\{V_n, n \geq 1\}$ of random elements in l_1 by requiring the $\{V_n, n \geq 1\}$ to be independent with

$$P\{V_n = v^{(n)}\} = P\{V_n = -v^{(n)}\} = \frac{1}{2}, \quad n \geq 1.$$

Let $a_n = 1$, $b_n = n$, $n \geq 1$. Then (2.2), (2.3), (2.4), and (2.5) all fail but (3.1) holds. Also, (2.1) (with $V = V_1$ and $D = 1$) and (2.6) both hold. Since

$$\left\| \sum_{j=1}^n V_j \right\| / n = 1 \text{ a.c.}, \quad n \geq 1,$$

the conclusion (2.7) of the theorem fails. It should be noted that l_1 is not of Rademacher type p for any $1 < p \leq 2$. To see this, observe for the random elements $\{V_n, n \geq 1\}$ considered above that

$$E \left\| \sum_{j=1}^n V_j \right\|^p = n^p, \quad \sum_{j=1}^n E \|V_j\|^p = n, \quad n \geq 1.$$

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