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Comparison of LR, Score, and Wald Tests in a Non-IID Setting

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Considering a large class of tests, we study higher order power in a possibly non-iid set-up. Optimum properties for the likelihood ratio and score tests are exhibited under the criteria of second-order local maximinity and third-order local average power, respectively. The issue of stringency with regard to third-order average power has been addressed. We also compare the power properties of various Bartlett-type adjustments for the tests. © 1997 Academic Press

1. INTRODUCTION

Higher order comparison of tests under contiguous alternatives has received a considerable attention over the last two decades; see Ghosh (1991) and Mukerjee (1993) for reviews. Most of the results in this area, including the differential geometric ones for the curved exponential family (Amari (1985), Ch. 6; Kumon and Amari (1983)), relate to the case of independently and identically distributed (iid) observations. In particular, in the iid case and under the absence of nuisance parameters, optimality results are now know for (a) the likelihood ratio (LR) test in terms of second-order local maximinity and (b) the score test in terms of third-order local average power (Mukerjee, 1994).

Under a possibly non-iid set-up involving an unknown scalar parameter, Taniguchi (1991) considered the problem of third-order comparison of tests. He worked with a large class of tests that includes the LR, Rao's score and

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Wald's tests (as described in Rao (1973, pp. 415–420), suggested a Bartlett-type adjustment for the tests in the class and then, on the basis of such adjusted versions, explored the point-by-point maximization of third-order power. Consequently, his optimal test is dependent on the alternative hypothesis and his results are not comparable with the ones mentioned in (a) and (b) above.

We attempt to bridge this gap and show that the results in (a) and (b) continue to hold in Taniguchi's (1991) possibly non-iid setting. In addition, we discuss the derivation of a most stringent test with respect to third-order average power. These results have been represented in Section 3. The necessary preliminaries are given in Section 2 where a simplified and compact expression for the third-order power function has been derived. This expression, more informative than what has been known so far even in the iid case, enables us to study the third-order average power not only locally but also in its entirety and hence to address the issue of stringency as indicated above. Finally, we note the availability of two more Bartlett-type adjustments for each test statistic in Taniguchi's (1991) class, in addition to the one proposed by Taniguchi himself, and in Section 4 compare such adjusted versions under the criteria of maximinity and average power. This substantially strengthens some of the earlier results reported in Rao and Mukerjee (1995) who compared various Bartlett-type adjustments for the score statistic. As in Taniguchi (1991), consideration of the possibly non-iid setting allows us to illustrate the results with models which are important in time series analysis. It may be noted that our class of tests is slightly larger than that in Taniguchi (1991). This is needed to cover not only the tests considered by Taniguchi (1991) but also the various available Bartlett-type adjustments thereof.

Earlier, Amari (1985, pp. 180–181) considered the issue of stringency with respect to third-order power. However, unlike him, we consider a possibly non-iid set-up, make no assumption regarding curved exponentiality and compare the tests in their original forms without any modification for local unbiasedness. Because of the last reason, our findings differ from those of Amari (1985); see e.g., Section 3 below. We refer to Madansky (1989) and Mukerjee (1993, 1994) for more discussion on the motivation for comparison of tests in their original forms.

2. PRELIMINARY COMPUTATIONS

Let $X^{(n)} = (X_1, ..., X_n)'$, $n \geqslant 1$, be a collection of possibly vector-valued random variables with density $f_n(x^{(n)}; \theta)$, where the parameter θ belongs to an open subset of $\mathscr{R}^1$. Consider the null hypothesis $H_0\colon \theta = \theta_0$ against the alternative $\theta \neq \theta_0$. We make the assumptions stated in Taniguchi (1991). In particular, it is assumed that for an appropriate sequence $\{c_n\}$, satisfying $c_n \to \infty$ as $n \to \infty$, the cumulants, up to the fourth order, of

$$Z_i(\theta) = c_n^{-1}\left[\frac{d^i}{d\theta^i}\log f_n(X^{(n)};\theta) - E_\theta\left\{\frac{d^i}{d\theta^i}\log f_n(X^{(n)};\theta)\right\}\right] \quad (i=1,2,3)$$

possess asymptotic expansion of the form

$$\begin{aligned} cum_\theta\{Z_i(\theta), Z_j(\theta)\} &= k_{ij}^{(1)}(\theta) + c_n^{-2}k_{ij}^{(2)}(\theta) + o(c_n^{-2}), \\ cum_\theta\{Z_i(\theta), Z_j(\theta), Z_l(\theta)\} &= c_n^{-1}k_{ijl}(\theta) + o(c_n^{-2}), \\ cum_\theta\{Z_i(\theta), Z_j(\theta), Z_l(\theta), Z_m(\theta)\} &= c_n^{-2}k_{ijlm}(\theta) + o(c_n^{-2}), \end{aligned}$$

where $k_{ij}^{(1)}(\theta)$, $k_{ij}^{(2)}(\theta)$, $k_{ijl}(\theta)$, $k_{ijlm}(\theta)$ do not involve n. Let

$$\begin{aligned} &I = k_{11}^{(1)}(\theta), \quad J = k_{12}^{(1)}(\theta_0), \quad K = k_{111}(\theta_0), \quad L = k_{13}^{(1)}(\theta_0), \\ &M = k_{22}^{(1)}(\theta_0), \quad \tilde{M} = M - I^{-1}J^2, \quad Z_i = Z_i(\theta_0) \quad (i=1,2,3), \\ &W_1 = I^{-1/2}Z_1, \quad W_2 = Z_2 - JI^{-1}Z_1, \quad W_3 = Z_3 - LI^{-1}Z_1. \end{aligned}$$

We shall consider alternatives of the form $\theta_n = \theta_0 + c_n^{-1}\varepsilon$. Let $\mathscr{F}$ be a class of test statistics for H_0 such that every statistic T in $\mathscr{F}$ admits an expansion of the form

$$\begin{aligned} T = {} & W_1^2 + c_n^{-1}(a_1 W_1^2 W_2 + a_2 W_1^3) \\ & + c_n^{-2}(b_1 W_1^2 + b_2 W_1^2 W_2^2 + b_3 W_1^4 + b_4 W_1^3 W_2 \\ & + b_5 W_1^3 W_3 + b_6 W_1^6) \\ & + o(c_n^{-2}), \end{aligned} \tag{2.1}$$

over a set with P_{θ_n}-probability $1 + o(c_n^{-2})$ (uniformly on compact subsets of ε), where a_1, a_2 and b_1, ..., b_6 are constants free from n. Taniguchi (1991) studied a subclass $\mathscr{F}_0$ of $\mathscr{F}$ consisting of those members of $\mathscr{F}$ for which b_6 equals zero. Consideration of the wider class $\mathscr{F}$ enables us to compare the various Bartlett-type adjustments available for the members of $\mathscr{F}_0$. As noted by Taniguchi (1991), the subclass $\mathscr{F}_0$ is a very natural one and includes, in particular, the LR, score, Wald's and modified Wald's statistics, the corresponding expressions for a_1 and a_2 being given respectively by

$$a_1(LR) = I^{-1}, \qquad a_2(LR) = -\tfrac{1}{3}I^{-3/2}K, \tag{2.2a}$$

$$a_1(\text{score}) = 0, \qquad a_2(\text{score}) = 0, \tag{2.2b}$$

$$a_1(\text{Wald}) = 2I^{-1}, \qquad a_2(\text{Wald}) = I^{-3/2}J, \tag{2.2c}$$

$$a_1(\text{mod Wald}) = 2I^{-1}, \qquad a_2(\text{mod Wald}) = -I^{-3/2}(J+K). \tag{2.2d}$$

Corresponding to any statistic T as in (2.1) we shall consider a critical region

$$T > z^2 + c_n^{-2} d_T,$$

where z^2 is the upper α-point of a central chi-square variate with 1 degree of freedom (d.f.) and d_T is a constant, free from n, to be so determined that the test has size $\alpha + o(c_n^{-2})$. For $\lambda \geqslant 0$ and positive integral v, let $h_{v,\lambda}(\cdot)$ and $G_{v,\lambda}(\cdot)$ denote respectively the probability density function and the cumulative distribution function of a possibly non-central chi-square variate with v d.f. and non-centrality parameter λ. Then, analogously to Theorem 1 in Taniguchi (1991) but with suitable modification so as to take care of the extra term $b_6 W_1^6$ in (2.1), it can be shown that

$$\begin{aligned} P_{\theta_n}(T > z^2 + c_n^{-2} d_T) = {} & 1 - G_{1,\delta}(z^2) - c_n^{-1} \sum_{j=0}^{3} B_j^{(T)} G_{1+2j,\delta}(z^2) \\ & - c_n^{-2} \left\{ h_{1,\delta}(z^2)\, d_T + \sum_{j=0}^{6} A_j^{(T)} G_{1+2j,\delta}(z^2) \right\} + o(c_n^{-2}), \end{aligned} \tag{2.3}$$

where $\delta = I\varepsilon^2$,

$$\begin{aligned} & B_0^{(T)} = -\tfrac{1}{6}(3J + K)\,\varepsilon^3, \qquad B_1^{(T)} = \tfrac{1}{2} J\varepsilon^3 - \tfrac{1}{2} I^{-1}(K + 3I^{3/2} a_2)\,\varepsilon \\ & B_2^{(T)} = -\tfrac{1}{2} I^{3/2} a_2 \varepsilon^3 + \tfrac{1}{2} I^{-1}(K + 3I^{3/2} a_2)\,\varepsilon, \\ & B_3^{(T)} = \tfrac{1}{6}(K + 3I^{3/2} a_2)\,\varepsilon^3, \end{aligned} \tag{2.4}$$

$$\begin{aligned} & A_0^{(T)} = C_0^{(T)}, \qquad A_1^{(T)} = C_1^{(T)}, \qquad A_2^{(T)} = C_2^{(T)} - \tfrac{15}{2} b_6, \\ & A_3^{(T)} = C_3^{(T)} + \tfrac{15}{2} b_6 (1 - 3\delta), \qquad A_4^{(T)} = C_4^{(T)} + \tfrac{15}{2} b_6 (3\delta - \delta^2), \\ & A_5^{(T)} = C_5^{(T)} + \tfrac{1}{2} b_6 (15\delta^2 - \delta^3), \qquad A_6^{(T)} = C_6^{(T)} + \tfrac{1}{2} b_6\, \delta^3, \end{aligned} \tag{2.5}$$

the $C_j^{(T)}$ $(0 \leqslant j \leqslant 6)$ being as in Taniguchi (1991, p. 228).

The quantities $C_j^{(T)}$ $(0 \leqslant j \leqslant 6)$ are somewhat involved and we require a simplified expression for $\sum_{j=0}^{6} A_j^{(T)} G_{1+2j,\delta}(z^2)$ in order to derive our results. To that effect, for positive integral v, we write $h_{v,\delta} = h_{v,\delta}(z^2)$, $G_{v,\delta} = G_{v,\delta}(z^2)$, and note that

$$\begin{aligned} G_{v,\delta} - G_{v+2,\delta} &= 2h_{v+2,\delta}, \\ \delta h_{v+4,\delta} + v h_{v+2,\delta} &= z^2 h_{v,\delta}. \end{aligned}$$

Then by (2.5) and the expressions in Taniguchi (1991, p. 228), the coefficient of, say, b_3 in $\sum_{j=0}^{6} A_j^{(T)} G_{1+2j,\delta}(z^2)$ equals

$$-\{\tfrac{3}{2}(G_{3,\delta}-G_{5,\delta})+3\delta(G_{5,\delta}-G_{7,\delta})+\tfrac{1}{2}\delta^2(G_{7,\delta}-G_{9,\delta})\}$$
$$=-\{3h_{5,\delta}+6\delta h_{7,\delta}+\delta^2 h_{9,\delta}\}$$
$$=-\{\delta(\delta h_{9,\delta}+5h_{7,\delta})+(\delta h_{7,\delta}+3h_{5,\delta})\}$$
$$=-z^2(\delta h_{5,\delta}+h_{3,\delta})=-z^4 h_{1,\delta}.$$

Similar calculations eventually yield

$$\sum_{j=0}^{6} A_j^{(T)} G_{1+2j,\delta}(z^2)=\Psi_0(\varepsilon, z^2)-\Psi_{1T}(z^2)\,h_{1,\delta}-R_T(\delta, z^2), \tag{2.6}$$

where $\Psi_0(\varepsilon, z^2)$ is free from $a_1, a_2, b_1, ..., b_6$ and hence is the same for all statistics in the family $\mathscr{F}$, $\Psi_{1T}(z^2)$ is a constant which is free from ε but dependent on T, and

$$R_T(\delta, z^2)=\tfrac{1}{2}\tilde{M}\,\delta z^2(I^{-1}h_{1,\delta}a_1-\tfrac{1}{2}h_{3,\delta}a_1^2)$$
$$+\tfrac{1}{2}I^{-3/2}\,\delta z^2\{z^2Jh_{1,\delta}-\tfrac{1}{3}\delta(K+3J)\,h_{3,\delta}\}\,a_2-\tfrac{1}{4}z^4\,\delta h_{3,\delta}a_2^2. \tag{2.7}$$

For positive integral v, let $h_v=h_{v,0}(z^2)$. Now, taking $\varepsilon=0$ in (2.3) and invoking the size condition up to $o(c_n^{-2})$, by (2.4), (2.6), and (2.7),

$$d_T h_1+\Psi_0(0, z^2)-\Psi_{1T}(z^2)\,h_1=0.$$

Solving for d_T, from (2.3) and (2.6), the third order power function of the test based on T is given by

$$B_T(\varepsilon, z^2)=P_{\theta_n}(T>z^2+c_n^{-2}d_T)$$
$$=1-G_{1,\delta}(z^2)-c_n^{-1}\sum_{j=0}^{3} B_j^{(T)}G_{1+2j,\delta}(z^2)$$
$$+c_n^{-2}\{R_T(\delta, z^2)-\Psi(\varepsilon, z^2)\}+o(c_n^{-2}), \tag{2.8}$$

where $\Psi(\varepsilon, z^2)$ is the same for all statistics in $\mathscr{F}$. By (2.5) and (2.7), the third order power function (2.8) depends on T only through a_1 and a_2.

3. RESULTS ON HIGHER ORDER POWER

First consider the second order power function which is given by the first three terms in (2.8). By (2.4), the coefficient of ε in each $B_j^{(T)}$ $(0\leqslant j\leqslant 3)$ is zero if and only if $a_2=-\frac{1}{3}I^{-3/2}K=a_2(LR)$ (vide (2.2a)). Hence exactly as in Mukerjee (1992, 1994), under the criterion of second-order local maximinity, the LR test is superior to the test given by any other statistic T in

$\mathscr{F}$ for which $a_2 \neq -\frac{1}{3}I^{-3/2}K$. On the other hand, if $a_2 = -\frac{1}{3}I^{-3/2}K$ for some statistic T then, by (2.2a), (2.4), the corresponding test will have the same second order power function as the LR test.

Turning to the criterion of average power, let

$$\bar{B}_T(\delta, z^2) = \tfrac{1}{2}\{B_T(\varepsilon, z^2) + B_T(-\varepsilon, z^2)\},$$

where $\delta = I\varepsilon^2$, and by (2.4), (2.7), (2.8), observe that

$$\bar{B}_T(\delta, z^2) = 1 - G_{1,\delta}(z^2) + c_n^{-2}\{R_T(\delta, z^2) - \bar{\Psi}(\delta, z^2)\} + o(c_n^{-2}), \quad (3.1)$$

$\bar{\Psi}(\delta, z^2)$ being the same for all statistics in $\mathscr{F}$. Thus, under this criterion, all statistics in $\mathscr{F}$ are equivalent up to the second order and a third-order comparison, on the basis of $R_T(\delta, z^2)$, is warranted. By (2.7),

$$R_T(\delta, z^2) = \delta V_T(z^2)\, h_1 + O(\delta^2), \quad (3.2)$$

where

$$V_T(z^2) = \tfrac{1}{2}\tilde{M}z^2(I^{-1}a_1 - \tfrac{1}{2}z^2a_1^2) + \tfrac{1}{2}z^4(I^{-3/2}Ja_2 - \tfrac{1}{2}z^2a_2^2). \quad (3.3)$$

In view of (3.1) and (3.2), up to the third order of approximation and for small δ, the behaviour of the average power function $\bar{B}_T(\delta, z^2)$ depends on T only through $V_T(z^2)$, i.e., $V_T(z^2)$ can be interpreted as a measure of third-order local average power associated with T. By (2.2b) and (3.3), $V_T(z^2) = 0$ for the score statistic. Now, as noted in Taniguchi (1991), the quantity $\tilde{M}$ can be interpreted in terms of Efron's (1975) statistical curvature and is non-negative. If $\tilde{M} = 0$ then, by (3.3), for any statistic T with $a_2 \neq 0$, $V_T(z^2)$ will be negative for large z^2 (i.e., for small α) provided a_2 is free from z^2 and, therefore, the test associated with T will be inferior to the score test, with regard to third-order local average power, for small test size. If $\tilde{M} > 0$ then, by (3.3), the same conclusion will hold for any statistic T with $(a_1, a_2) \neq (0, 0)$ and a_1, a_2 free from z^2. In particular, then by (2.2) for small α the score test will be superior to the LR, Wald's and modified Wald's tests in terms of third-order local average power.

Note that unlike Amari (1985) or Taniguchi (1991), we are comparing the tests in their original forms and not via their bias-corrected or Bartlett-type adjusted versions. Consequently, under the present approach and up to the third order of comparison, one can discriminate among the members of $\mathscr{F}$ even under models with $\tilde{M} = 0$. This may be contrasted with the findings in Amari (1985) or Taniguchi (1991). In the last two paragraphs we have noted some desirable properties of the LR and score tests which were earlier known to hold in the iid case (Mukerjee, 1994). The expression for $R_T(\delta, z^2)$, as in (2.7), is in fact more informative than the corresponding expression known so far in the iid setting when the tests are compared in their original forms—see e.g., Mukerjee (1994) who obtained only an expression analogous to (3.2). This enables us to study the third-order average power function (3.1) from another standpoint.

Specifically, we bring in the criterion of stringency with respect to third-order average power and, in view of (3.1), for each T in $\mathscr{F}$ consider the function

$$Q_T(\delta, z^2) = \{\sup_T R_T(\delta, z^2)\} - R_T(\delta, z^2),$$

where, for fixed δ and z^2, the supremum is over the class $\mathscr{F}$. By (2.7),

$$Q_T(\delta, z^2) = \tfrac{1}{4}\tilde{M}\, \delta z^2 \{I^{-1}(h_{1,\delta}/h_{3,\delta}) - a_1\}^2 h_{3,\delta} + \tfrac{1}{4}\delta z^4 [I^{-3/2}\{J(h_{1,\delta}/h_{3,\delta}) - \tfrac{1}{3}\delta z^{-2}(K+3J)\} - a_2]^2 h_{3,\delta}. \quad (3.4)$$

A most stringent test in $\mathscr{F}$, with respect to third order average power, is one which minimizes the supremum of $Q_T(\delta, z^2)$, with respect to $\delta > 0$, over $\mathscr{F}$. The analytical derivation of such a test is difficult but, as illustrated below, for a given model, using (3.4) and well-known closed-form expression for $h_{1,\delta}$ and $h_{3,\delta}$, a numerical solution can be obtained. One can as well employ (2.2) and (3.4) to study the performance of standard tests under this criterion.

Interestingly, the first term in (3.4) agrees with the corresponding expression in Amari (1985, pp. 180–181) while the second term is attributable to consideration of tests in their original forms as done here. Because of this second term, the "universal" nature of the most stringent solution that was observed in Amari's (1985) set-up holds no more in the present context.

EXAMPLE 3.1. Let $X^{(n)}$ represent a stretch of a Gaussian autoregressive process of order 1 with mean zero and spectral density

$$\Phi_\theta(\mu) = \sigma^2/\{2\pi(1 - 2\theta\cos\mu + \theta^2)\}, \qquad \mu \in [-\pi, \pi],$$

where σ^2 (>0) is known and $\theta(|\theta| < 1)$ is an unknown parameter. With $C_n = n^{1/2}$, from Proposition 1 in Taniguchi (1986) here

$$I = (1-\theta_0^2)^{-1}, \qquad K = -3J = 6\theta_0(1-\theta_0^2)^{-2}, \qquad \tilde{M} = 2(1-\theta_0^2)^{-2}. \quad (3.5)$$

(i) By (2.2) and (3.5), the LR and Wald tests have identical second-order power and, if $\theta_0 \neq 0$, they are superior to the score and modified Wald tests in terms of second-order local maximinity. On the other hand, if $\theta_0 = 0$ then all the four tests are equivalent up to the second order of comparison.

(ii) Consider next the criterion of third-order local average power. Let $V_{\text{LR}}(z^2)$, $V_{\text{Wald}}(z^2)$ and $V_{m\,\text{Wald}}(z^2)$ denote the expressions for $V_T(z^2)$ when T represents the LR, Wald and modified Wald statistics respectively. Using (2.2) and (3.5) in (3.3), it can be seen that in this example, each of these three quantities is negative whenever $z^2 > 2$. Thus for $\alpha = 0.10$, 0.05 or 0.01, the score is better than the LR, Wald and modified Wald tests with regard to third-order local average power.

(iii) We next illustrate the issue of stringency taking $\alpha = 0.05$ and $\theta_0 = 0$. Then by (3.5), $I = 1$, $K = J = 0$, $\tilde{M} = 2$ so that by (2.2),

$$a_1(\text{LR}) = 1, \qquad a_1(\text{Score}) = 0, \qquad a_1(\text{Wald}) = a_1(\text{mod Wald}) = 2, \tag{3.6}$$

and a_2 equals zero for each of these statistics. Also, by (3.4),

$$Q_T(\delta, z^2) = \tfrac{1}{2}\delta z^2 \{(h_{1,\delta}/h_{3,\delta}) - a_1\}^2\, h_{3,\delta} + \tfrac{1}{4}\delta z^4 a_2^2 h_{3,\delta} \tag{3.7}$$

where $z^2 = 3.8416$. Hence, in order to find a most stringent test with respect to third-order average power, it is enough to restrict to statistics for which $a_2 = 0$. Numerical methods show that for such statistics the optimal choice of a_1 so as to minimize sup $Q_T(\delta, z^2)$ over $\delta > 0$ is $a_1 = 1.24$. Table 3.1 summarizes some more related calculations. In this table, δ^* represents the value of δ which maximizes $Q_T(\delta, z^2)$ for given T.

The entries in column (4) of Table 3.1 are proportional to the corresponding entries in Table 6.1 of Amari (1985). In view of the discussion just preceding this example, this is expected for $\theta_0 = 0$ since then we are effectively ignoring the second term in the expression for $Q_T(\delta, z^2)$ and the findings can be quite different for $\theta_0 \neq 0$. Incidentally, if $\alpha = 0.05$, $\theta_0 = 0$, then by (2.7), (3.5) and (3.6), one can check that the score test in superior, with regard to third-order average power, to the LR test for $0 < \delta \leqslant 0.86$ and to Wald or modified Wald tests for $0 < \delta \leqslant 3.83$. Consequently, Table 3.1 (see column under δ^*) is not in conflict with what has been stated under (ii) above.

TABLE 3.1

Values of sup $Q_T(\delta, z^2)$ over $\delta > 0$ under the AR(1) Model
($\alpha = 0.05$, $\theta_0 = 0$)

T	a_1	a_2	$\sup_{\delta > 0} Q_T(\delta, z^2)$	δ^*
LR	1	0	0.253	12.41
Score	0	0	1.569	8.82
Wald/modified Wald	2	0	0.804	2.82
Most stringent	1.24	0	0.131	14.04

EXAMPLE 3.2. Let $X^{(n)}$ represent a stretch of a Gaussian moving average process of order 1 with mean zero and spectral density

$$\Phi_\theta^*(\mu) = \{\sigma^2/(2\pi)\}(1 - 2\theta\cos\mu + \theta^2), \qquad \mu \in [-\pi, \pi],$$

where σ^2 (>0) is known and $\theta(|\theta| < 1)$ is an unknown parameter. With $c_n = n^{1/2}$, by Proposition 1 in Taniguchi (1986) here

$$I = (1 - \theta_0^2)^{-1}, \quad J = -\tfrac{2}{3}K = 4\theta_0(1 - \theta_0^2)^{-2}, \quad \tilde{M} = (6 - 2\theta_0^2)(1 - \theta_0^2)^{-3}. \tag{3.8}$$

(i) By (2.2) and (3.8), the LR and modified Wald tests have identical second-order power and, if $\theta_0 \neq 0$, they are superior to the score and Wald tests in terms of second-order local maximinity. On the other hand, if $\theta_0 = 0$ then these four tests are equivalent up to the second order of comparison.

(ii) By (2.2), (3.4), and (3.8), it can be seen that $V_{\mathrm{LR}}(z^2)$, $V_{\mathrm{Wald}}(z^2)$ and $V_{m\,\mathrm{Wald}}(z^2)$ are all negative for $\alpha = 0.05$ or 0.01. Thus for these values of α, the superiority of the score test, with regard to third-order local average power, follows.

(iii) If $\alpha = 0.05$ and $\theta_0 = 0$ then by (3.4) and (3.8),

$$Q_T(\delta, z^2) = \tfrac{3}{2}\delta z^2\{(h_{1,\delta}/h_{3,\delta}) - a_1\}^2 h_{3,\delta} + \tfrac{1}{4}\delta z^4 a_2^2 h_{3,\delta}.$$

Comparing the above with (3.7), as before a statistic with $a_1 = 1.24$, $a_2 = 0$ gives a most stringent test with respect to third-order average power. Furthermore, Table 3.1 continues to remain valid provided each entry in its last but one column is multiplied by 3.

4. COMPARISON OF BARTLETT-TYPE ADJUSTMENTS

As indicated earlier, Taniguchi (1991) considered a subclass $\mathscr{F}_0$ of $\mathscr{F}$ consisting of those statistics T for which b_6 equals zero (vide (2.1)). For each member of $\mathscr{F}_0$ he suggested a Bartlett-type adjustment. At about the same time, Chandra and Mukerjee (1991) and Cordeiro and Ferrari (1991) proposed Bartlett-type adjustments which are applicable to each statistic in $\mathscr{F}_0$. For any statistic T in $\mathscr{F}_0$ the Bartlett-type adjusted versions due to Chandra and Mukerjee (1991), Cordeiro and Ferrari (1991) and Taniguchi (1991) will be denoted by T_1, T_2 and T_3 respectively. While the null distribution, up to $o(c_n^{-2})$, of each of T_1, T_2 and T_3 is central chi-square with 1 d.f., their power properties may not be identical up to that order. Therefore, given T, a comparative power study will help in choosing one of the adjusted versions depending on the context. Rao and Mukerjee (1995) considered this problem when T represents the score statistic. The present general framework enables us to handle all members of $\mathscr{F}_0$.

Given any statistic T in $\mathscr{F}_0$, it can be seen that each of T_1, T_2 and T_3 belongs to $\mathscr{F}$ and that the expressions for a_1 and a_2 associated with them are given by

$$a_1(T_1)=a_1(T), \qquad a_2(T_1)=-\tfrac{1}{3}I^{-3/2}K, \tag{4.1a}$$

$$a_1(T_2)=a_1(T), \qquad a_2(T_2)=a_2(T), \tag{4.1b}$$

$$a_1(T_3)=a_1(T), \qquad a_2(T_3)=-\tfrac{1}{3}I^{-3/2}K, \tag{4.1c}$$

where $a_1(T)$ and $a_2(T)$ correspond to T. The relations (4.1b) and (4.1c) follow from Cordeiro and Ferrari's (1991) equation (14) and Taniguchi's (1991) equation (3.5) respectively. The relation (4.1a) can be proved essentially along the line of Chandra and Mukerjee's (1991) Theorem 2.1 and we omit the details. By (4.1), T_2 differs from T by $O(c_n^{-2})$ while this may not be the case with T_1 or T_3 in general. Hence, compared to T_1 or T_3 the adjustment T_2 perturbs T to a smaller extent (cf. Rao and Mukerjee, 1995). From Cordeiro and Ferrari's (1991) equation (14), note that T_2 may belong to $\mathscr{F}-\mathscr{F}_0$ and this justifies our consideration of the wider class $\mathscr{F}$.

From (2.4), (2.7) and (2.8), recall that the third-order power function depends on any statistic only via a_1 and a_2. Hence by (4.1), the third-order power functions corresponding to T_3 and T are identical with those associated with T_1 and T_2 respectively. Thus, it will suffice to compare T_1 and T_2. In fact, if $a_2(T)=-\frac{1}{3}I^{-3/2}K$ then, by (4.1), T, T_1, T_2 and T_3 will lead to identical power functions up to the third order. Therefore, to avoid trivialities, let $a_2(T)\neq -\frac{1}{3}I^{-3/2}K$. Then by (2.2a), (4.1), $a_2(T_1)=a_2(LR)\neq a_2(T_2)$ so that, as in the first paragraph of Section 3, T_1 (or equivalently, T_3) is superior to T_2 with regard to second-order local maximinity. Furthermore, by (3.3) and (4.1),

$$V_{T_1}(z^2)-V_{T_2}(z^2)=\tfrac{1}{36}z^4I^{-3}s_T(z^2),$$

where

$$s_T(z^2)=z^2\{9I^3a_2^2(T)-K^2\}-6J\{K+3I^{3/2}a_2(T)\}, \tag{4.2}$$

and, under the criterion of third-order local average power, T_1 is superior (inferior) to T_2 if $s_T(z^2)$ is positive (negative). One can as well employ (2.7), in conjunction with (4.1), to compare the entire third order average power functions of T_1 and T_2.

Some of the results in Rao and Mukerjee (1995), which relate to the scalar parameter case, follow from the above considerations. For further elucidation, let T represent the Wald statistic for the rest of this section. Then by (2.2c) and (4.2), $a_2(T) \neq -\frac{1}{3}I^{-3/2}K$ if and only if $K+3J \neq 0$, and

$$s_T(z^2) = z^2(9J^2 - K^2) - 6J(K+3J). \tag{4.3}$$

EXAMPLE 3.1 (continued). By (3.5), here $K+3J=0$ and T, T_1, T_2 and T_3 lead to identical power functions whatever θ_0 might be.

EXAMPLE 3.2 (continued). By (3.8), here $K+3J \neq 0$ for $\theta_0 \neq 0$. Hence for $\theta_0 \neq 0$, T_1 (or T_3) is better than T_2 in terms of second-order local maximinity. Also, then by (3.8) and (4.3), $s_T(z^2) = 9J^2(\frac{3}{4}z^2 - 1)$, which is positive for $\alpha = 0.10$, 0.05 or 0.01. Thus, for these values of α, T_1 (or T_3) is superior to T_2 also under the criterion of third-order local average power. In fact, by (2.2c), (2.7), (3.8) and (4.1),

$$R_{T_1}(\delta, z^2) - R_{T_2}(\delta, z^2) = \delta z^2 \theta_0^2 (1-\theta_0^2)^{-1} \{(2\delta + 3z^2)\, h_{3,\delta} - 4z^2 h_{1,\delta}\}$$

and if $\alpha = 0.10$, 0.05 or 0.01 and $\theta_0 \neq 0$ then this is seen to be positive for every $\delta > 0$. Hence, for these values of α, T_1 (or T_3) dominates T_2 uniformly in δ (>0) with respect to third order average power. Consequently, in this example, if $\theta_0 \neq 0$ then there is enough reason to prefer the adjustment T_1 (or T_3) for the Wald statistic to T_2.

In the present paper, we have worked under the framework of contiguous alternative. It will be of interest to compare the tests from consideration of Bahadur efficiency. It is, however, likely that such a study will involve the use of entirely different tools.

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